A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

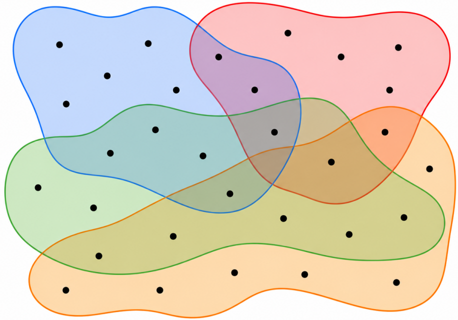
Romain Bourneuf
LaBRI & LIP

Joint work with Pierre Charbit (IRIF)
and Stéphan Thomassé (ENS de Lyon)

July 9, 2026

Hitting set

Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a collection of nonempty subsets of a set V.



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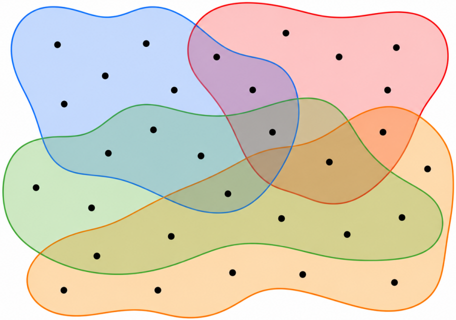

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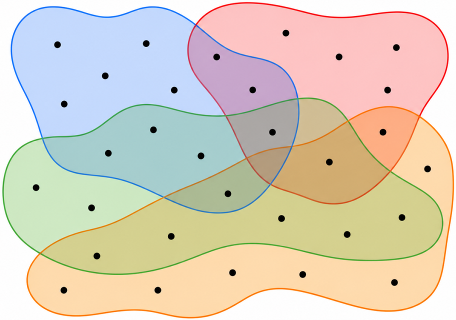
$\tau(\mathcal{S}) =$ minimum size of such a set X.



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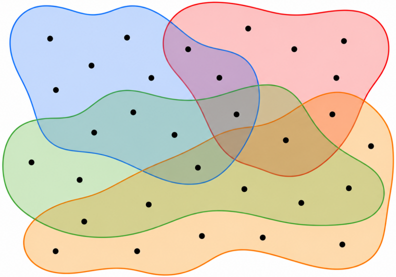

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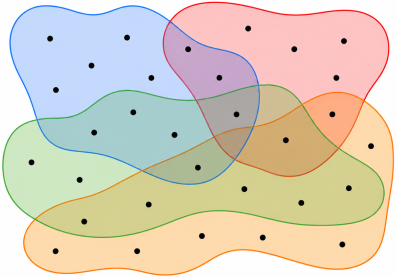
Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a collection of nonempty subsets of a set V. A *fractional hitting set* is a function $w : V \to [0, 1]$ such that $w(S_i) \geq 1$ for every $S_i \in \mathcal{S}$.



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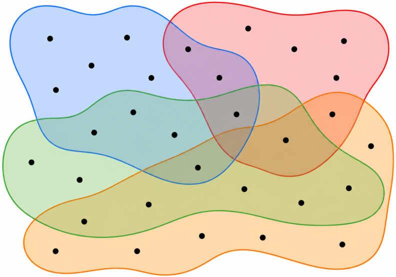

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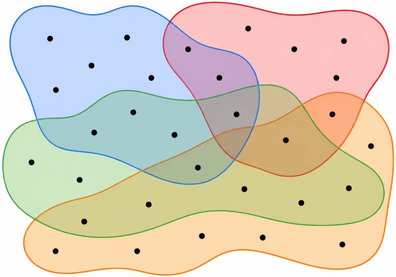

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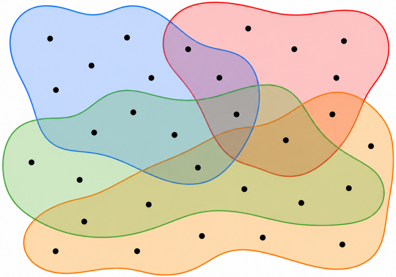
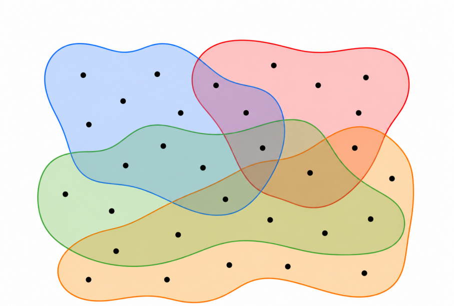
Do we have $\tau(\mathcal{S}) \leq f(\tau^*(\mathcal{S}))$?



