A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

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LaBRI, LIP

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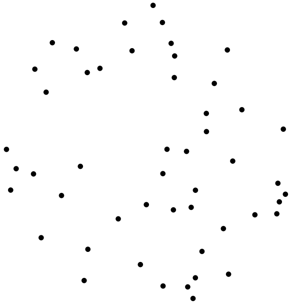
Dense Neighborhood Lemma

Theorem

Finite set $V \subseteq \mathbb{R}^N$ with $|B(v,1) \cap V| \geq \delta|V|$ for every $v \in V$.



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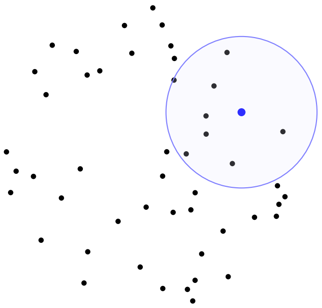

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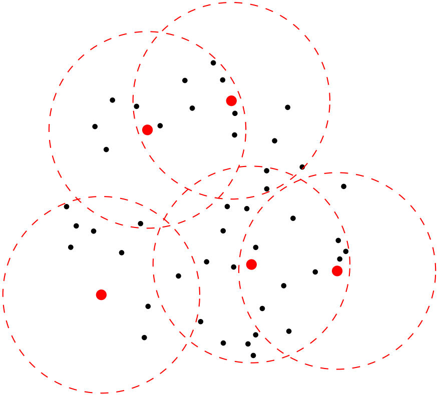

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- Similar statements in many other settings.

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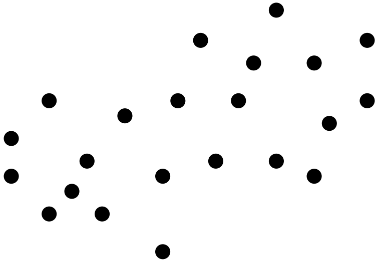

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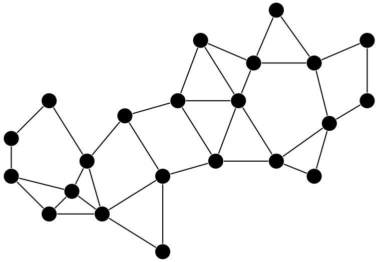

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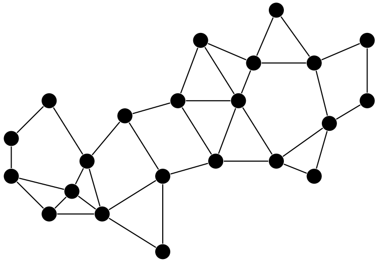

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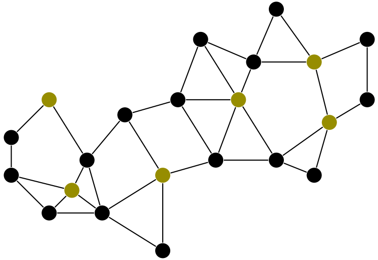

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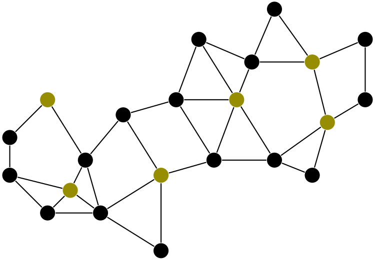

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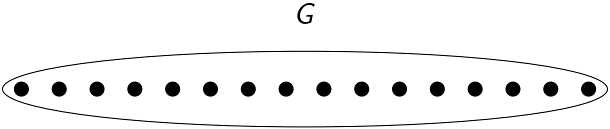

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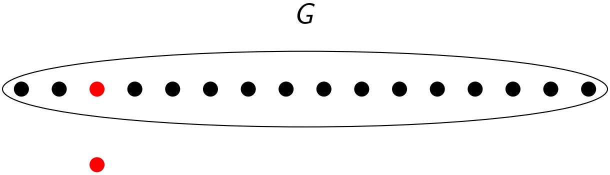

