An approximate Tutte-decomposition for all connectivities I: Capturing cyclic structure with flowers

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Joint work with Johannes Carmesin, Joseph Devine, Jan Kurkofka and Tim Planken (TU Freiberg)

BWAG'25

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Main question: Given a k-connected graph, how to describe its structure with respect to k-separations?

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What should we expect for general k?
What should be the basic building blocks?

Angry graphs

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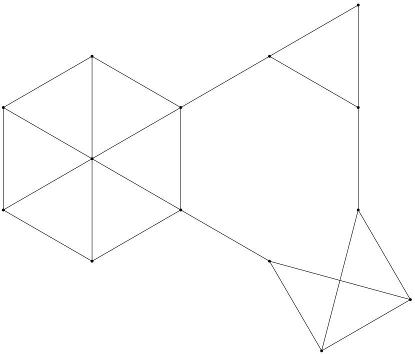
This work: characterization of approximately k-angry graphs.

Tutte-decomposition



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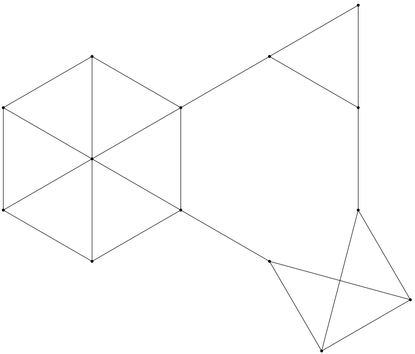

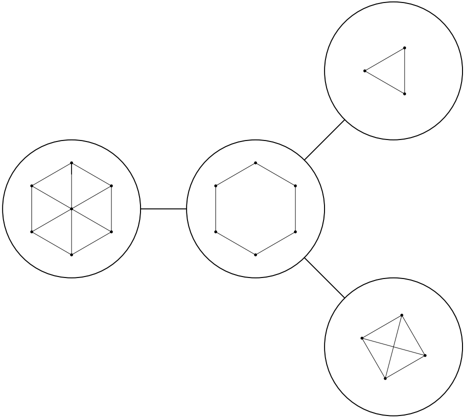

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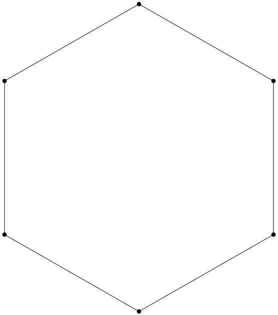

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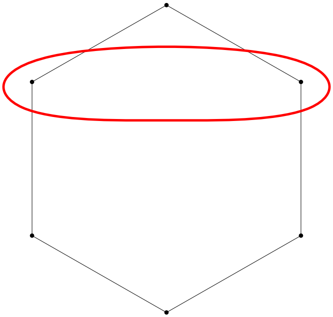

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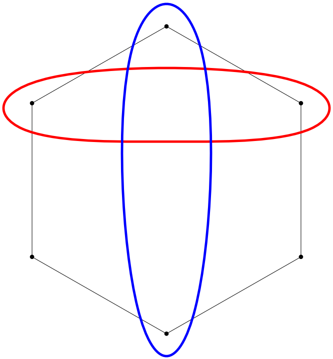

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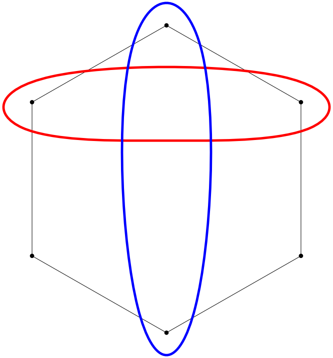

Theorem [Tutte '61]

If G is 2-connected and every 2-separation of G is crossed by another 2-separation then either G is 3-connected or G is a cycle.

The 2-angry graphs are exactly the cycles and the 3-connected graphs.

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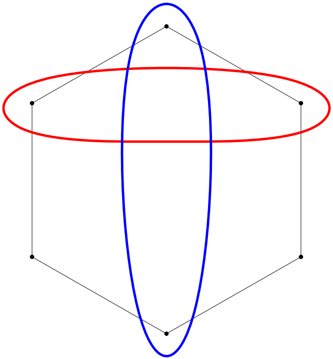

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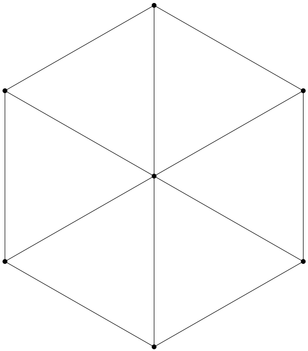
The 2-angry graphs are exactly the cycles and the 3-connected graphs. These are the basic building blocks for the Tutte-decomposition.

