A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

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Theorem

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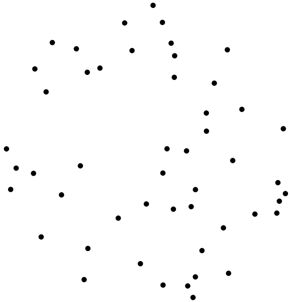

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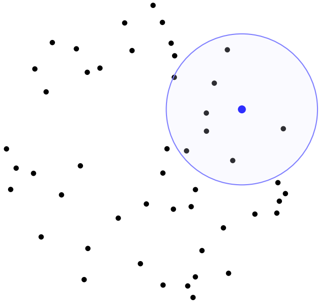

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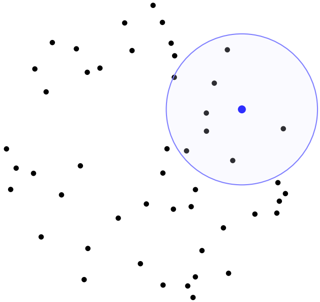

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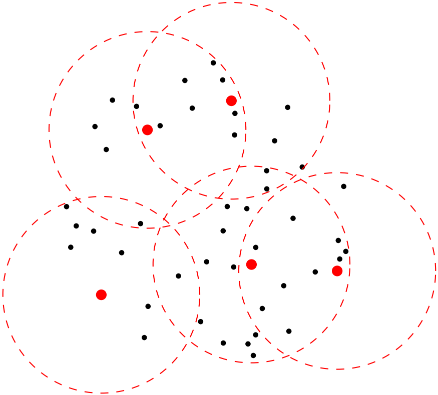

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- Several similar statements in various contexts.

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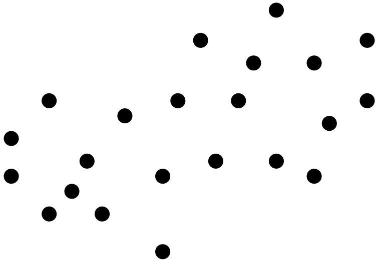

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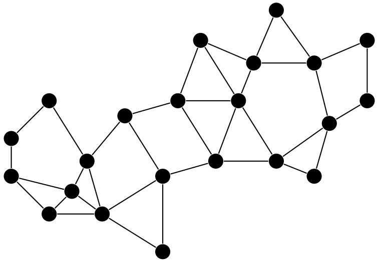

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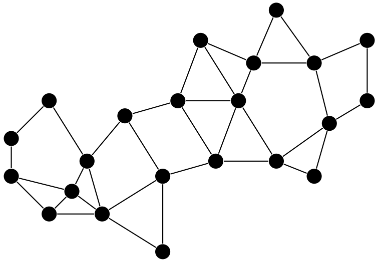

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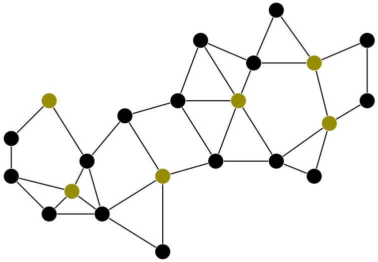

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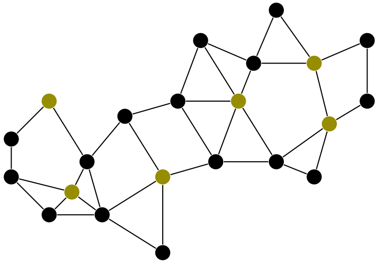

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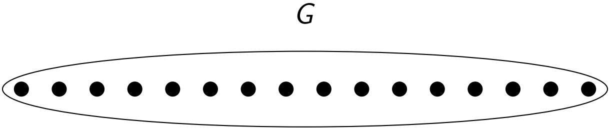