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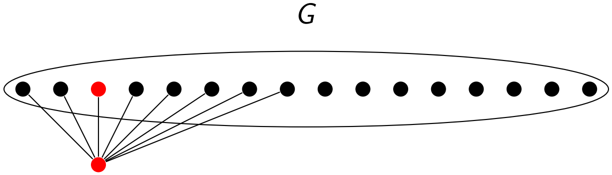

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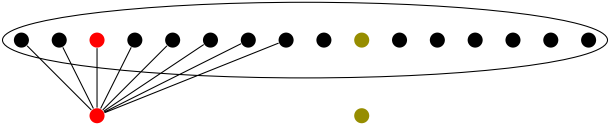

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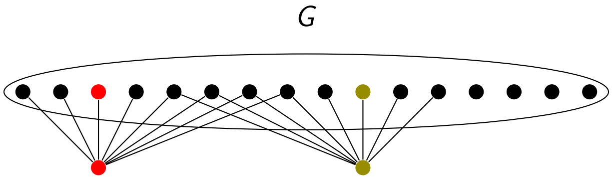

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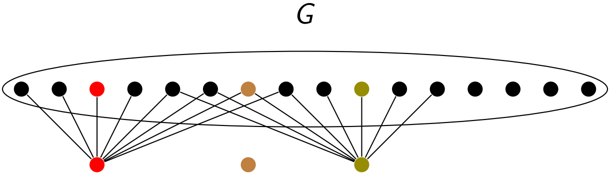

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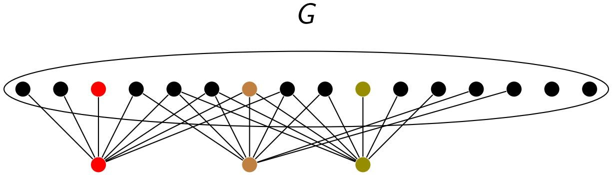

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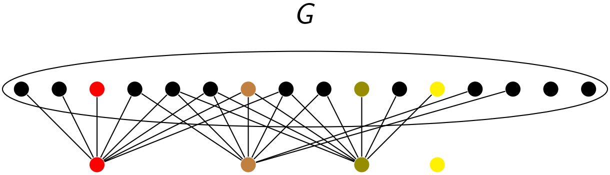

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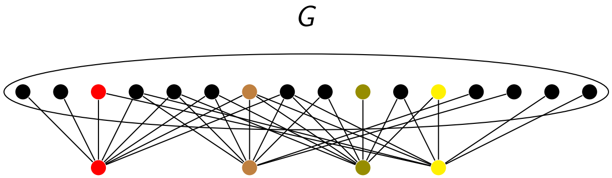

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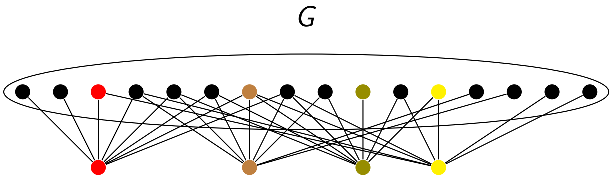

If $G \sim G(n/2, n/2, 1/2)$ then $\delta(G) \approx n/4$ but $\gamma(G) \approx \log(n)$.