Why wheels and $K_{3,m}$'s?



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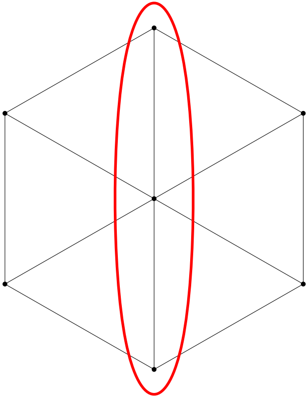

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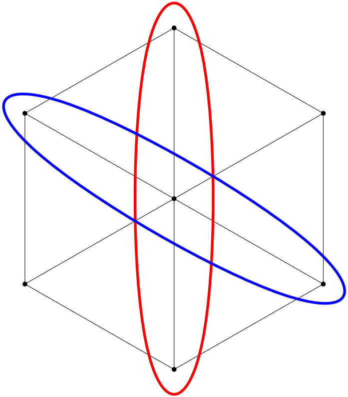

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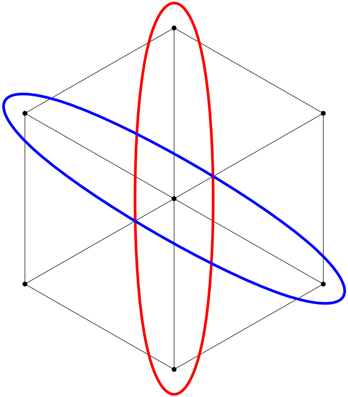

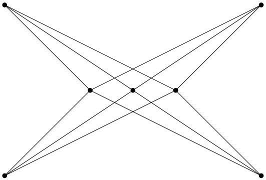

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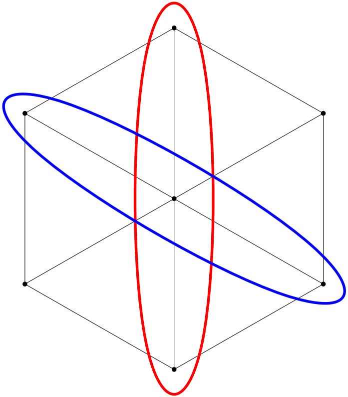

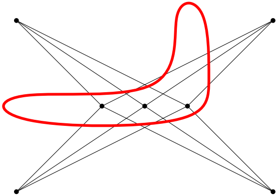

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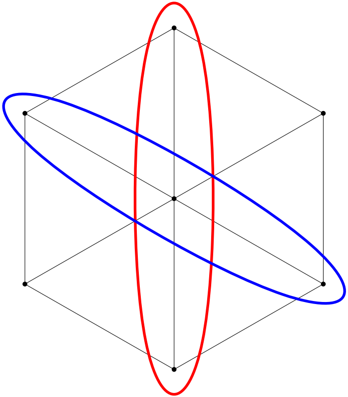

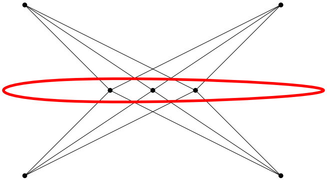

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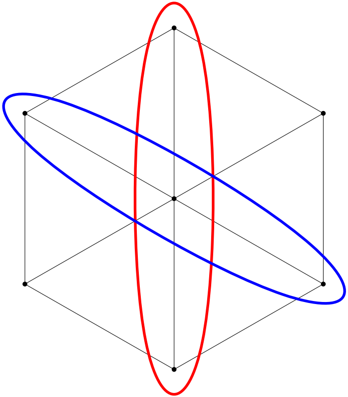

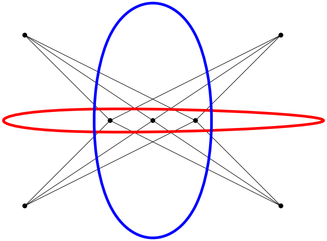

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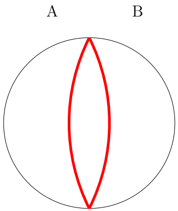

A separation (A, B) is *h-huge* if $|A \setminus B| \geq h$ and $|B \setminus A| \geq h$.

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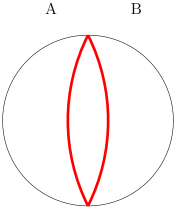

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G is *a-almost k-connected* if every $< k$-separation has a side of size $\leq a$.