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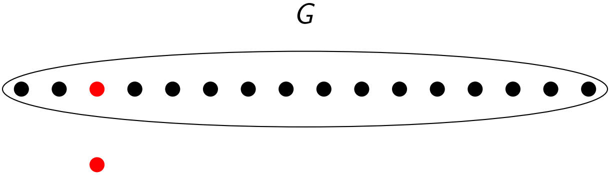

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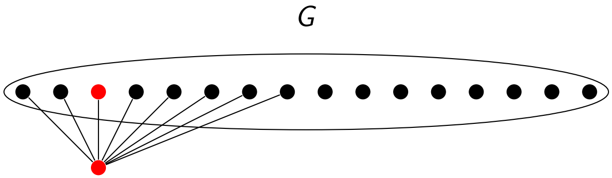

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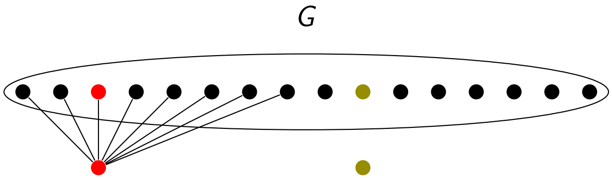

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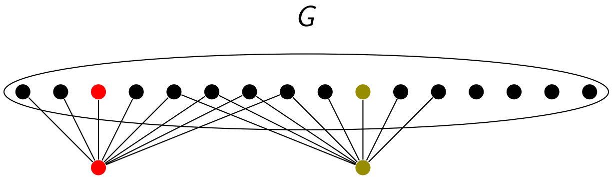

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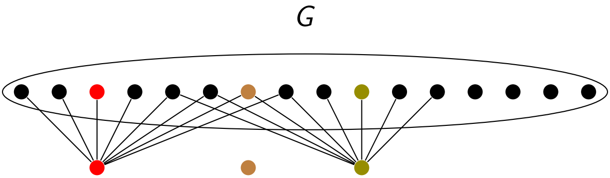

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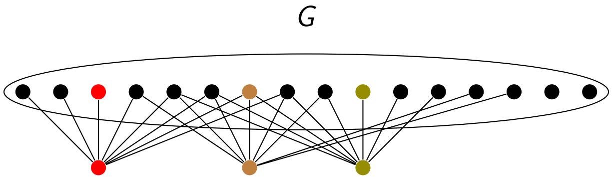

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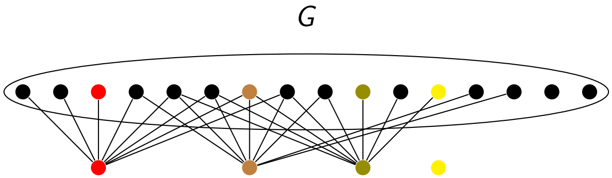

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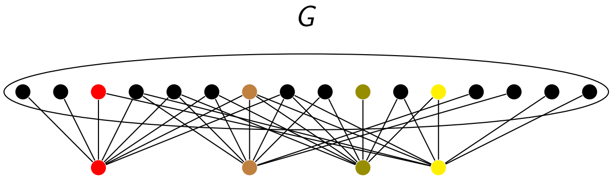

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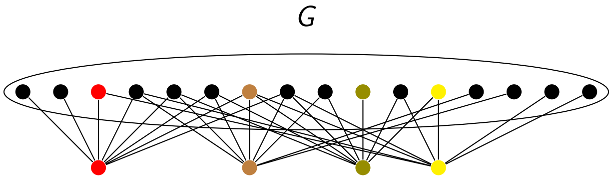

If $G \sim G(n/2, n/2, 1/2)$ then $\delta(G) \approx n/4$ but $\gamma(G) \approx \log(n)$.