Integrality gap

Let $\mathcal{S}$ consist of all sets of size $n/2$.



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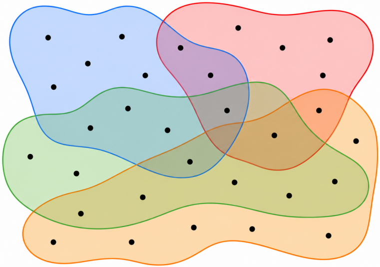

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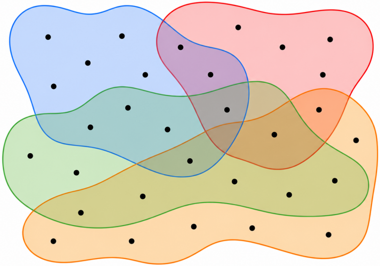

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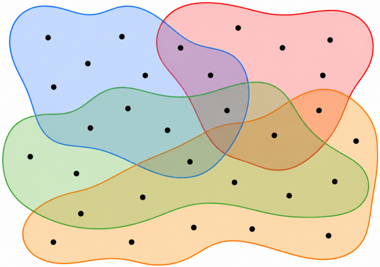

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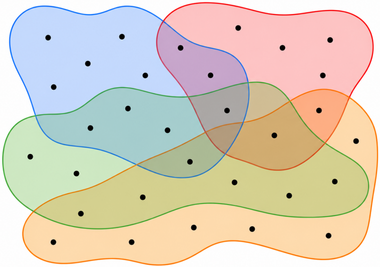

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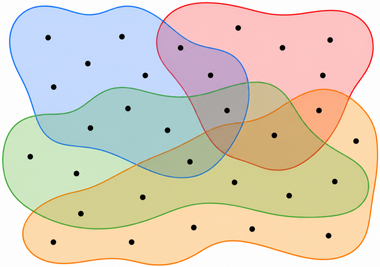

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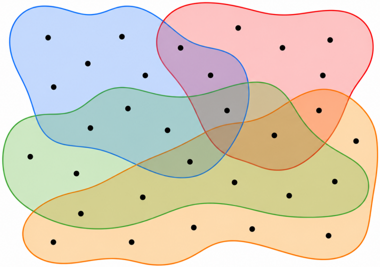
Classes of set systems are closed under vertex deletion and set deletion.



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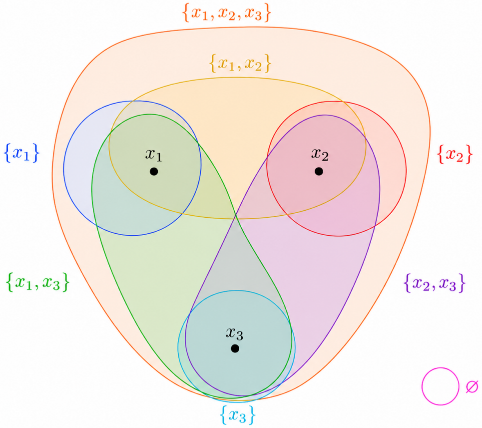

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Introduced by Vapnik & Chervonenkis in 71 in the context of learning. Multiple applications in combinatorics, computational geometry...

Small hitting sets

Theorem [Vapnik, Chervonenkis '71; Haussler,Welzl '89]

Let $\mathcal{S}$ be a set system of VC-dimension at most d. Then,

$$\tau = \mathcal{O}(d \cdot \tau^* \cdot \log(\tau^*)).$$

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Theorem [Vapnik, Chervonenkis '71]

Let $\mathcal{C}$ be a class of set systems. Then,
$\mathcal{C}$ satisfies $\tau \leq f(\tau^*) \iff \mathcal{C}$ is not the class of all set systems.

Geometric application

Theorem [Folklore]

Finite set $V \subseteq \mathbb{R}^N$ with $|B(v,1) \cap V| \geq \delta |V|$ for every $v \in V$.