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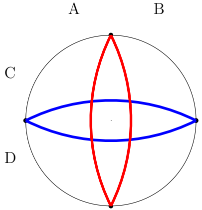

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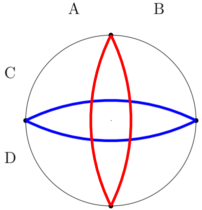

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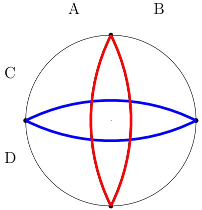

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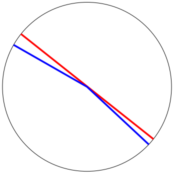

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Two separations (A, B) and (C, D) are *c-close* if $|A \Delta C| \leq c$ and $|B \Delta D| \leq c$.

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Theorem [Carmesin, Kurkofka '23]

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These are the basic building blocks for the tri-decomposition.

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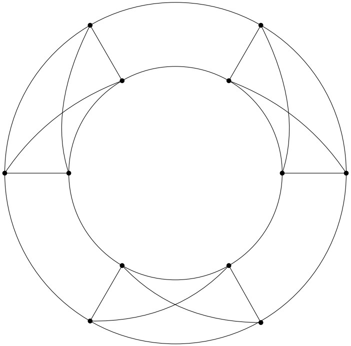

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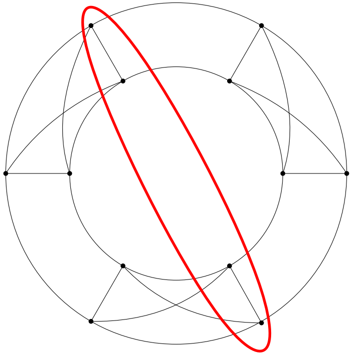

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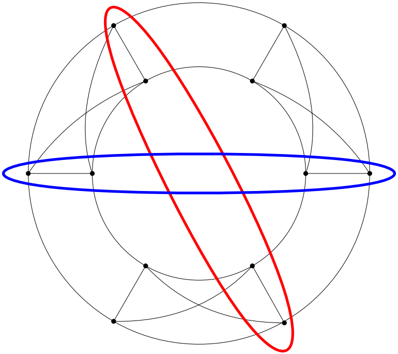

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If G is 4-connected and every 2-huge 4-separation of G is 1-crossed by another 2-huge 4-separation, then one of the following holds:

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What about general k?

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Theorem

If G is k-connected and every huge k-separation of G is crossed substantially by another huge k-separation, then G looks like one of the following:

- ??
- ??

Anemones

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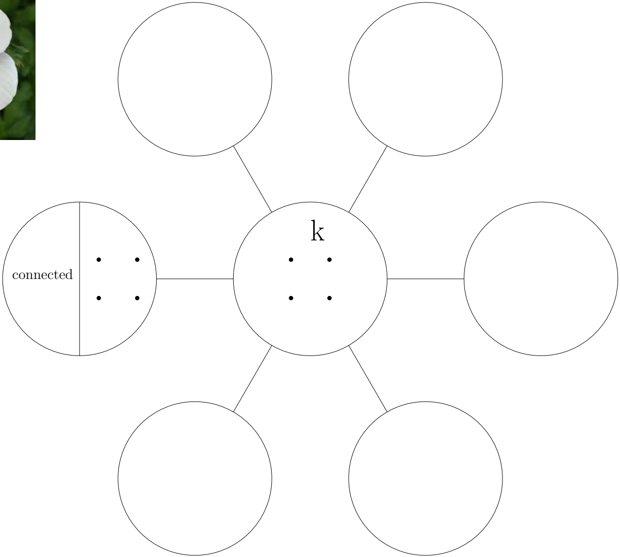

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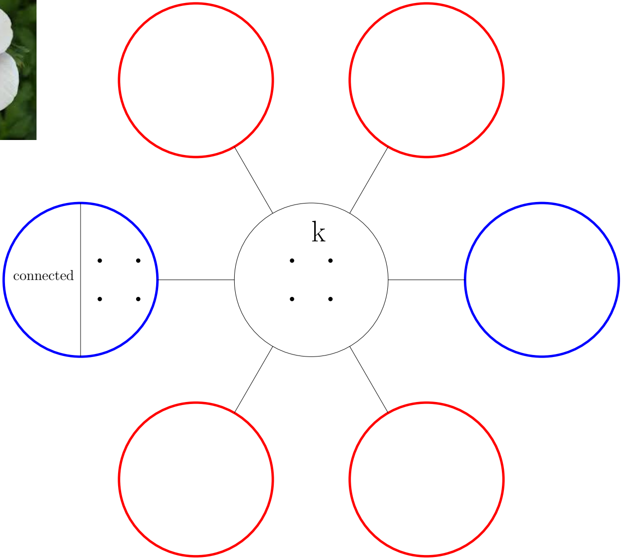