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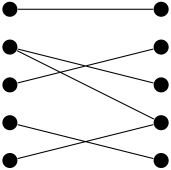

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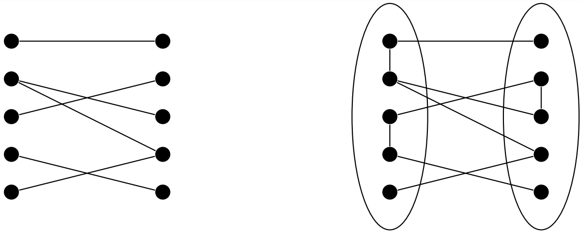

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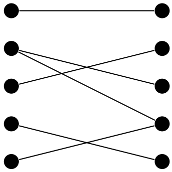

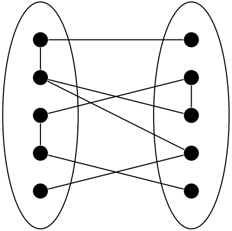

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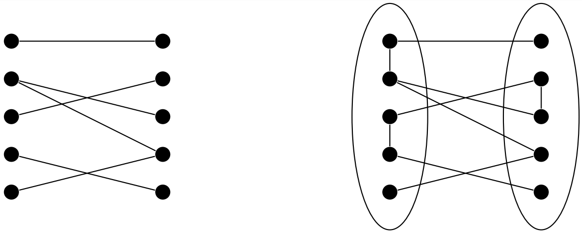

$\text{VC-dim}(\mathcal{C}) =$ "size" of the smallest missing subgraph.

Theorem [Haussler, Welzl '89]

If G has VC-dimension d and minimum degree δn then G has a dominating set of size $f(\delta, d)$.

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Every n-vertex graph is a threshold graph in $\mathbb{R}^n \Rightarrow$ needs to depend on N.

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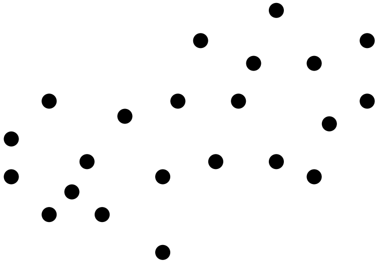

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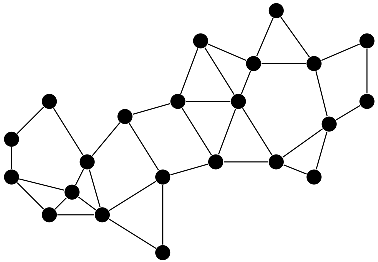

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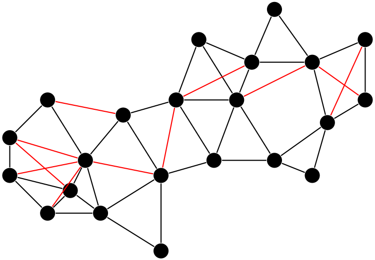

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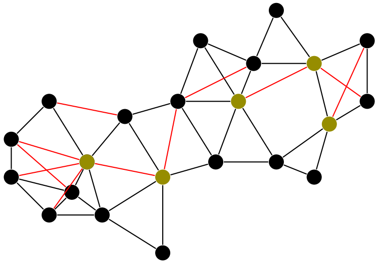

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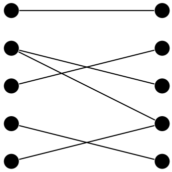

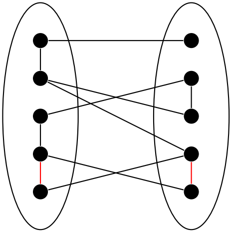

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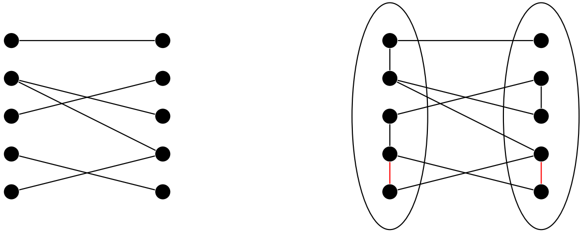

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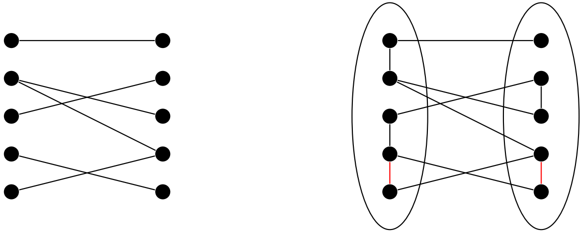

