Chromatic Number and Twin-Width

Romain Bourneuf
Supervised by Stéphan Thomassé
École Normale Supérieure de Lyon

Joint work with Édouard Bonnet, Julien Duron,
Colin Geniet and Nicolas Trotignon

July 10, 2023

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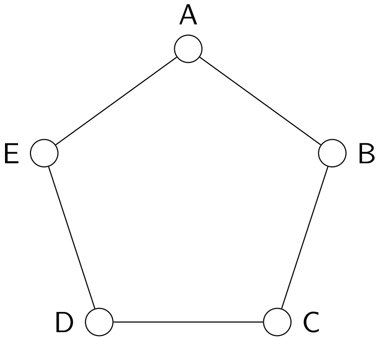

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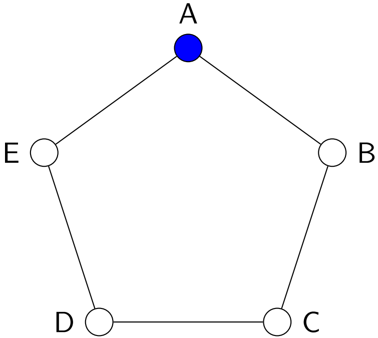

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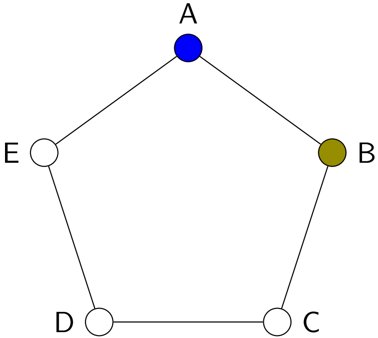

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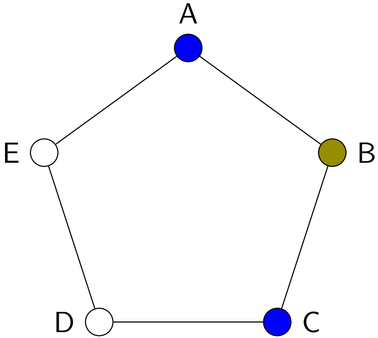

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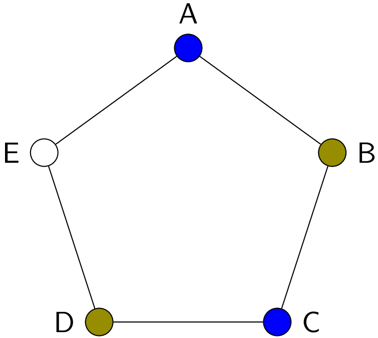

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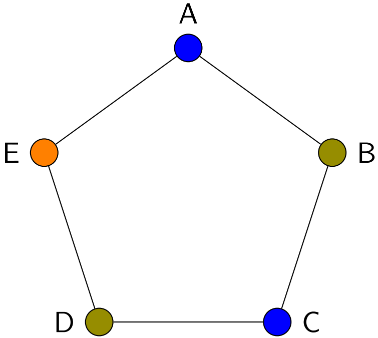
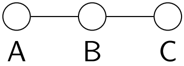


Twinning & Gluing



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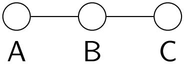

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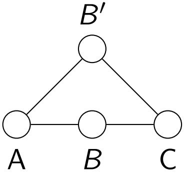

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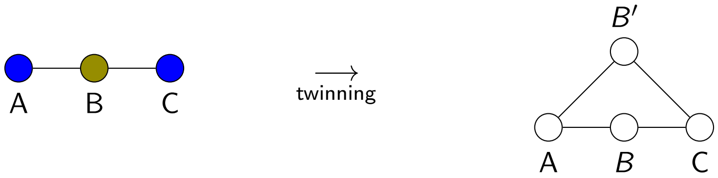