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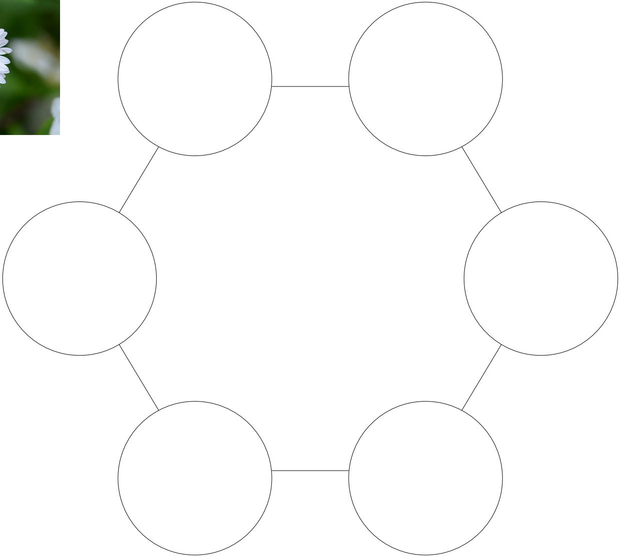

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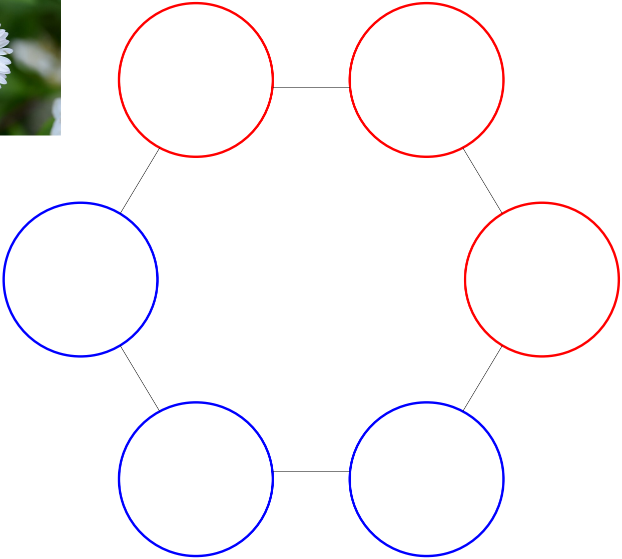

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Theorem [B., Carmesin, Devine, Kurkofka, Planken 25+]

Let G be a k-connected graph and $\mathcal{S}$ be the set of all h-huge k-separations of G.
Suppose that $\mathcal{S}$ is not empty and that every separation in $\mathcal{S}$ is $\gamma(h,k)$-crossed by another separation of $\mathcal{S}$.
Then, there exists a k-flower $F(\mathcal{S})$ of G such that every separation in $\mathcal{S}$ is $c(h,k)$-close to some induced separation of $F(\mathcal{S})$.

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- We just need G to be h-almost k-connected.

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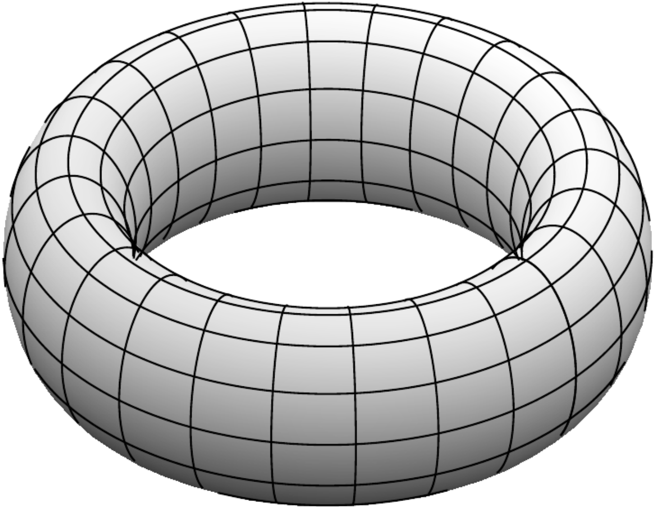