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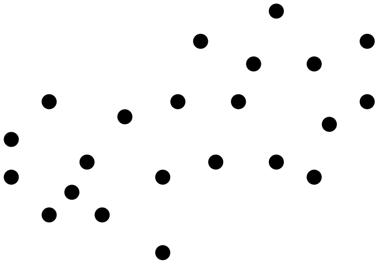

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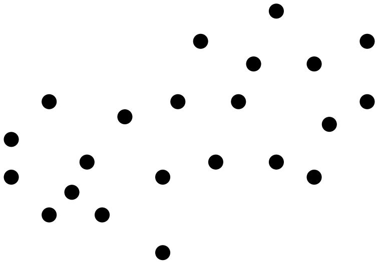

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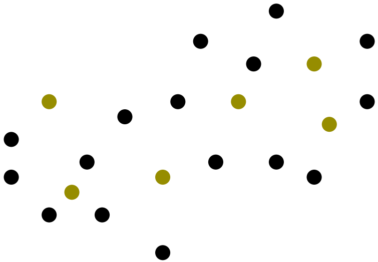

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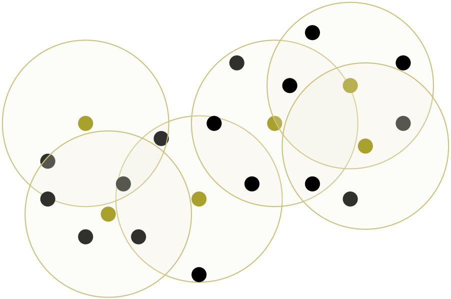

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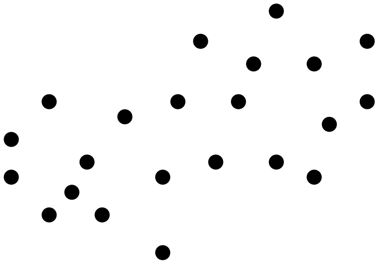

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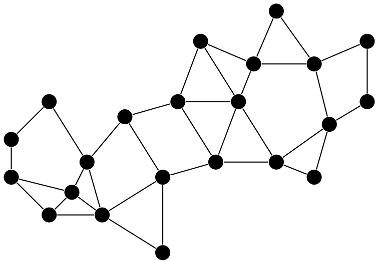

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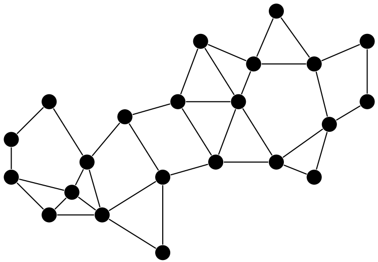

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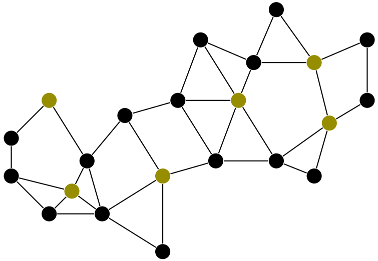

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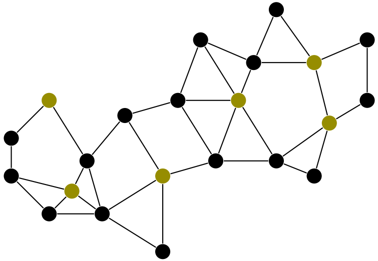

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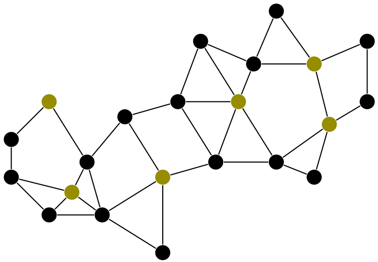

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VC-dim$(\mathcal{S}) \leq N + 1$.

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Every set system on n elements can be represented in $\mathbb{R}^n \Rightarrow$ needs to depend on N.

