# Correspondences between codensity and coupling-based liftings, a practical approach

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# Introduction: coalgebras, bisimulations, and quantitative generalisations

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### We need a quantitative generalisation !

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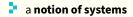
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#### **Presentation outline:**

- 1. giving more details on the two liftings,
- 2. exposing our results on the correspondences between the two constructions.

# Categorical generalisation: codensity and coupling-based liftings



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### If one knows how to compare the states of a system before a transition, then the liftings pave the way towards comparisons after a transition.

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### **Categorical generalisation ?**

We add the different ingredients we mentioned.

### **Codensity lifting:**

functor  $\mathcal{D}$ , with pseudometric  $d: X \times X \rightarrow [0, 1]$ , and  $\mu, \nu \in \mathcal{D}X$ ,

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#### **Codensity lifting:** adding quantales

functor  $\mathcal{D}$ , with pseudometric  $d: X \times X \to \mathcal{V}$ , and  $\mu, \nu \in \mathcal{D}X$ , (de is a kind of Euclidean distance on  $\mathcal{V}$ ),

$$\mathcal{D}^{\uparrow}d(\mu,\nu) = \bigwedge \qquad \mathsf{d}_{\mathsf{e}}\left(\mathbb{E}_{\mu}[f],\mathbb{E}_{\nu}[f]\right)$$

f: d→depseudometric morphism

### **Categorical generalisation ?**

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$$F^{\uparrow}d(x,y) = \bigwedge d_{e}(\mathbb{E}_{x}[f],\mathbb{E}_{y}[f])$$

 $f: X \rightarrow V$  pseudometric morphism

**Codensity lifting:** replacing the expected values by arbitrary modalities functor *F*, with pseudometric  $d: X \times X \rightarrow V$ , and  $x, y \in FX$ ,  $(\tau: FV \rightarrow V)$  is a kind of modality replacing  $\mathbb{E}: \mathcal{D}[0, 1] \rightarrow [0, 1]$ .

$$d_{e}(\tau \circ Ff(x), \tau \circ Ff(y))$$

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#### **Codensity lifting:**

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$$F^{\uparrow}d(x,y) = \bigwedge_{f: X \to \mathcal{V} \text{pseudometric morphism}} \mathsf{d}_{\mathsf{e}} \left( \tau \circ Ff(x), \tau \circ Ff(y) \right)$$

#### **Coupling-based lifting:**

$$F^{\downarrow}d(x,y) = \bigvee_{\sigma \in \Omega_F(x,y)} \tau \circ Fd(\sigma)$$

**Important:** expected values  $\mathbb{E}: \mathcal{D}[0, 1] \to [0, 1]$  are replaced by some modality  $\tau: F\mathcal{V} \to \mathcal{V}$ .

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Question: for which functors, quantales, and modalities does duality hold ?

# Correspondences between the codensity and coupling-based liftings



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**Question:** can we find general results on the generalised Kantorovich-Rubinstein duality ?

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There is a "duality-like" result for DFA with  $\Gamma$  containing one modality per letter in the alphabet plus one modality for terminal states.

### From duality to correspondence

**Correspondence:** 

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**Correspondence:** equality between the coupling-based and codensity liftings with:

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$$G ::= A \mid \operatorname{Id}_{\tau} \mid A \times G \mid \coprod G$$
$$F ::= A \mid \operatorname{Id}_{\tau} \mid \mathcal{P} \circ G \mid \mathcal{D} \circ G \mid \prod F_i \mid \coprod F_i$$

where the subscripts  $\tau$  indicate some choices of modality to obtain initial duality results.

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Our work also includes some limitation: we show that some usual notion of bisimulation for conditional transition systems cannot be retrieved using our construction.

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#### Thank you for your attention !