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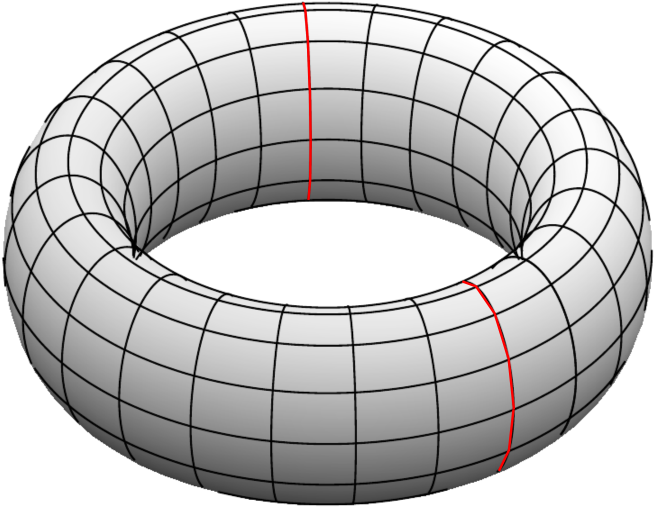

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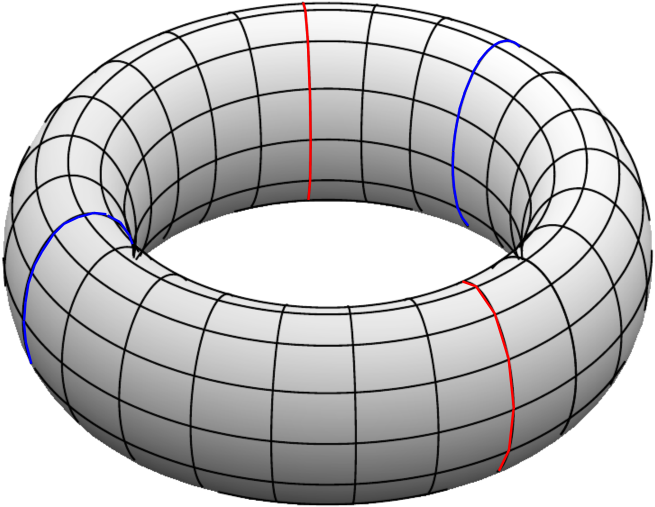

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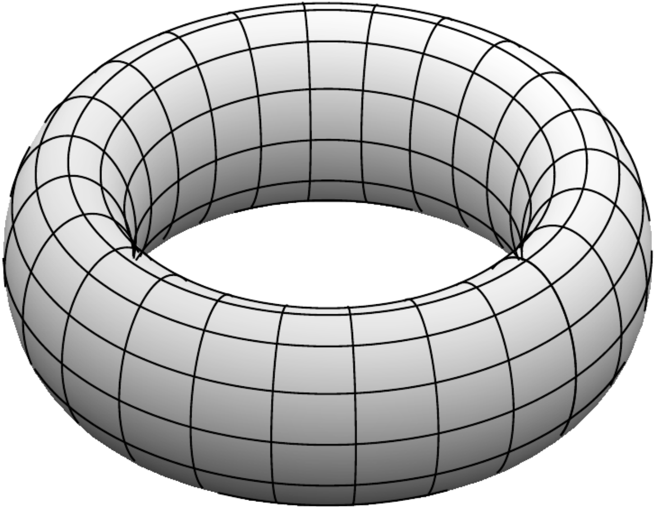

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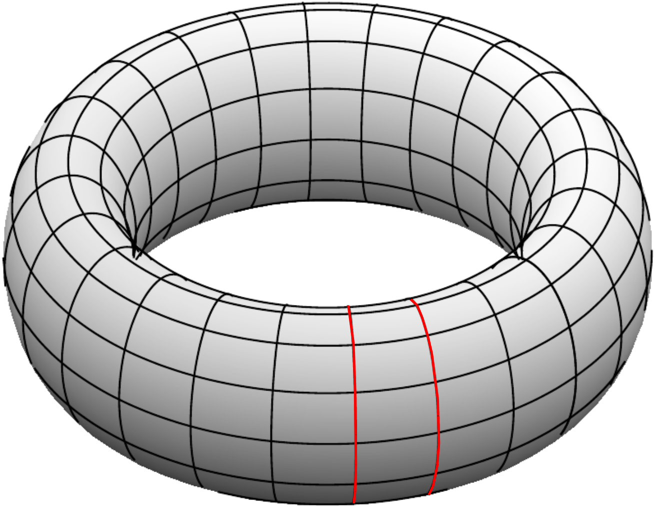

