A Dense Neighborhood Lemma, with Applications to Domination and Chromatic Number

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Joint work with Pierre Charbit (IRIF)
and Stéphan Thomassé (LIP)

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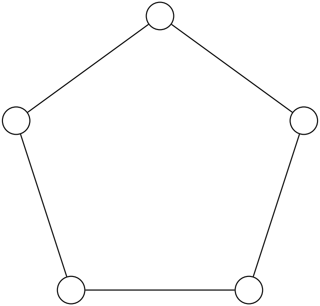
Domination



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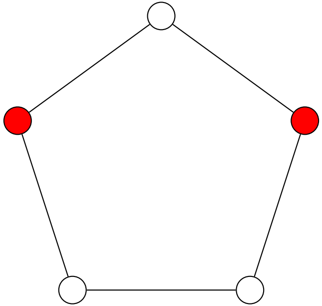

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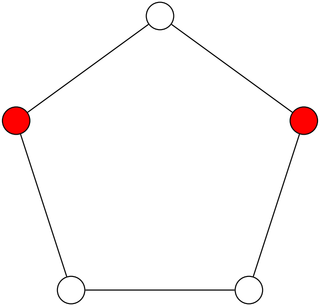

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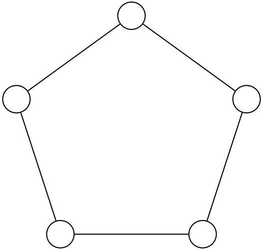
$\gamma(G) :=$ minimum size of a dominating set

Fractional domination



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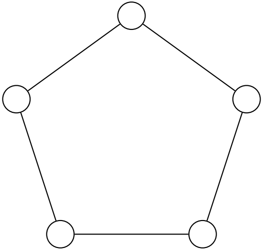

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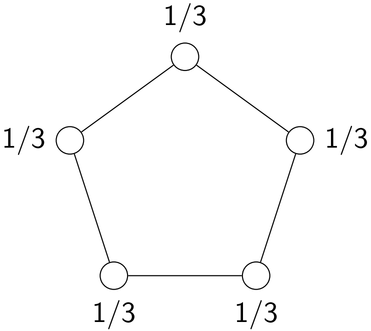

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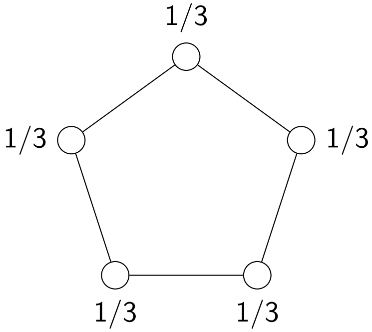

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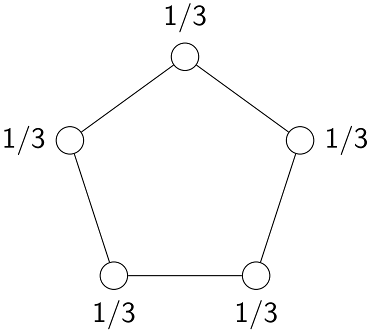

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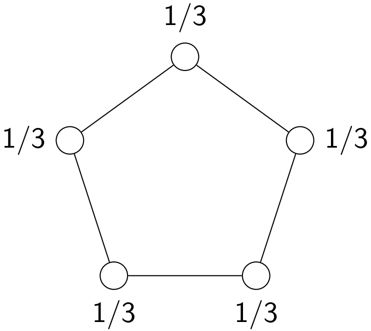

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Question

When do we have $\gamma(G) \leq f(\gamma^*(G))$?

Some negative examples

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If $G \sim G(n, 1/2)$ then whp. $\gamma^*(G) \leq 3$ and $\gamma(G) \geq \log(n)/2$.

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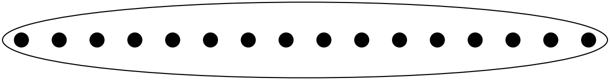

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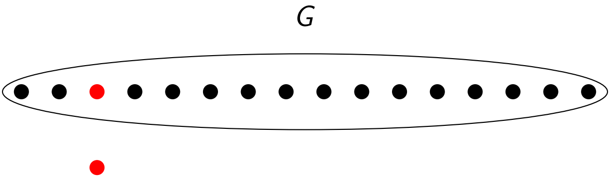