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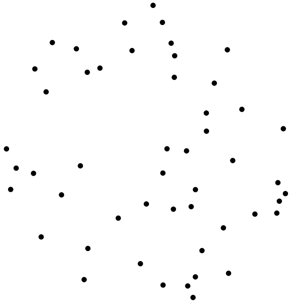

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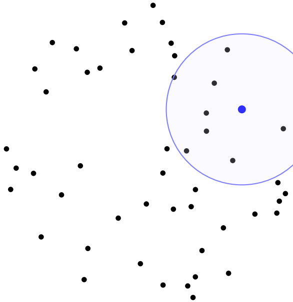

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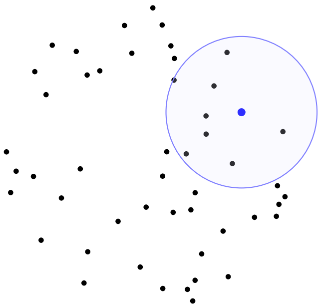

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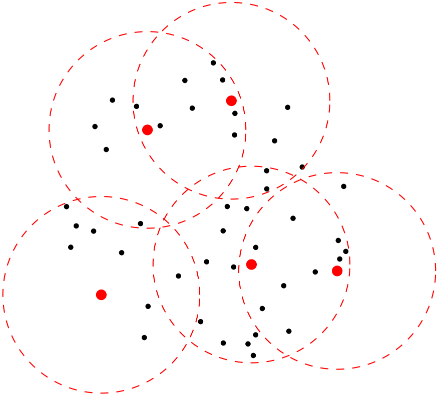

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First observations

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- Several similar statements in various contexts.

Trigraphs

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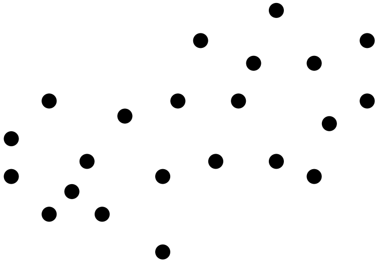

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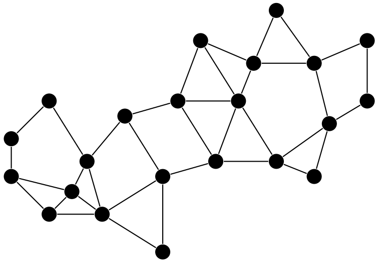

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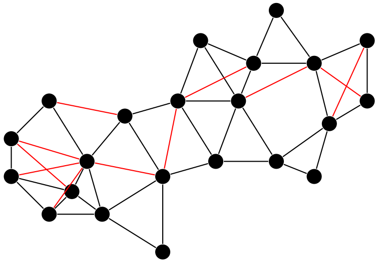

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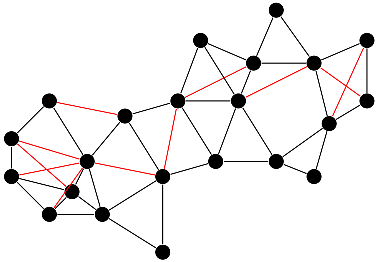

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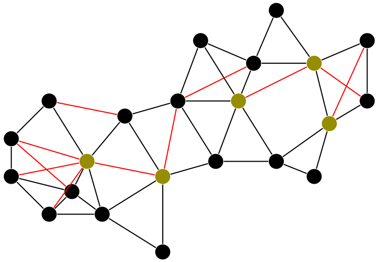

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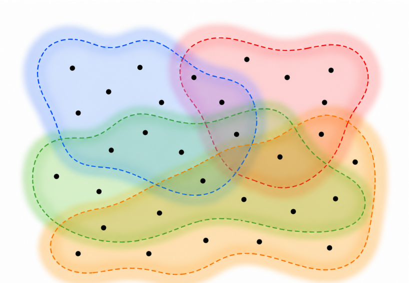
Want: black/red dominating set X of size $f(\delta, \varepsilon)$.

Partial set systems

A *partial set* is a tripartition (B, R, W) of V.



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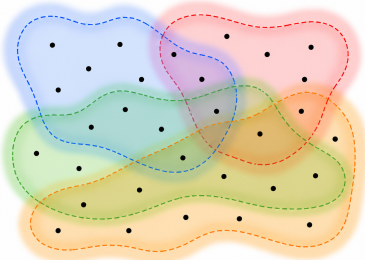

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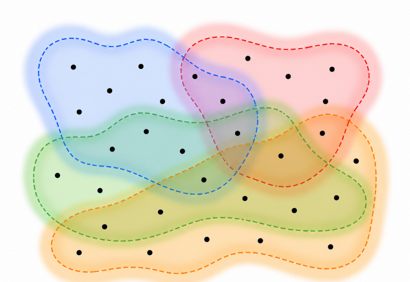