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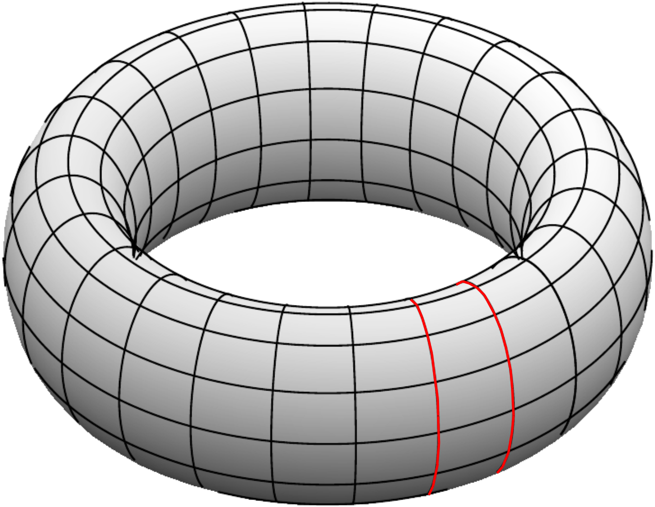

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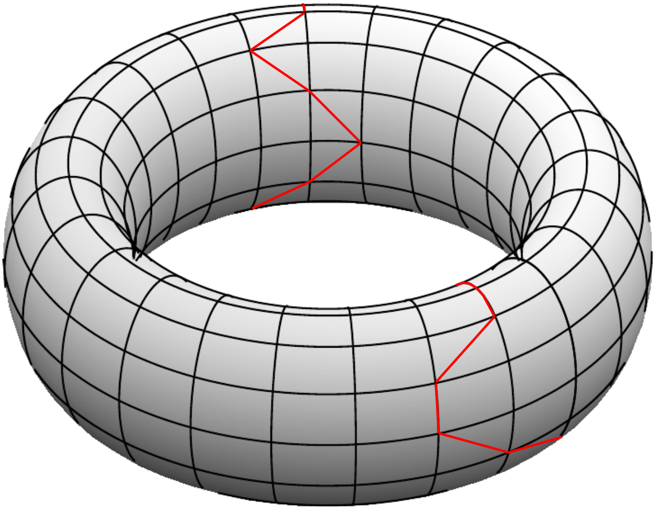

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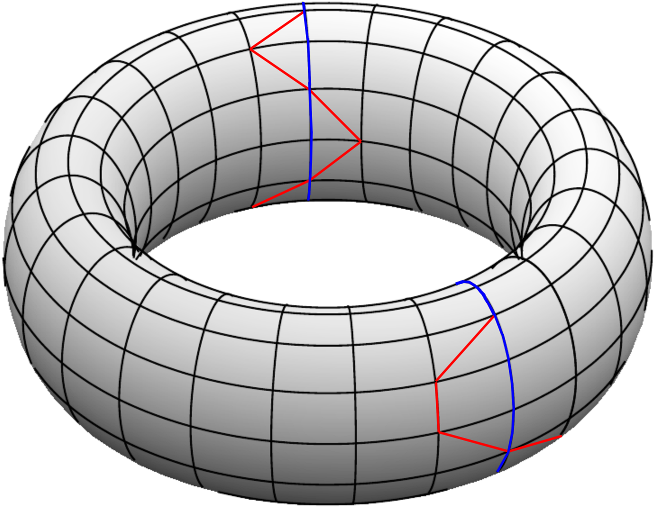

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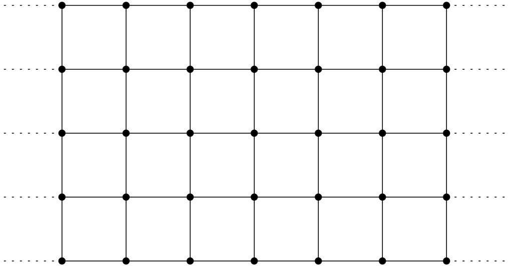

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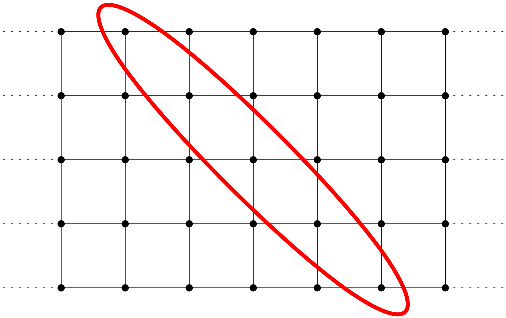

