Correspondences between codensity and coupling-based liftings, a practical approach

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Introduction: coalgebras, bisimulations, and quantitative generalisations

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"Simulation": w.r.t. the transitions.

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Categorical generalisation: codensity and coupling-based liftings

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If one knows how to compare the states of a system before a transition, then the liftings pave the way towards comparisons after a transition.

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Important: expected values $\mathbb{E}\colon \mathcal{D}[0,1] \to [0,1]$ are replaced by some modality $\tau\colon F\mathcal{V} \to \mathcal{V}$.

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Question: for which functors, quantales, and modalities does duality hold ?

Correspondences between the codensity and coupling-based liftings

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Question: can we find general results on the generalised Kantorovich-Rubinstein duality ?

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There is a “duality-like” result for DFA with Γ containing one modality per letter in the alphabet plus one modality for terminal states.

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$$G ::= A \mid \mathrm{Id}_\tau \mid A \times G \mid \coprod G$$
$$F ::= A \mid \mathrm{Id}_\tau \mid \mathcal{P} \circ G \mid \mathcal{D} \circ G \mid \prod F_i \mid \coprod F_i$$

where the subscripts τ indicate some choices of modality to obtain initial duality results.

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Our work also includes some limitation: we show that some usual notion of bisimulation for conditional transition systems cannot be retrieved using our construction.

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Thank you for your attention !