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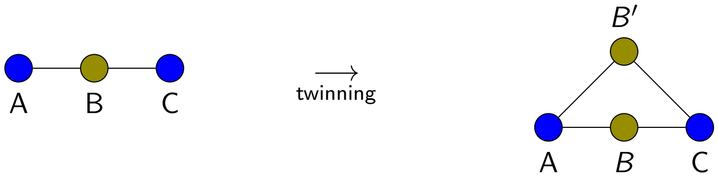

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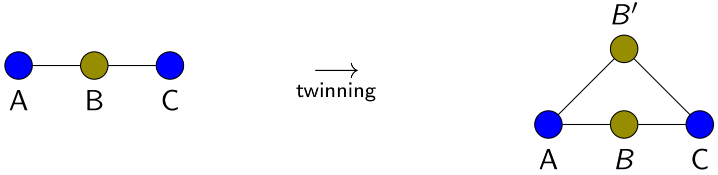

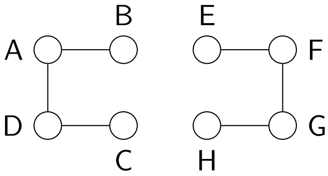

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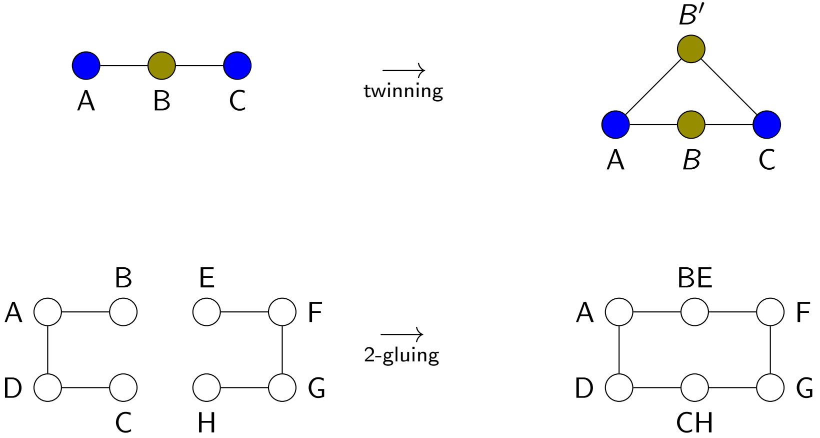

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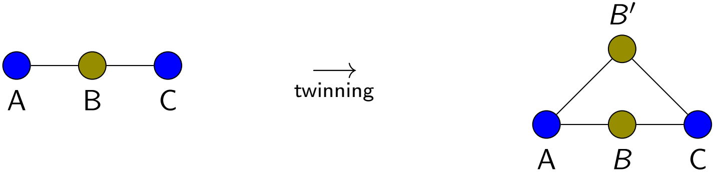

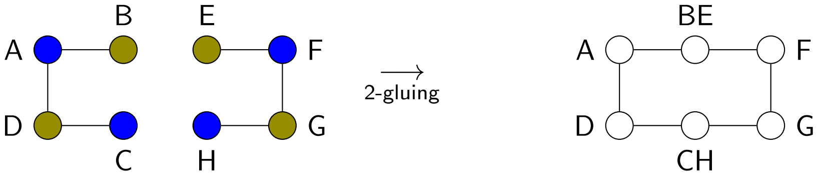

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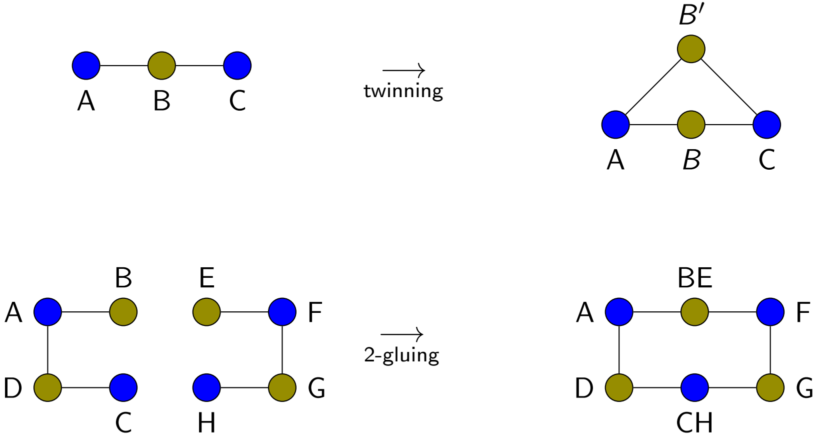