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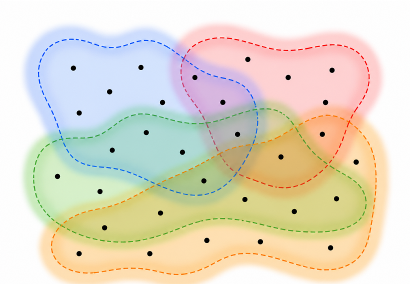

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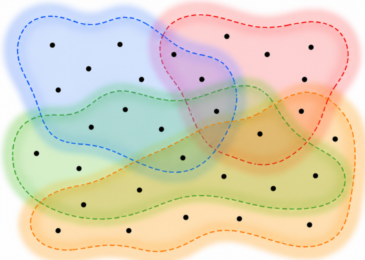

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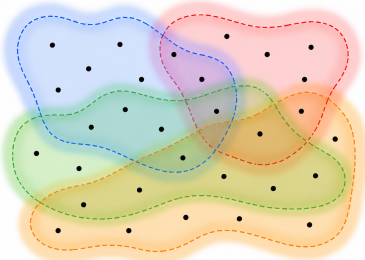

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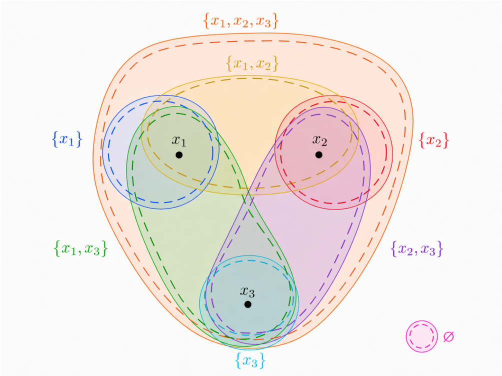

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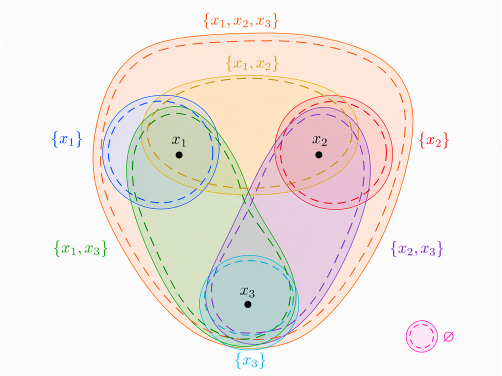

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Theorem [Alon, Hanneke, Holzman, Moran '21], [BCT '25]

Let $\mathcal{S}$ be a partial set system of VC-dimension at most d. Then,

$$\tau = \mathcal{O}(d \cdot \tau^* \cdot \log(\tau^*)).$$

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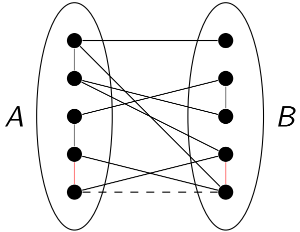

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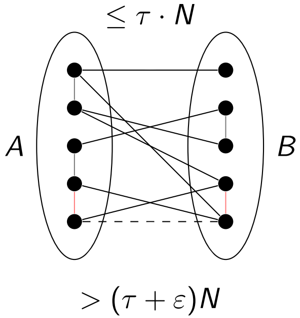

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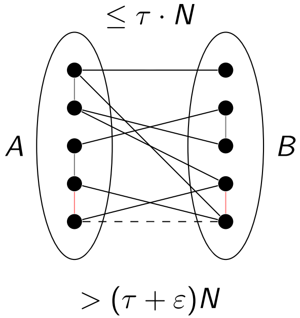

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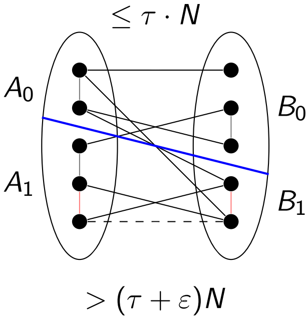

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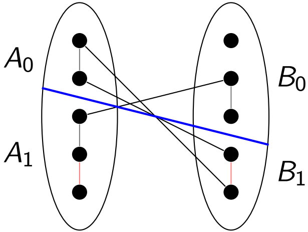