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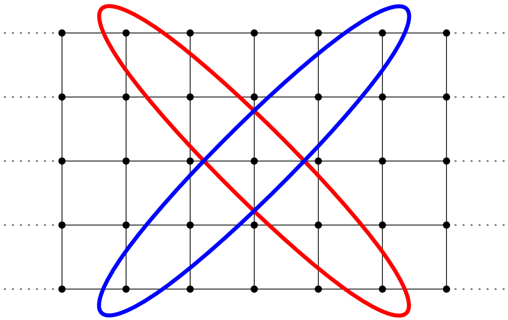
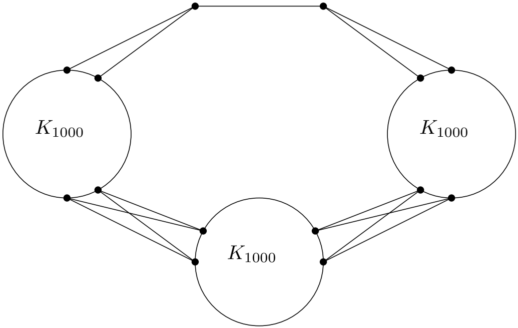


Example 3



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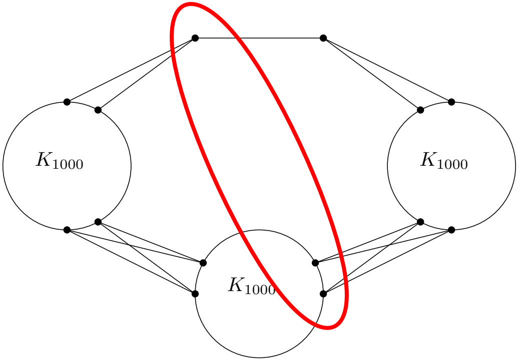

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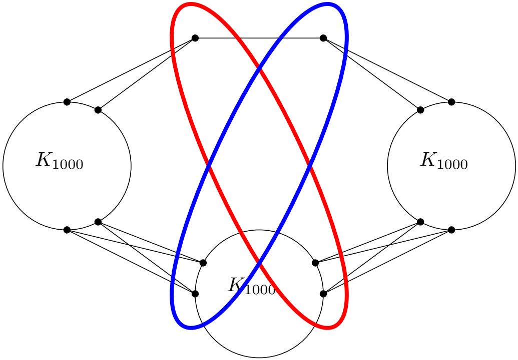

About the proof

A set $\mathcal{S}$ of separations is *uniformly γ-crossing* if for every $(A, B) \in \mathcal{S}$, all separations in $\mathcal{S}$ which γ-cross (A, B) induce the same tri-partition of $A \cap B$.

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Key Lemma 1

Let G be a k-connected graph and $\mathcal{S}$ be the set of all h-huge k-separations of G.
Then, $\mathcal{S}$ is uniformly $\gamma(h, k)$-crossing.

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Key Lemma 1

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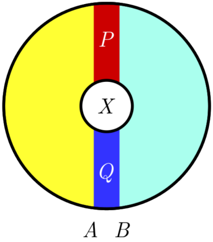
Key Lemma 2

Let G be a k-connected graph and $\mathcal{S}$ be a set of k-separations of G which is uniformly γ-crossing.
Then, there is a k-flower $F(\mathcal{S})$ of G such that every separation in $\mathcal{S}$ is $c(\gamma, k)$-close to some induced separation of $F(\mathcal{S})$.

Proof of Key Lemma 2



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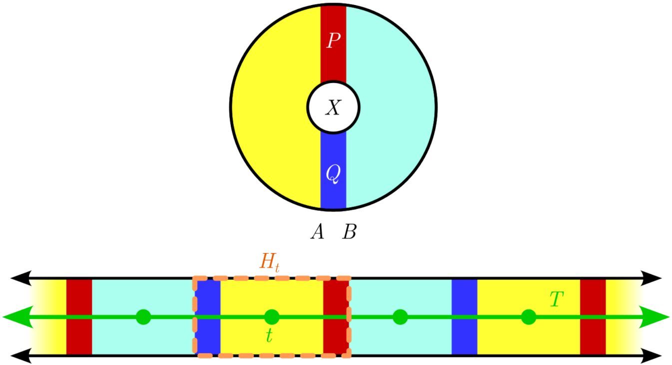

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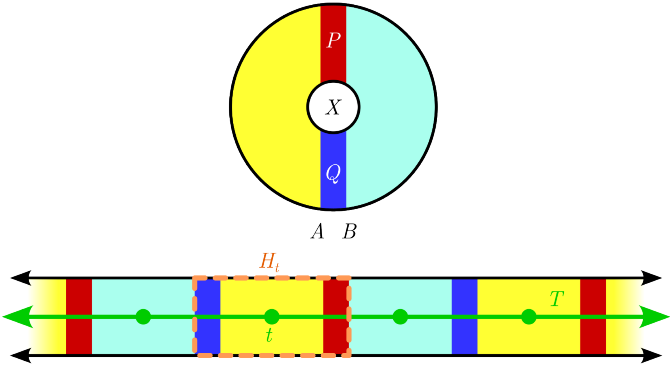