Twinning, Gluing & Chromatic Number

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Question (Chudnovsky, Penev, Scott, Trotignon '13)

Let $\mathcal{C}$ be a class of k-colorable graphs.
Can we build graphs of arbitrary chromatic number from $\mathcal{C}$ using only twinning and 2-gluings?

Twincut graphs

Theorem (Bonnet, B., Duron, Geniet, Thomassé, Trotignon)

Let $\mathcal{C} = \{K_1, K_2, \overline{K_2}\}$.
Starting from $\mathcal{C}$, we can build graphs of arbitrary chromatic number using only twinning and 2-gluings.

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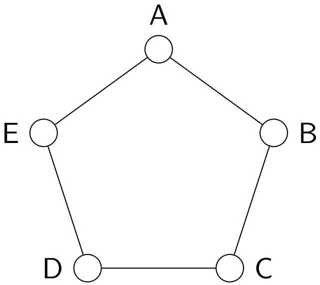

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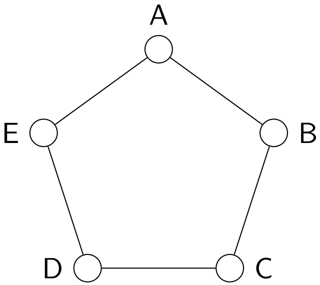
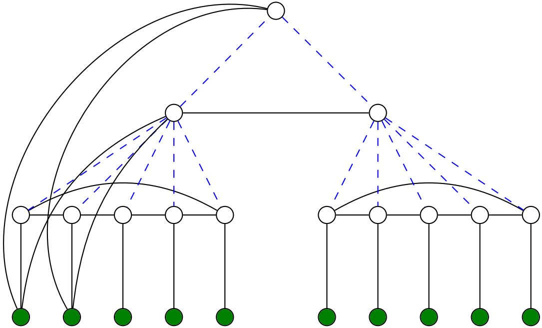
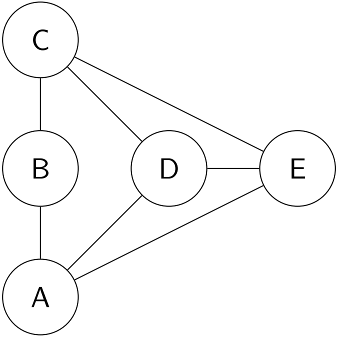




Contraction sequence



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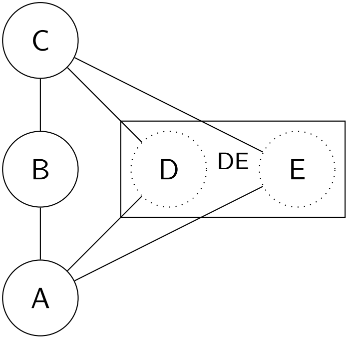

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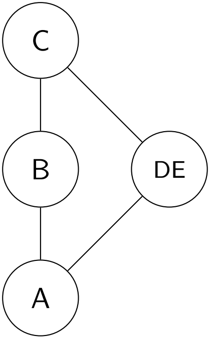

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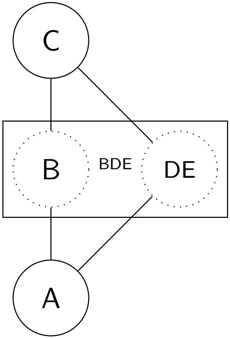

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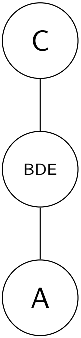

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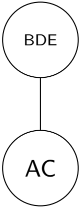

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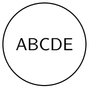

