Easy and Hard Problems on Bounded-Degree Boolean Automata Networks

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Bounded-degree automata networks. *Boolean automata networks* (BANs) are discrete models of interacting entities. A BAN is defined as a function $f : \{0,1\}^n \to \{0,1\}^n$. It is interpreted as n automata, each holding a Boolean state, updated synchronously according to *local functions* $f_i : \{0,1\}^n \to \{0,1\}$ for $i \in [n]$ (where $[n] = \{1,\ldots,n\}$ and $f(x) = (f_1(x),\ldots,f_n(x))$ for all $x \in \{0,1\}^n$). The graph of f is denoted $\mathcal{G}_f$. The architecture of a BAN f of size n is captured by its *interaction digraph* G_f on vertex set $[n]$, which contains an arc from i to j if and only if f_j effectively depends on its i-th component. When G_f has bounded degree, it enforces the system to be distributed, and the behavioral richness to emerge from local interactions.

Little is know on the potential dynamics of *bounded-degree Boolean automata networks* (BDBANs). For example, denoting d the degree [2]:

- it is impossible to have strictly between 2^{n-d} and 2^n fixed points in $\mathcal{G}_f$;
- it is impossible to have a cycle of length 2^n in $\mathcal{G}_f$ when $d = 2$;
- it is possible to have a cycle of length 2^{n-1} and one fixed point when $d = 2$.

However, it is open whether a cycle of length 2^n can be constructed with BDBANs, and recognizing whether the dynamics of some BAN can be obtained by some BDBAN (up to isomorphism) is known to be coNP-hard and in PSPACE.

The intuitive "complexity" of their dynamics is naturally formalized through the lens of the computational complexity of various questions one may ask on $\mathcal{G}_f$, given its definition in terms of local functions. The latter are encoded as propositional formulas, or more generally as Boolean circuits. Some problems remain hard even when restricted to BDBAN of small degree:

Theorem 1 ([1,4]). *Given a* BDBAN f *of degree* d*, the following problems are* NP*-complete:*

- *does a given configuration x has a preimage under f ? For any fixed $d \geq 3$.*
- *does $\mathcal{G}_f$ has a fixed point? For any fixed $d \geq 2$.*
- *does $\mathcal{G}_f$ has a limit cycle of length ℓ? For any fixed $\ell \geq 2$ and $d \geq 2$.*

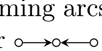
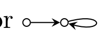
It is important to underline that, with the promise that G_f has degree bounded by d, one can compute it in polynomial time ($n^{\mathcal{O}(d)}$) from a circuit encoding of the local functions (otherwise it is as hard as DP) [4]. As a consequence, deciding whether f is a constant, or the identity, is easy on BDBANs.

Functional graphs. It is possible to generalize the constructions of Theorem 1 in order to prove that finding any *functional digraph* (i.e. of out-degree exactly one) in $\mathcal{G}_f$ is complete for NP. Given D a functional digraph, we write $D \sqsubseteq \mathcal{G}$ when D appears in $\mathcal{G}$, that is when D is an induced subgraph of $\mathcal{G}$.

Theorem 2. *Let D be any non-empty functional digraph. Given a BDBAN f of degree $d = 2$, it is* NP*-complete to decide whether $D \sqsubseteq \mathcal{G}$.*

Paths. A simple relaxation of the functional constraint is given by paths. Let P_ℓ be the directed path of length ℓ on $\ell+1$ vertices (only the last vertex has out-degree zero). Given a BDBAN f, one can parse G_f to find $i, j, k \in [n]$ (through local constraints) and design $x, y, z \in \{0, 1\}^n$ such that $x_i \neq y_i$, $x_j \neq z_j$ and $y_k \neq z_k$ in order to obtain a P_2 in $\mathcal{G}_f$. Generalizing this idea, we obtain:

Theorem 3. *Given G_f, there is an algorithm in time $\mathcal{O}(d^\ell n^{\ell+1})$ to decide whether $P_\ell \sqsubseteq \mathcal{G}_f$. Hence, with ℓ and d fixed, it is polynomial-time tractable to decide, given a Boolean circuit for a BDBAN f of degree d, whether $P_\ell \sqsubseteq \mathcal{G}_f$. However, if ℓ is given in the input (in unary), the problem is* NP*-complete.*

Bijectivity. With two in-coming arcs, we can express the bijectivity of f as the fact that $\mathcal{G}_f$ contains neither  nor  . Surprisingly, we have:

Theorem 4. *Given a BDBAN f of degree $d = 2$, deciding bijectivity is in* P.

This result is obtained in terms of the *influence (hyper)graph I_f*, having vertex set $[n]$ and edges the predecessor sets of individual vertices in the interaction graph. See Figure 1 for an example when $d = 2$. One can prove that the bijectivity of f depends only on its influence graph, and, when $d = 2$, that f is bijective if and only if I_f is acyclic and has exactly one loop in each connected component.

Fig. 1. Interaction digraph G_f (left) and influence graph I_f (right), with degree $d = 2$.

Easy/Hard dichotomy for first-order (FO)? The questions above are expressible in FO graph logics. When $d = 1$, one can prove that all FO expressible graph problems are tractable. In the spirit of [3], is it possible to characterize the hard problems? From what precedes, we expect a subtler dichotomy.

References

1. Akutsu, T., Hayashida, M., Zhang, S.Q., Ching, W.K., Ng, M.: Analyses and Algorithms for Predecessor and Control Problems for Boolean Networks of Bounded Indegree. Information and Media Technologies **4**(2), 338–349 (2009)
2. Aracena, J., Bridoux, F., Gadouleau, M., Guillon, P., Perrot, K., Richard, A., Theyssier, G.: On the Dynamics of Bounded-Degree Automata Networks. Natural Computing **25**(6) (2026)
3. Gamard, G., Guillon, P., Perrot, K., Theyssier, G.: Rice-Like Theorems for Automata Networks. In: STACS'2021. LIPIcs, vol. 187, pp. 32:1–32:17 (2021)
4. Perrot, K.: Études de la complexité algorithmique des réseaux d'automates. HDR thesis, Aix-Marseille Université (2022)