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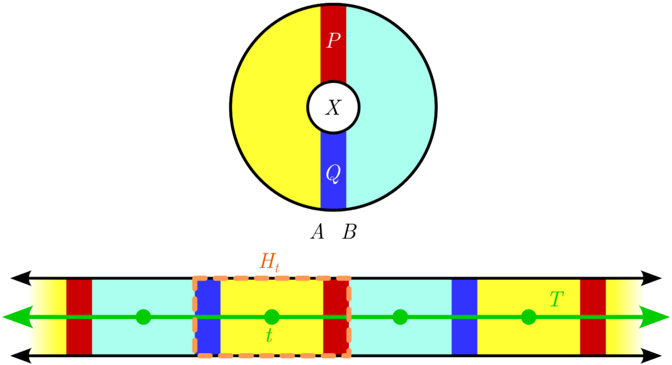

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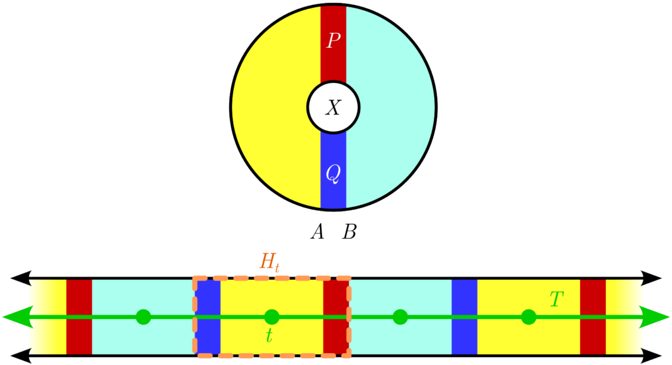

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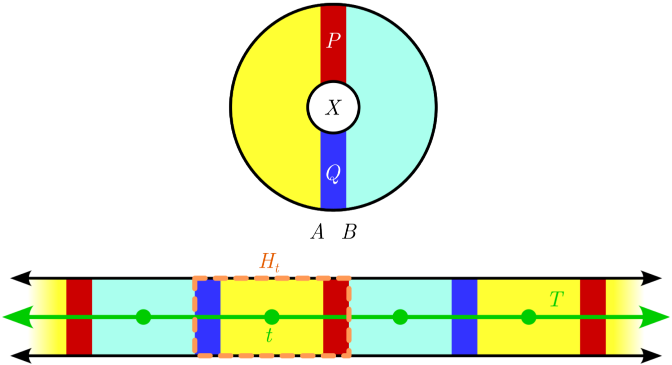

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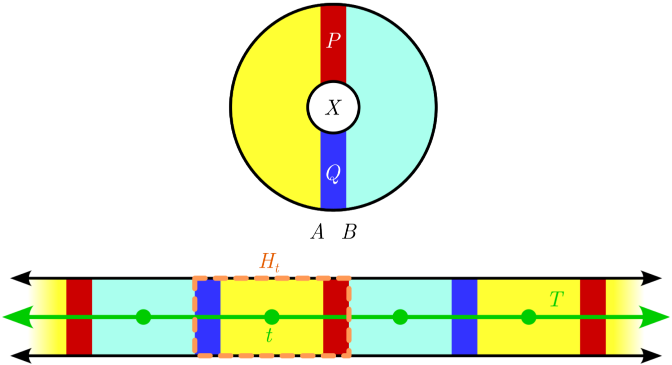

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The separations in L induce a tree-decomposition of the covering, which lifts to a cycle-decomposition of $G - X$.

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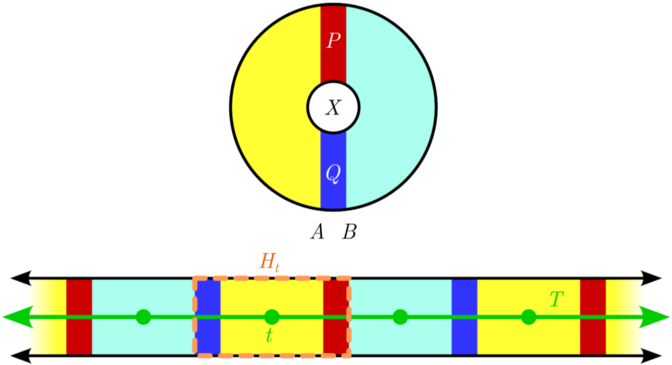

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Thank you!