Functor liftings and the Kantorovich-Rubinstein duality

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Abstract

This work's aim is to define the Wasserstein and Kantorovich liftings of functors in a categorical setting, using fibrations and quantales enriched categories, and to compare both liftings, generalizing the Kantorovich-Rubinstein duality. The latter is studied for both relations and pseudometrics defined on a quantale.

On relations we generalize a well-known inequality between the two liftings and prove that duality either cannot hold either is trivial for polynomial functors.

Some of these results are extended to pseudometrics. For the latter we devise a general but non-systematic method to prove that duality holds in particular cases. We apply it on different functors solving duality for the constant, coproduct involving, identity, powerset, and diagonal functors. Our method rests on a particular lemma that we name "technical lemma".

Contents

Introduction

Systems. Different kinds of systems are used throughout computer science. In many cases a system is made of states, represented by a set, and of different other data used in transition structures or as observables. Those data are encapsulated in diverse mathematical structures and are regarded as the syntax of systems. Syntax implies semantics. A way of giving a semantic to a state-based system is by considering equivalent states. The idea is that two states with equivalent data attached to them will be considered the same, because information one has on the system cannot distinguish the two states. Another way of stating this is that a minimal version of the system will send both states to the same minimal state.

Category theory. All those notions are formalized in category theory. (Transition) systems through coalgebras, i.e. maps of the form $c \colon X \to FX$ for some functor F. The object X is interpreted to be the state space, and F to be the type of the system, i.e. a mathematical structure on X that potentially contains other data. Coalgebras can be used to model a large variety of transition systems. For examples, nondeterministic systems are modeled by $X \to \mathcal{P}X$ with $\mathcal{P}$ the powerset functor; finite deterministic automata are modeled by $X \to X^\Sigma \times \{0,1\}$ where Σ is the alphabet, and $\{0,1\}$ an observable encoding whether the state is final or not. Equivalence of states are modeled by bisimulations. They can be defined using final coalgebras (see Section 1.1 or [7, Chapter 3]). Coalgebra theory was introduced by the seminal paper [11].

Quantitative systems. More recently a great interest was shown toward quantitative systems. If the observables are not discrete but continuous, one want to replace equivalences of states by pseudometrics expressing that two states can be close even though they are not equivalent. The papers [3, 4] introduce two ways of defining pseudometrics on coalgebras through two liftings of functors from the category **Set** of sets and functions to the category of pseudometric spaces with nonexpansive maps: the Kantorovich and the Wasserstein liftings. Those liftings have been generalized to a more general setting using fibrations in respectively [2] and [5]. This generalization is suited to model equivalences of states, pseudometrics, and more exotic notions of distance.

Relations on a quantale. The general framework we use represents distances between states as particular relations on quantales. The origin of this representation is Lawvere's work [10]. This will be presented in full generality in the sequel. For now, let us show how usual pseudometrics can be represented as categories enriched over the quantale $(\overline{\mathbb{R}}_+, +, 0)$. A pseudometric $r \colon X \times X \to \overline{\mathbb{R}}_+$ is a map that is reflexive, symmetric, and that verifies the triangle inequality. A category enriched over $\overline{\mathbb{R}}_+$ is a set of object X, and for each pair $(x,y) \in X \times X$ a homobject in $\overline{\mathbb{R}}_+$. Those homobjects define a map $r \colon X \times X \to \overline{\mathbb{R}}_+$ and are subject to usual category axioms, replacing identities by $r(x,x)$, and composition by the operation $+$. Thus, unit laws translate to $r(x,x) + r(x,y) = r(x,y) + r(y,y) = r(x,y)$, and thus for all $x \in X$, $r(x,x) = 0$.

Definition of composition gives $r(x, y) \leqslant r(x, z) + r(z, y)$ for all $x, y, z \in X$. These $\overline{\mathbb{R}}_+$-enriched categories are called generalized metrics. In particular pseudometrics are generalized metric that are symmetric. In general simple maps $r \colon X \times X \to \mathcal{V}$ replacing $\overline{\mathbb{R}}_+$ by any quantale are called $\mathcal{V}$-relations, and transitive, reflexive, and symmetric such relations are calles $\mathcal{V}$-pseudometrics.

