A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

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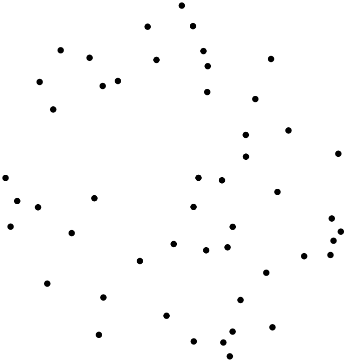
Joint work with Pierre Charbit (IRIF)
and Stéphan Thomassé (LIP)

December 16, 2025

Euclidean Setting

Finite $V \subseteq \mathbb{R}^N$



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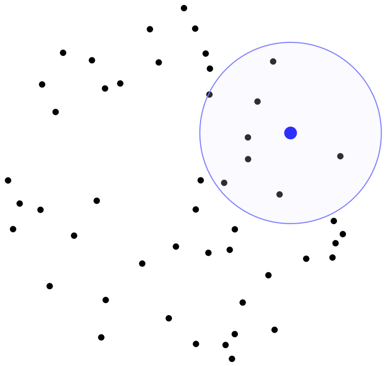

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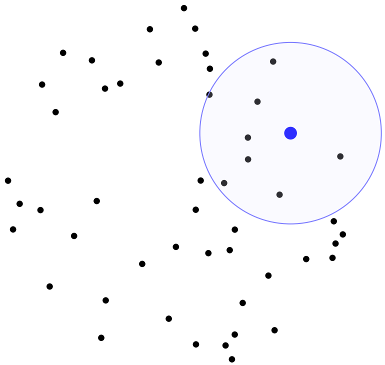

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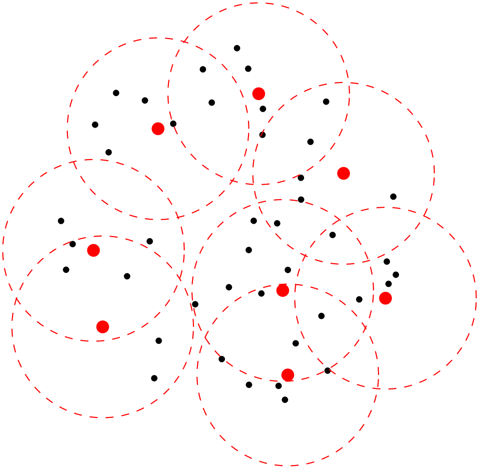

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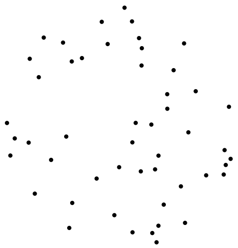

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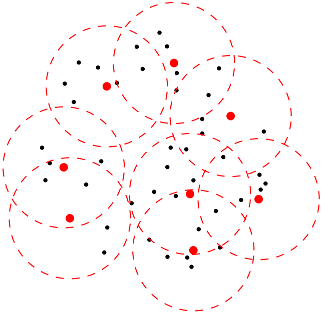

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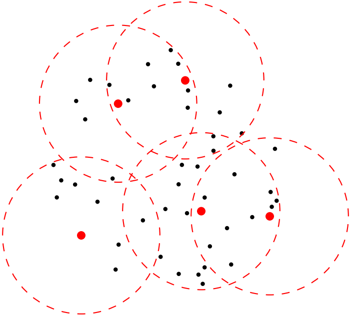

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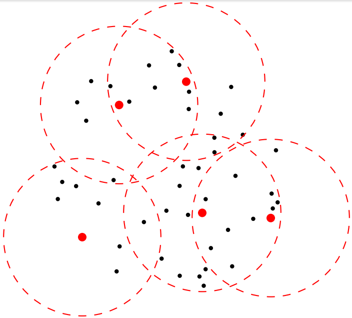

Hitting Set

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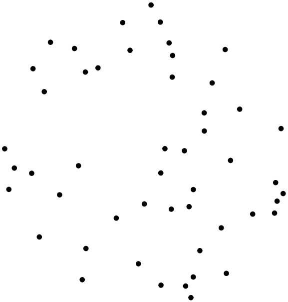