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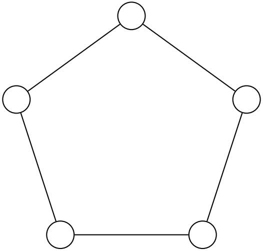

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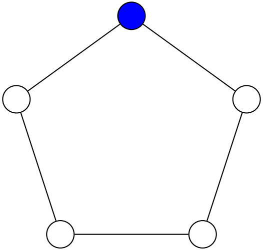

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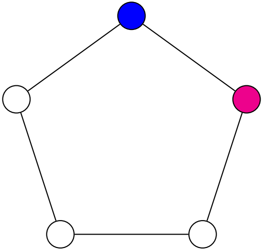

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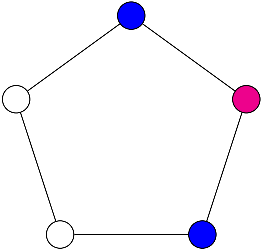

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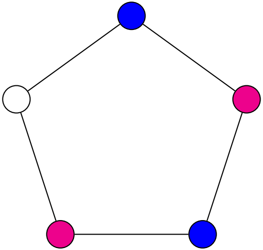

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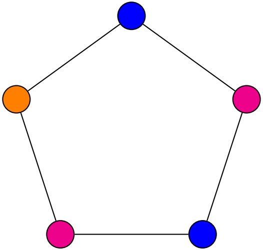

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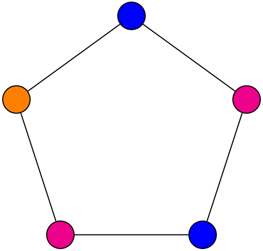

Theorem [Mycielski, Zykov, Tutte, Erdős...]

There exist triangle-free graphs with arbitrarily large χ.

The chromatic threshold of triangle-free graphs (1/2)

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Conjecture [Erdős, Simonovits '73]

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Theorem [Brandt, Thomassé '04]

Every triangle-free graph with minimum degree $> n/3$ has chromatic number at most 4.

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If u, v are dominated by $x \in X$ and are neighbors in G then $|N(x) \cup N(u) \cup N(v)| \geq 3 \cdot (1/3 + \varepsilon) \cdot n - 2 \cdot \varepsilon n > n$, a contradiction.

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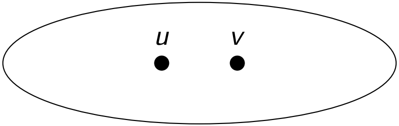
$V \subseteq \mathbb{R}^N$, set of n points. There is a partition of V into $2^{\text{poly}(1/\varepsilon)}$ clusters such that if u, v are in the same cluster,



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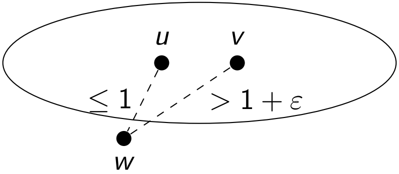

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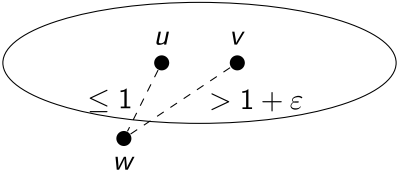

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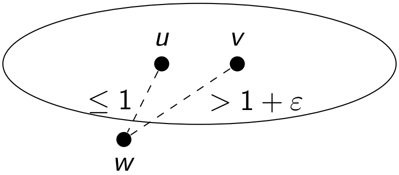

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Then, $|N(u) \cup N(v) \cup N(w) \cup N(w')| \geq 4 \cdot (1/4 + \varepsilon) \cdot n - \binom{4}{2} \cdot \varepsilon \cdot n/2 > n$, a contradiction.

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There is always an ε-fair set of recipients of size $O(1/\varepsilon^2)$.

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Thank you!