

Correspondences between codensity and coupling-based liftings, a practical approach

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Introduction: coalgebras, bisimilarity, and quantitative generalisations

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“Simulation”: w.r.t. the transitions.

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as coalgebras for an alphabet A :

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Our work: how do these two constructions relate to one another ?

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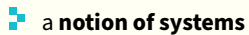
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1. giving more details on the two liftings,
2. exposing our results on the correspondences between the two constructions.

Categorical generalisation: codensity and coupling-based liftings

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If one knows how to compare the states of a system before a transition, then the liftings pave the way towards comparisons after a transition.

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Codensity lifting:

functor \mathcal{D} , with pseudometric $d: X \times X \rightarrow [0, 1]$, and $\mu, \nu \in \mathcal{D}X$,

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Codensity lifting: adding quantales

functor \mathcal{D} , with pseudometric $d: X \times X \rightarrow \mathcal{V}$, and $\mu, \nu \in \mathcal{D}X$, (d_e is a kind of Euclidean distance on \mathcal{V}),

$$\mathcal{D}^\uparrow d(\mu, \nu) = \bigwedge_{f: d \rightarrow d_e \text{ pseudometric morphism}} d_e(\mathbb{E}_\mu[f], \mathbb{E}_\nu[f])$$

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Codensity lifting: adding arbitrary functors

functor F , with pseudometric $d: X \times X \rightarrow \mathcal{V}$, and $x, y \in FX$,

$$F^\uparrow d(x, y) = \bigwedge_{f: d \rightarrow d_e \text{ pseudometric morphism}} d_e(\mathbb{E}_x[f], \mathbb{E}_y[f])$$

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Codensity lifting: replacing the expected values by arbitrary modalities

functor F , with pseudometric $d: X \times X \rightarrow \mathcal{V}$, and $x, y \in FX$, ($\tau: F\mathcal{V} \rightarrow \mathcal{V}$ is a kind of modality replacing $\mathbb{E}: \mathcal{D}[0, 1] \rightarrow [0, 1]$).

$$F^\uparrow d(x, y) = \bigwedge_{f: d \rightarrow d_e \text{ pseudometric morphism}} d_e(\tau \circ Ff(x), \tau \circ Ff(y))$$

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Important: expected values $\mathbb{E}: \mathcal{D}[0, 1] \rightarrow [0, 1]$ are replaced by some modality $\tau: F\mathcal{V} \rightarrow \mathcal{V}$.

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Question: for which functors, quantales, and modalities does duality hold ?

Correspondences between the codensity and coupling-based liftings

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There is a “duality-like” result for DFA with Γ containing one modality per letter in the alphabet plus one modality for terminal states.

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$$\begin{aligned} G &::= A \mid \text{Id}_\tau \mid A \times G \mid \coprod G \\ F &::= A \mid \text{Id}_\tau \mid \mathcal{P} \circ G \mid \mathcal{D} \circ G \mid \prod F_i \mid \coprod F_i \end{aligned}$$

where the subscripts τ indicate some choices of modality to obtain initial duality results.

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Our work also includes some limitation: we show that some usual notion of bisimulation for conditional transition systems cannot be retrieved using our construction.

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Thank you for your attention !