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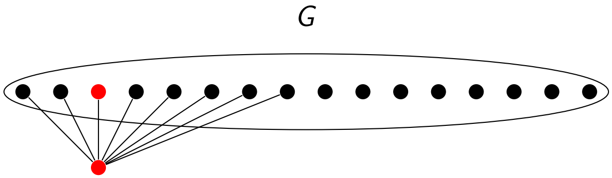

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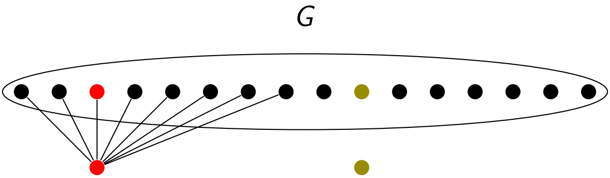

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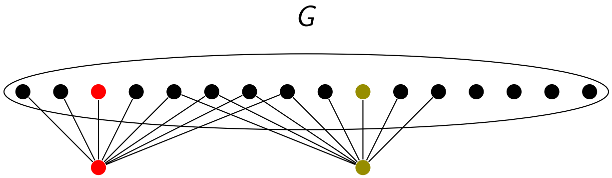

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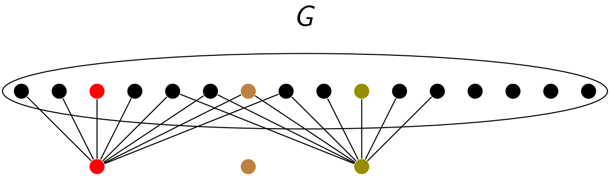

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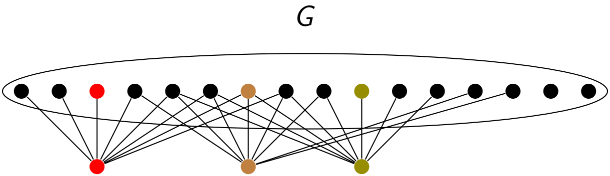

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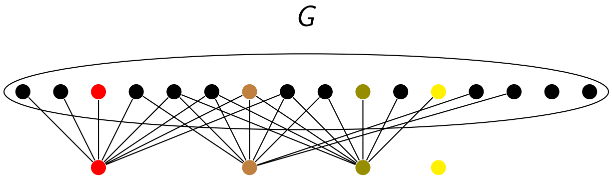

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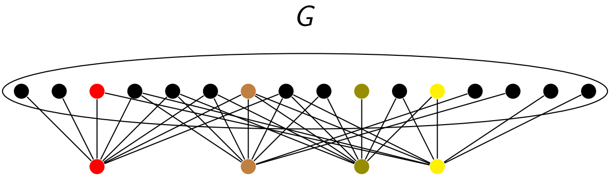

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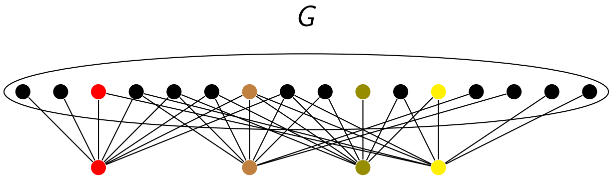

Proposition [Folklore]

If $G \sim G(n/2, n/2, 1/2)$ then whp. $\gamma^*(G) \leq 6$ and $\gamma(G) \geq \log(n)/2$.

A positive result

Theorem [Vapnik, Cervonenkis '71]

If $\mathcal{C}$ is a monotone class of graphs then the following are equivalent:

- $\mathcal{C}$ does not contain all bipartite graphs.
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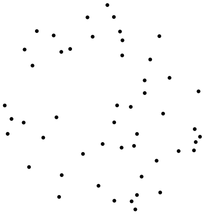
Theorem [Vapnik, Cervonenkis '71], [Haussler, Welzl '89]

If G has VC-dimension d then $\gamma(G) \leq f(d, \gamma^*(G))$.

