# PPP-Completeness and Extremal combinatorics

**Romain Bourneuf**, Lukáš Folwarczný, Pavel Hubáček, Alon Rosen and Nikolaj I. Schwartzbach

European Research Council
Established by the European Commission

# PWPP and Extremal combinatorics

**Definition (WeakPigeon)**

Input : Circuit $C : \{0,1\}^n \to \{0,1\}^{n-1}$.
Solution : $x \neq y \in \{0,1\}^n$ s.t. $C(x) = C(y)$.

Defines the class PWPP.

# PWPP and Extremal combinatorics

**Definition (WeakPigeon)**

Input : Circuit $C : \{0,1\}^n \to \{0,1\}^{n-1}$.
Solution : $x \neq y \in \{0,1\}^n$ s.t. $C(x) = C(y)$.

Defines the class PWPP.

Extremal combinatorics: If the size of some object is **large enough** then some structure must appear.

# PWPP and Extremal combinatorics

**Definition (WeakPigeon)**

Input : Circuit $C : \{0,1\}^n \to \{0,1\}^{n-1}$.
Solution : $x \neq y \in \{0,1\}^n$ s.t. $C(x) = C(y)$.

Defines the class PWPP.

Extremal combinatorics: If the size of some object is **large enough** then some structure must appear.

➔ Characterize PWPP & PPP: new complete problems from extremal combinatorics.

# The Sperner Antichain problem

**Theorem (Sperner, 1928)**

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

# The Sperner Antichain problem

**Theorem (Sperner, 1928)**

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

**Definition (Weak Sperner Antichain problem)**

Input: $C : \{0,1\}^{\alpha+1} \to \{0,1\}^{2n}$, with $\alpha = \log \binom{2n}{n}$.
Solution: $x \neq y \in \{0,1\}^{\alpha+1}$, s.t. $C(x) \subseteq C(y)$.

# The Sperner Antichain problem

**Theorem (Sperner, 1928)**

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

**Definition (Weak Sperner Antichain problem)**

Input: $C : \{0,1\}^{\alpha+1} \to \{0,1\}^{2n}$, with $\alpha = \log \binom{2n}{n}$.
Solution: $x \neq y \in \{0,1\}^{\alpha+1}$, s.t. $C(x) \subseteq C(y)$.

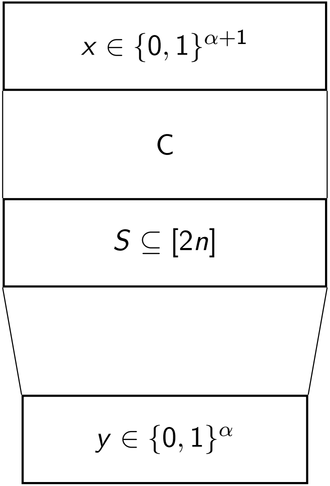
**Theorem**

Weak Sperner Antichain is PWPP-complete.

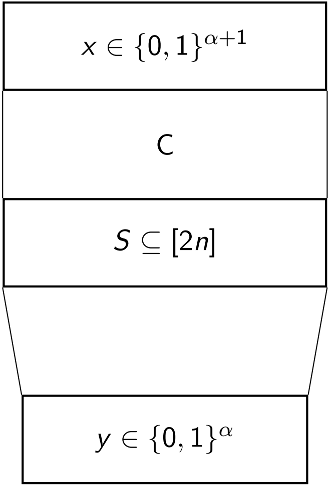
# PWPP-Hardness

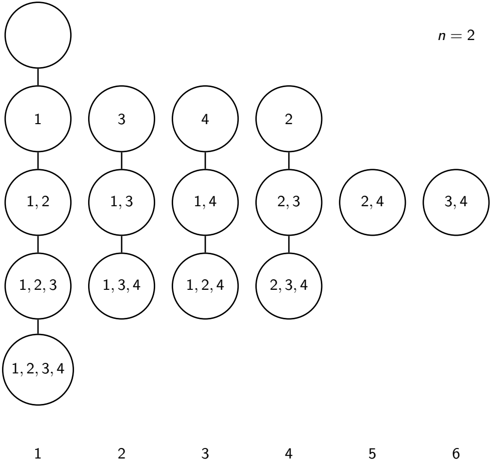
Variant of the graph-hash product (Komargodski, Naor, Yogev).