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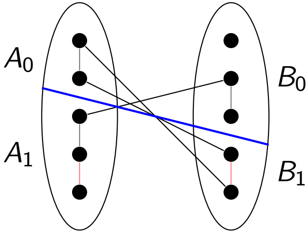

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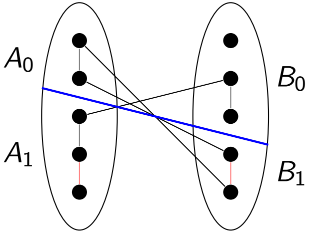

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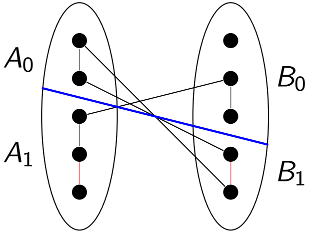

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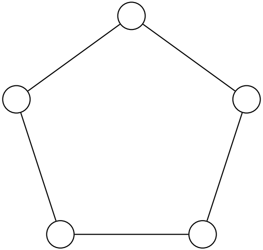

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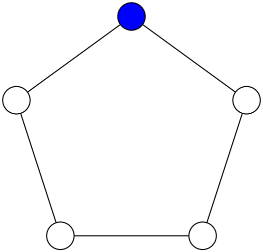

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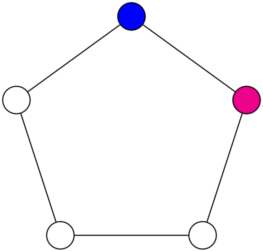

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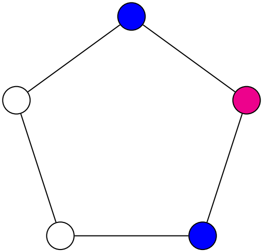

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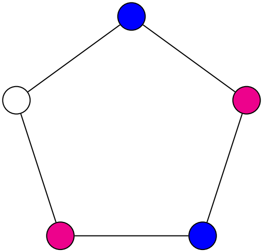

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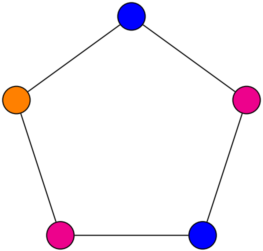

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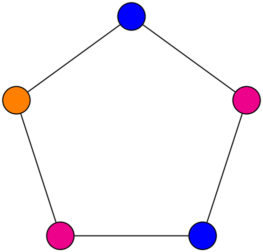

Theorem [Mycielski, Zykov, Tutte, Erdős...]

There exist triangle-free graphs with arbitrarily large χ.

The chromatic threshold of triangle-free graphs (1/2)

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What is the smallest δ such that every triangle-free G with minimum degree at least δn satisfies $\chi(G) \leq f(\delta)$?

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Conjecture [Erdős, Simonovits '73]

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$\mathcal{F}$ – family of subsets of $[n]$ s.t.:
$\forall S \in \mathcal{F}$, at least $\delta|\mathcal{F}|$ sets $S' \in \mathcal{F}$ s.t. $S \cap S' = \emptyset$.
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If $|N(u) \cap N(x)|, |N(v) \cap N(x)| \leq \varepsilon \cdot n$ and $uv \in E(G)$ then $|N(x) \cup N(u) \cup N(v)| \geq 3 \cdot (1/3 + \varepsilon) \cdot n - 2 \cdot \varepsilon n > n$, a contradiction.

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"Proof by obfuscation"

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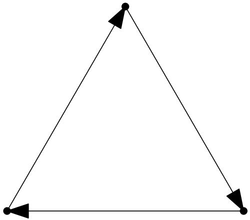

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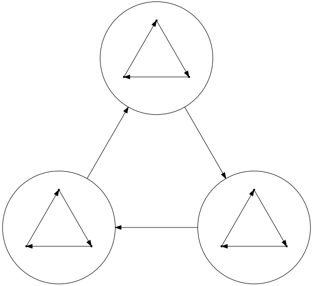

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There is a coalition of size 5.

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Every tournament has a fractional dominating set of weight at most 2.

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Then, there exists a black/red dominating set X of size $f(\varepsilon)$.

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Thank you!