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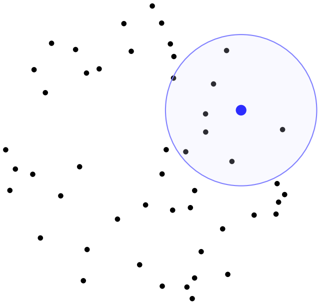

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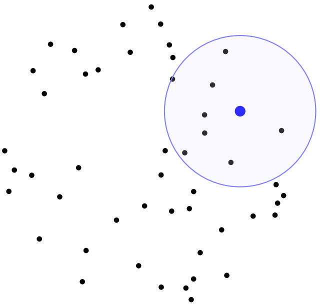

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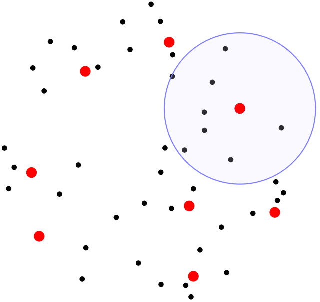

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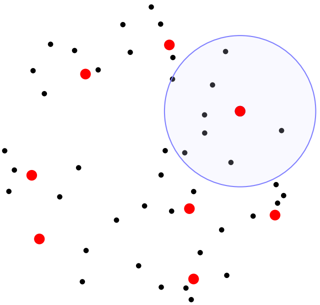

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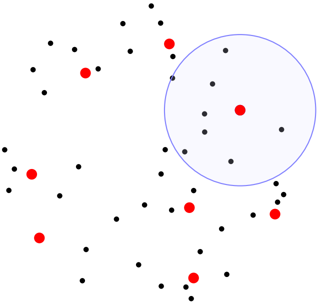
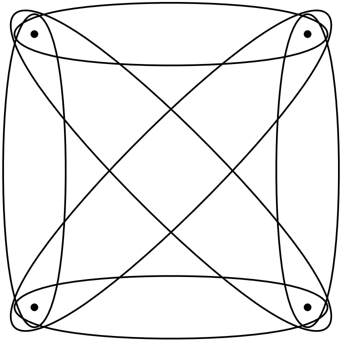
Question: If $|S| \geq \delta|V|$ for every $S \in \mathcal{S}$, do we have $\tau(\mathcal{S}) \leq f(\delta)$?



A counterexample

$V = [n]$ and $\mathcal{S} = \binom{[n]}{n/2}$.



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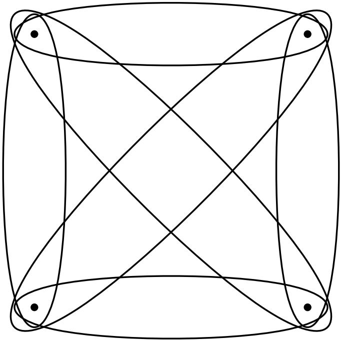

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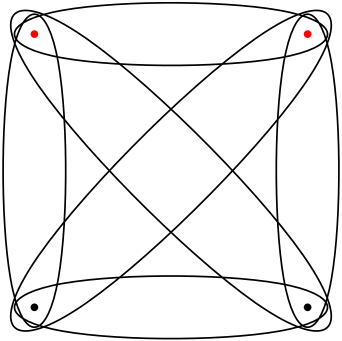

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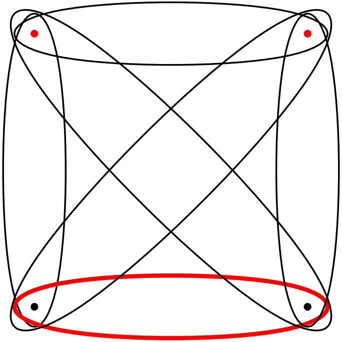

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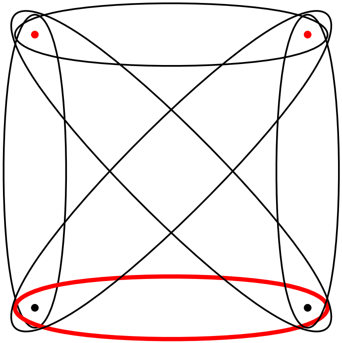

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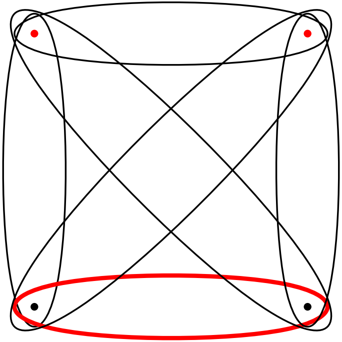
$\mathcal{S}$ is complex: $\mathcal{S}_{|[n/2]} = \mathcal{P}([n/2])$.