A geometric variant

$V \subseteq \mathbb{R}^N$



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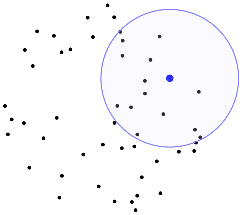

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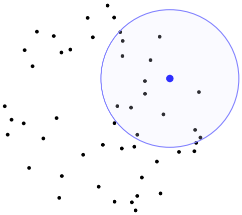

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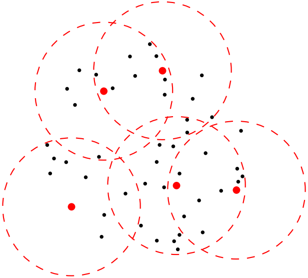

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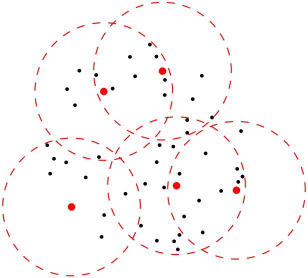

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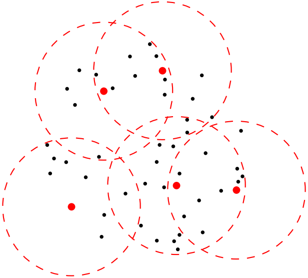

Question

For $\tau > 0$, is there a function f such that every $V \subseteq \mathbb{R}^N$ satisfies $\gamma_\tau(V) \leq f(\gamma^*_\tau(V))$?

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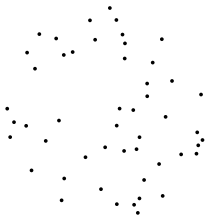

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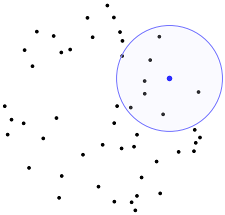

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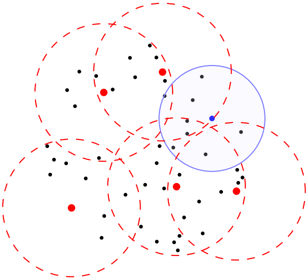

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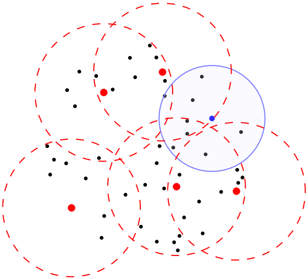
Theorem [BCT '25], [Alon, Hanneke, Holzman, Moran '21]

For every $\varepsilon > 0$, every $V \subseteq \mathbb{R}^N$ satisfies $\gamma_{(1+\varepsilon)\tau}(V) \leq \text{poly}(1/\varepsilon, \gamma_\tau^*(V))$.



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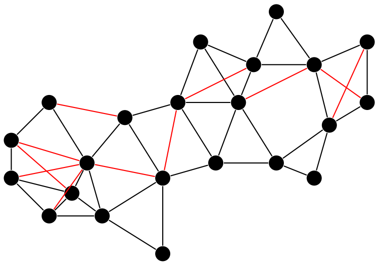

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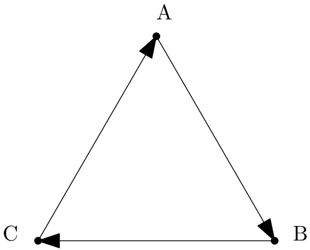

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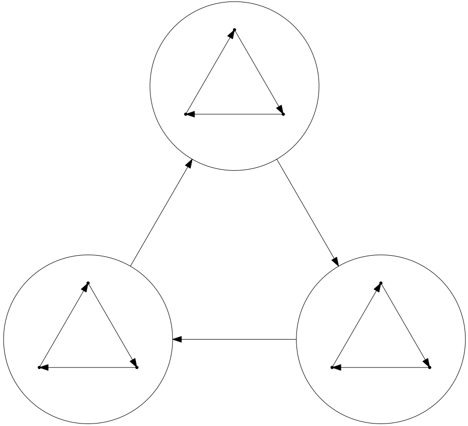

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For $2k+1$ voters, there is always a set of at most $f(k)$ candidates which are not all simultaneously beaten by any candidate.

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Theorem [BCT '25], [Charikar, Ramakrishnan, Wang '25]

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Proof sketch

Theorem [BCT '25], [Charikar, Ramakrishnan, Wang '25]

For every $\varepsilon > 0$, there is always a set of at most $O(1/\varepsilon^2)$ candidates which are not all simultaneously $(1/2 + \varepsilon)$-beaten by any candidate.

Form a tri-directed graph $T = (V, A, R)$ with

- $u \to v \in A$ if u $1/2$-beats v,
- $u \to v \in R$ if u $(1/2 - \varepsilon)$-beats v.

Claim [Folklore]

T has a black fractional dominating set of weight at most 2.

Claim

T has VC-dimension $O(1/\varepsilon^2)$.

Then, there exists a black/red dominating set X of size $f(\varepsilon)$.

Open questions

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Thank you!