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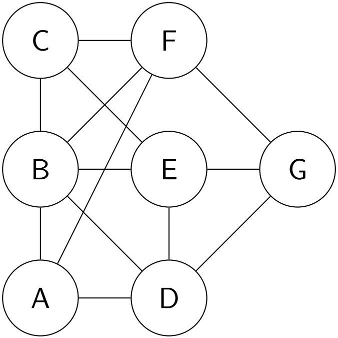

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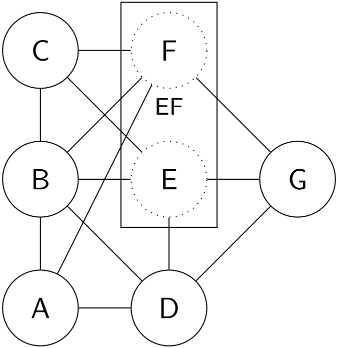

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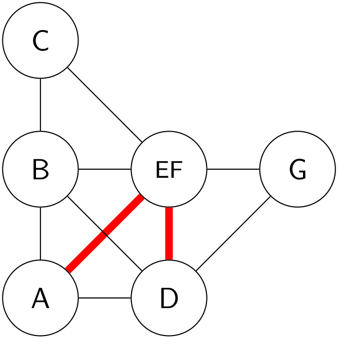

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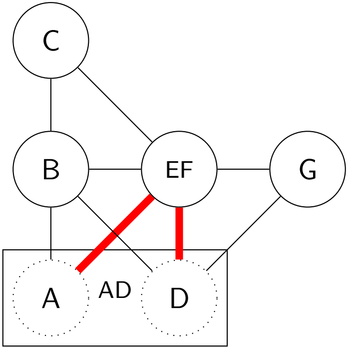

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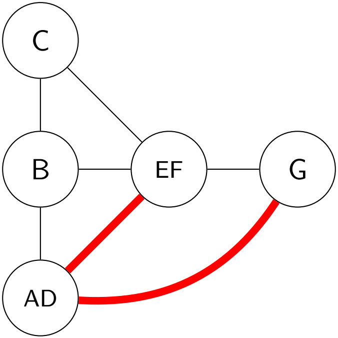

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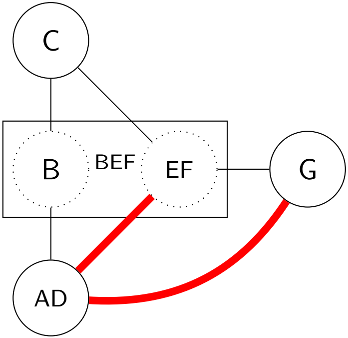

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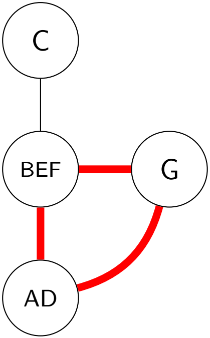

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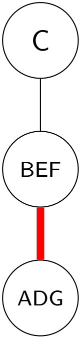

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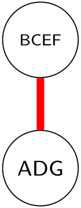

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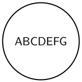

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- Enumerative Combinatorics (exponential growth vs. factorial growth)
- Structural Graph Theory (generalize bounded treewidth, cliquewidth)

Twin-Width & Chromatic Number

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- If G is triangle-free and $\chi(G) = k+1$ then $tww(G) \geq k-1$.
- If G has twin-width t and has no clique of size k, then $\chi(G) \leq (t+2)^{k-2}$.

χ-Boundedness

Definition (χ-boundedness (Gyárfás '87))

A class of graphs $\mathcal{C}$ is *χ-bounded* if there is a function f such that for every $G \in \mathcal{C}$, we have $\chi(G) \leq f(\omega(G))$.

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- The class of perfect graphs is the class of graphs with $\chi(G) = \omega(G)$.

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Theorem [Bonnet, Geniet, Kim, Thomassé, Watrigant '20]

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The End

Thank you for your attention :)

Questions?