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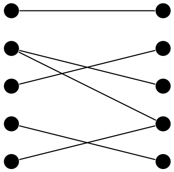

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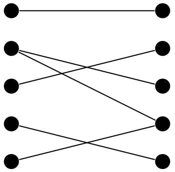

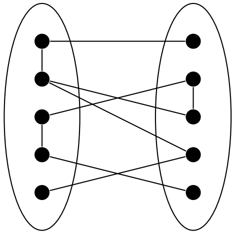

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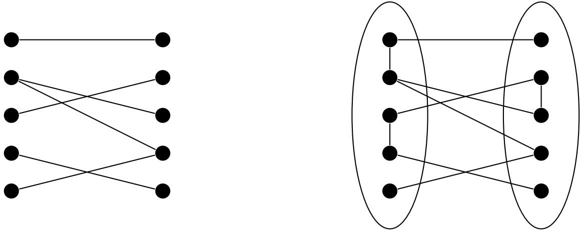

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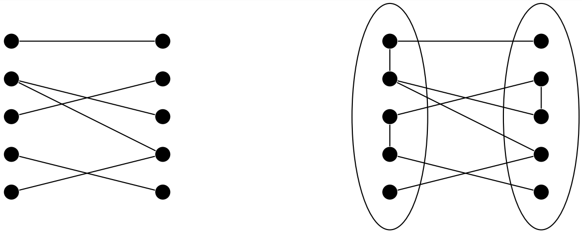

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Theorem [Haussler, Welzl '89]

If G has VC-dimension d and minimum degree δn then G has a dominating set of size $f(\delta, d)$.

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Proposition [Folklore]

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Every n-vertex graph is a distance-threshold graph in $\mathbb{R}^n$ $\Rightarrow$ needs to depend on N.

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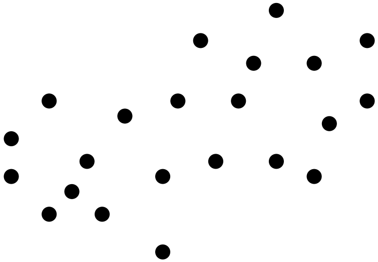

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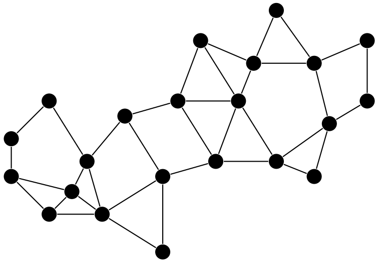

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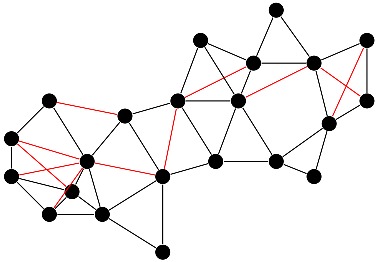

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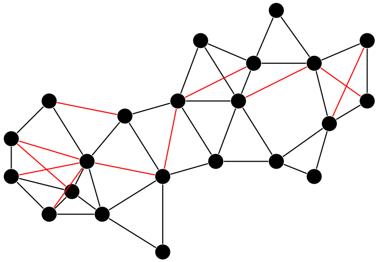

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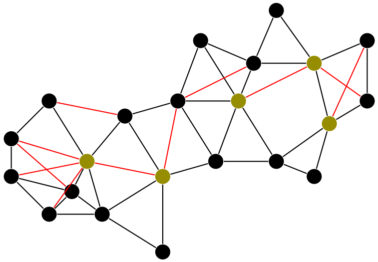

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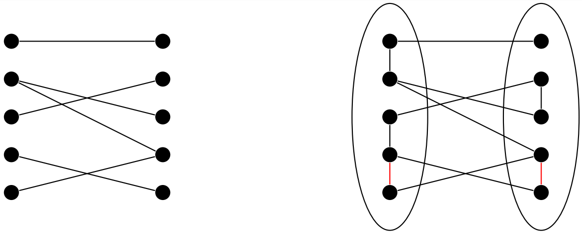