# Weak Sperner Antichain reduces to Weak Pigeon



# Weak Sperner Antichain reduces to Weak Pigeon



# From PWPP to PPP

**Theorem (Weak Pigeonhole Principle)**

If $f : [2k] \to [k]$, $\exists x \neq y, f(x) = f(y)$.

# From PWPP to PPP

**Theorem (Weak Pigeonhole Principle)**

If $f : [2k] \rightarrow [k]$, $\exists x \neq y, f(x) = f(y)$.

**Theorem (Strong Pigeonhole Principle)**

If $f : [k] \rightarrow [k]$, either $\exists x \neq y, f(x) = f(y)$, or $\exists x, f(x) = 1$.

# From PWPP to PPP

**Theorem (Weak Pigeonhole Principle)**

If $f : [2k] \to [k]$, $\exists x \neq y, f(x) = f(y)$.

**Theorem (Strong Pigeonhole Principle)**

If $f : [k] \to [k]$, either $\exists x \neq y, f(x) = f(y)$, or $\exists x, f(x) = 1$.

**Definition (Pigeon)**

Input : Circuit $C : \{0,1\}^n \to \{0,1\}^n$.
Solution :

- $x \neq y \in \{0,1\}^n$, s.t. $C(x) = C(y)$.
- $x \in \{0,1\}^n$, s.t. $C(x) = 0^n$.

This defines PPP.

# Strong version of Sperner

**Theorem (Weak Sperner Theorem)**

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

# Strong version of Sperner

**Theorem (Weak Sperner Theorem)**

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

**Theorem (Strong Sperner Theorem)**

If we have exactly $\binom{2n}{n}$ subsets of $[2n]$, then either one of them is contained in another, or one of them is $[n]$.

# Strong version of Sperner

**Theorem (Weak Sperner Theorem)**

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

**Theorem (Strong Sperner Theorem)**

If we have exactly $\binom{2n}{n}$ subsets of $[2n]$, then either one of them is contained in another, or one of them is $[n]$.

**Definition (Sperner Antichain problem)**

Input: $C : \{0,1\}^\alpha \to \{0,1\}^{2n}$, with $\alpha = \log \binom{2n}{n}$.
Solution: $x \neq y$, s.t. $C(x) \subseteq C(y)$, or $x$, s.t. $C(x) = [n]$.

# Strong version of Sperner

## Theorem (Weak Sperner Theorem)

If we have more than $\binom{2n}{n}$ subsets of $[2n]$, then one of them is contained in another.

## Theorem (Strong Sperner Theorem)

If we have exactly $\binom{2n}{n}$ subsets of $[2n]$, then either one of them is contained in another, or one of them is $[n]$.

## Definition (Sperner Antichain problem)

Input: $C : \{0,1\}^{\alpha} \to \{0,1\}^{2n}$, with $\alpha = \log \binom{2n}{n}$.
Solution: $x \neq y$, s.t. $C(x) \subseteq C(y)$, or $x$, s.t. $C(x) = [n]$.

## Theorem

The problem Sperner Antichain is PPP-complete.

# Other results & Open problems

PWPP & PPP-completeness results for problems related to:
- Erdős-Ko-Rado's Theorem
- Cayley's Formula for trees

A hierarchy of problems between PWPP and PPP based on Turán's Theorem.

# Other results & Open problems

PWPP & PPP-completeness results for problems related to:
- Erdős-Ko-Rado's Theorem
- Cayley's Formula for trees

A hierarchy of problems between PWPP and PPP based on Turán's Theorem.

Containment of Ramsey in PWPP?