Contributions. It is noted in [3, 4] that those two liftings have very different properties. However in some cases they coincide generalizing the Kantorovich-Rubinstein duality, and giving a canonical definition of metrics on coalgebras as in [3, 4]). The goal of the present paper is to investigate this generalized Kantorovich-Rubinstein duality. We studied it for both $\mathcal{V}$-relations and $\mathcal{V}$-pseudometrics. On $\mathcal{V}$-relations we generalize a well-known inequality between the two lifting, prove that for some functors (constant map and weak-pullbacks preserving) duality is trivial, and generalize this result for polynomial functors. Understanding that duality on $\mathcal{V}$-relations is not that interesting, we restrict the liftings to $\mathcal{V}$-pseudometrics and stufy duality there. Some properties we showed on $\mathcal{V}$-relations still hold, but overall duality is much more difficult to treat for $\mathcal{V}$-pseudometrics. We give a method to study it on some cases. This method is general though not systematic. We use it to solve the duality problem on diverse functors: constant, identity, powerset, diagonal functors, and functors involving coproducts.

Outline. The liftings will be defined in [5]'s setting. This fibrational setting is introduced in section 1 along with some categorical refreshers and useful lemmas, such as what we call the Technical lemma. This lemma will be used in Section 2 to prove the Kantorovich lifting is fibred and in Section 3 on different duality results. Section 1 concludes with the general definition of liftings and of well-behaved evaluation maps. The Kantorovich and Wasserstein liftings are defined in Section 2. Some elementary properties are given. Finally the corresponding duality is studied in Section 3.

Notations. Thereafter, the word "metric" will often be understood as "pseudometric". When considering a **Set** endofunctor F, X a set, and $t_1, t_2 \in X$, we name *coupling* of t_1 and t_2 an element $t \in F(X \times X)$ such that $F\pi_1(t) = t_1$ and $F\pi_2(t) = t_1$.

Given a set X and a unital quantale $\mathcal{V}$, that is a structured set with a distringuished unit object I, we will denote by $\kappa_X \colon X \to \mathcal{V}$ the function constant and equal to I.

1 Categorical preliminaries

1.1 Systems with coalgebras

As mentioned above, a great number of systems can be modeled using coalgebras. This is not actually of use in the rest of this work. Still it is important to remember the liftings of Section 2 are defined to lift the functor of some coalgebras as in [3], and that functors of Section 3 are used to model particular systems using coalgebras. For a more complete account on coalgebras see [7] or [11]. Here we define coalgebras in

general, their morphisms, final coalgebras, and we explain how the latter can be used to define bisimulations. Throughout this Section, $\mathcal{C}$ will denote an arbitrary category and $F\colon \mathcal{C} \to \mathcal{C}$ an arbitrary endofunctor.

Definition 1. An *F-coalgebra* is a data made of:

- an object $X \in \mathcal{C}$;

- an arrow $c\colon X \to FX$.

Coalgebras are generally represented using only the arrow $c\colon X \to FX$.

With $\mathcal{C} = \mathbf{Set}$ the category of sets and functions, we recover the presentation made of coalgebras in the introduction.

As most objects considered in category theory, there is a corresponding notion of morphism:

Definition 2. Let $c\colon X \to FX$ and $d\colon Y \to FY$ be two *F-coalgebras*. A *morphism of F-coalgebras* from c to d is a morphism $f\colon X \to Y$ in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\
FX & \xrightarrow[Ff]{} & FY
\end{array}
$$

in $\mathcal{C}$. It is noted $f\colon c \to d$.

Proposition 1. *The data made of:*

- *F-coalgebras as objects;*

- *F-coalgebra morphisms as arrows,*

is a well-defined category. It is called the category of *F*-coalgebras *and noted F*-**Coalg**.

A notion of behavior equivalence can be defined using final coalgebras :

Definition 3. Let $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor. A *final F-coalgebra* is a terminal object of *F*-**Coalg**, i.e. an *F*-coalgebra γ such that for any other *F*-coalgebra $c\colon X \to FX$ there is a unique coalgebra morphism $[\![-]\!]_c \colon c \to \gamma$.

Relations, bisimulations, and bisimilarity relations are defined on a category under mild assumptions. To make things simpler and because bisimulations are not the point here, we only consider the case of **Set**. The interested reader may look at Chapter 3 and Chapter 4 of [7].

Definition 4. Suppose that $\mathcal{C} = \mathbf{Set}$ and a final F-coalgebra γ exists. Let $c\colon X \to FX$ an F-coalgebra. A *bisimulation relation* on c is a relation $b \subseteq X \times X$ such that,

$$(x, y) \in b \Rightarrow [\![x]\!]_c = [\![y]\!]_c$$

The *bisimilarity relation* on c is the greatest of all bisimulations for the inclusion order.