VC-dimension

Definition (Vapnik, Cervonenkis '71)

The *VC-dimension* of $\mathcal{S}$ is the maximum size of a set $Y \subseteq V$ such that $\mathcal{S}_{|Y} = \mathcal{P}(Y)$.

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Theorem [Vapnik, Cervonenkis '71], [Haussler, Welzl '89]

If $\mathcal{S}$ has VC-dimension d and $|S| \geq \delta|V|$ for every $S \in \mathcal{S}$ then

$$\tau(\mathcal{S}) = O\left(\frac{d}{\delta}\log\frac{1}{\delta}\right).$$

Back to the Euclidean Setting

Proposition [Folklore]

If $\mathcal{S}$ is a set system of balls in $\mathbb{R}^N$ then VC-dim$(\mathcal{S}) \leq N + 1$.

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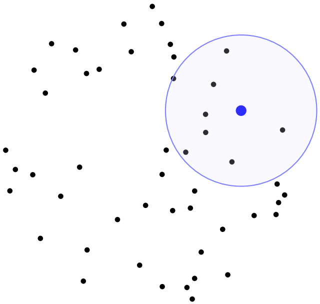
Partial set systems

Subset of $V \leftrightarrow$ Bipartition (B, W) of V.



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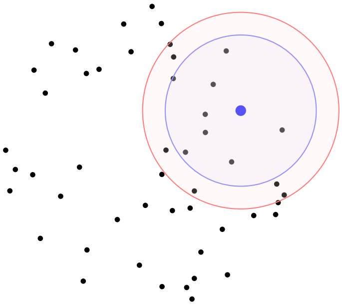

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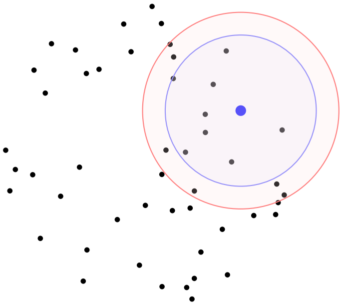

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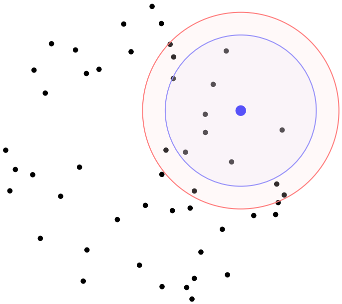
A *partial set system* on V is a collection of partial subsets of V.



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Proposition [Alon, Hanneke, Holzman, Moran '21], [BCT '25]

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Structure	Small set corresponding to v	Large set corresponding to v
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$V \subseteq \{0,1\}^N$	$\left\{u : d_H(u,v) \geq \tau \cdot N\right\}$	$\left\{u : d_H(u,v) \geq (\tau - \varepsilon) \cdot N\right\}$
Set system $\mathcal{S}$ on V	$\{u : \mathcal{S}_u \cap \mathcal{S}_v = \emptyset\}$	$\{u : \lvert\mathcal{S}_u \cap \mathcal{S}_v\rvert \leq \varepsilon \lvert\mathcal{S}\rvert\}$
Graph $G = (V, E)$	$\{u : N(u) \cap N(v) = \emptyset\}$	$\{u : \lvert N(u) \cap N(v)\rvert \leq \varepsilon n\}$
Digraph $D = (V, A)$	$\{u : N^-(u) \cap N^+(v) = \emptyset\}$	$\{u : \lvert N^-(u) \cap N^+(v)\rvert \leq \varepsilon n\}$
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Majority voting on V	$\{u : v$ is $1/2$-preferred to $u\}$	$\{u : v$ is $(1/2 - \varepsilon)$-preferred to $u\}$

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This work: Combinatorial consequences of the theory of VC-dimension for partial set systems.

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Theorem [BCT '25], [Charikar, Ramakrishnan, Wang '25]

For every $\varepsilon > 0$, every election admits a set of $O(1/\varepsilon^2)$ candidates which are not simultaneously $(1/2 + \varepsilon)$-beaten by any single candidate.

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Thank you!