Correspondences between codensity and coupling-based liftings, a practical approach

Samuel Humeau, Jurriaan Rot, Daniela Petrisan

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LIP, ENS de Lyon

Introduction: coalgebras, bisimilarity, and quantitative generalisations

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"Simulation": w.r.t. the transitions.

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Our work: how do these two constructions relate to one another ?

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2. exposing our results on the correspondences between the two constructions.

Categorical generalisation: codensity and coupling-based liftings

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If one knows how to compare the states of a system before a transition, then the liftings pave the way towards comparisons after a transition.

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functor $\mathcal{D}$, with pseudometric $d\colon X \times X \to [0, 1]$, and $\mu, \nu \in \mathcal{D}X$,

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Codensity lifting: adding quantales

functor $\mathcal{D}$, with pseudometric $d\colon X \times X \to \mathcal{V}$, and $\mu, \nu \in \mathcal{D}X$, ($\mathsf{d_e}$ is a kind of Euclidean distance on $\mathcal{V}$),

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Codensity lifting: adding arbitrary functors
functor F, with pseudometric $d\colon X \times X \to \mathcal{V}$, and $x, y \in FX$,

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Codensity lifting: replacing the expected values by arbitrary modalities

functor F, with pseudometric $d\colon X \times X \to \mathcal{V}$, and $x, y \in FX$, ($\tau\colon F\mathcal{V} \to \mathcal{V}$ is a kind of modality replacing $\mathbb{E}\colon \mathcal{D}[0,1] \to [0,1]$).

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Important: expected values $\mathbb{E}\colon \mathcal{D}[0,1] \to [0,1]$ are replaced by some modality $\tau\colon F\mathcal{V} \to \mathcal{V}$.

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Question: for which functors, quantales, and modalities does duality hold ?

Correspondences between the codensity and coupling-based liftings

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There is a “duality-like” result for DFA with Γ containing one modality per letter in the alphabet plus one modality for terminal states.

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$$G ::= A \mid \mathsf{Id}_\tau \mid A \times G \mid \coprod G$$

$$F ::= A \mid \mathsf{Id}_\tau \mid \mathcal{P} \circ G \mid \mathcal{D} \circ G \mid \prod F_i \mid \coprod F_i$$

where the subscripts τ indicate some choices of modality to obtain initial duality results.

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Our work also includes some limitation: we show that some usual notion of bisimulation for conditional transition systems cannot be retrieved using our construction.

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Thank you for your attention !