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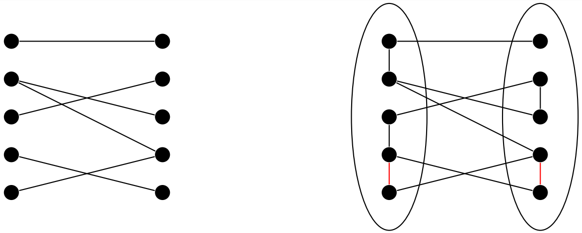

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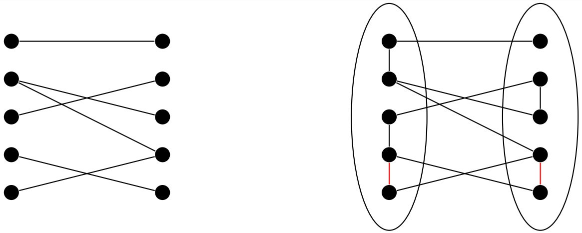

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Theorem [Rosenblatt '58]

Every $(\tau \cdot N, (\tau + \varepsilon) \cdot N)$-distance-threshold trigraph in $\{0,1\}^N$ has VC-dimension $O(1/\varepsilon^2)$.

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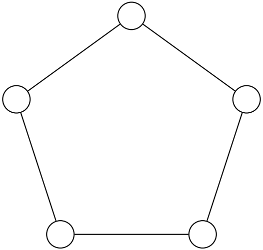

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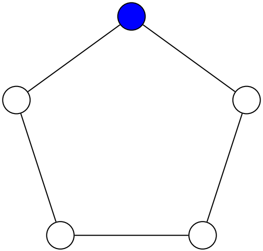

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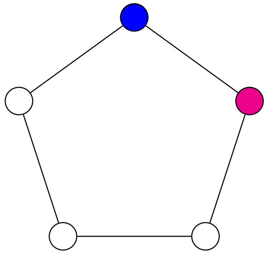

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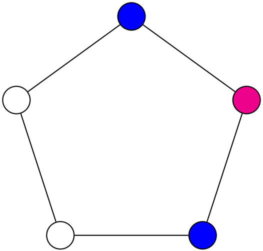

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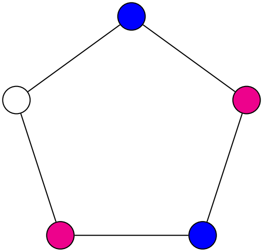

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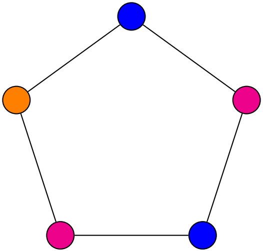

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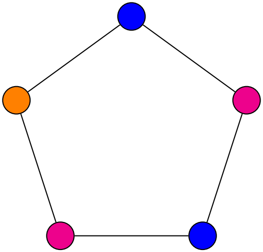

Theorem [Mycielski, Zykov, Tutte, Erdős...]

There exist triangle-free graphs with arbitrarily large χ.

The chromatic threshold of triangle-free graphs (1/2)

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Conjecture [Erdős, Simonovits '73]

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Every triangle-free graph with minimum degree $(1/3 + \varepsilon) \cdot n$ has chromatic number $\leq f(\varepsilon)$.

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A quick proof

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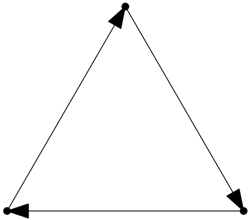

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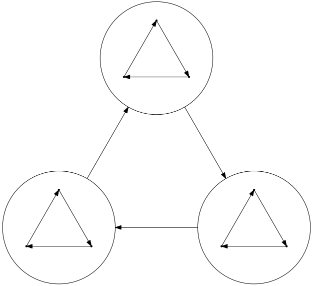

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Theorem [Jiang, Munagala, Wang '20] , [Nguyen, Song, Lin '25]

There is a coalition of order 5.

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Then, there exists a black/red dominating set X of size $f(\varepsilon)$.

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Thank you!