As mentioned in the introduction, two elements linked by a bisimulation are interpreted to have the same behavior. Still here we are interested not in behavior equivalences but in behavioral metrics. Instead of considering usual relations, we will consider relations using more exotic truth values through quantales (Sections 1.2 and 1.3). Expressing how relations on a quantale relate to coalgebras modeling systems (Section 1.1) is done using fibrations (Section 1.4) and then by lifting functors from **Set** to the category of relations on the said quantale (Section 1.6). Well-behaved liftings are defined using evaluation functions (Section 1.5).

1.2 Quantales

As mentioned above, we will consider not bisimulations nor behavioral pseudometrics, but relations using exotic truth values defined by quantales, that is complete lattices with some more structure. In this Section quantales are defined and two important examples are given. Then we retrieve intuitive properties we may want on quantales and give conditions for them to hold. Note that thereafter the word "suplattice" replaces "complete lattice". That is because morphisms of complete lattices preserve meets and morphisms of suplattices preserve joins.

Definition 5. A *quantale* $\mathcal{V}$ is a suplattice with an associative operation $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is distributive over arbitrary joins.

When $\otimes$ is commutative, $\mathcal{V}$ is called *symmetric*. When $\otimes$ has a *unit element*, $\mathcal{V}$ is said *unital*. The unit object is generally noted I.

In this context, given $r \in \mathcal{V}$, we note:

- $\uparrow r = \{v \in \mathcal{V} \mid v \geqslant r\}$, i.e. the subset of $\mathcal{V}$ made of elements "more true" than r;

- $\mathrm{true}_r\colon \uparrow r \hookrightarrow \mathcal{V}$ the inclusion morphism;

- $u \in F(\uparrow r)$ for $F\colon \mathbf{Set} \to \mathbf{Set}$ a functor, when u is in the image of $F(\mathrm{true}_r)$.

Being posets, suplattices are preorder categories. In this interpretation $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a bifunctor and $\bot, \top \in \mathcal{V}$ are respectively initial and final objects. Following this categorical interpretation:

Proposition 2. *A unital quantale is equivalently a closed monoidal suplattice $(\mathcal{V}, \otimes, I)$. In particular, for all $x \in \mathcal{V}$, $x \otimes -$ then has a right adjoint $[x, -]$ called the left internal hom functor, and so does $- \otimes x$ whose adjoint $[\![x, -]\!]$ is called the right internal hom functor. In the case $\mathcal{V}$ is symmetric, the internal hom functors coincide and are equal to $[-, -]\colon \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ a bifunctor that is contravariant (resp. covariant) in its first (resp. second) argument and called the internal hom functor.*

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Example 1. Two quantales are of importance as they are used to retrieve equivalence relations and usual pseudometrics. Both are symmetric and unital.

- $(\mathbf{2}, \wedge, 1)$ with $\mathbf{2} = \{0, 1\}$ the usual *boolean algebra*. Here $[x, -] = x \to -$. Usual relations are maps of the form $f \colon X \times X \to \mathbf{2}$. Requiring those maps to be symmetric, reflexive, and transitive gives equivalence relations.

- $(\overline{\mathbb{R}}_+, +, 0)$ the *extended reals with reversed order*. Here $[x, y] = y \dot{-} x = \max \{y - x, 0\}$ is the *truncated substraction*. Similarly, "relations" on this quantale are maps $f \colon X \times X \to \overline{\mathbb{R}}_+$. Requiring them to be symmetric, reflexive, and that the triangle inequality holds gives pseudometrics. We will see in Section 1.3 that the triangle inequality is a kind of transitivity. The quantale order will always be denoted $\leqslant$ whereas the usual order on the reals will be noted $\leqslant_{\mathbb{R}}$. By definition $\leqslant \, = \, \geqslant_{\mathbb{R}}$.

  More generaly one can consider quantales of the form $([0, \top], \min \{+, \top\}, 0)$, giving similar though slightly different properties.

In the rest of this Section, $\mathcal{V}$ will always denote a unital symmetric quantale.

An easy result to prove is the following. Its interpretation is that $\otimes$ and $[-, -]$ are "pseudo-inverse" to one another. It generalizes the inequality $(y \dot{-} x) + x \geqslant_{\mathbb{R}} y$ on reals. Its categorical meaning is simply that the counit of the adjunction between $[x, -]$ and $- \otimes x$ is a natural transformation from $[x, -] \otimes x$ to the identity.

Lemma 1. *For all $x, y \in \mathcal{V}$,*

$$[x, y] \otimes x \leqslant y$$

The next result is still simple but already more interesting. Its interpretation is that $\otimes$ adds falseness, or equivalently that $[x, y]$ is like adding x's truthness to y.

Lemma 2. *The following propositions are equivalent:*

1. $I = \top$;

2. $\forall x, y \in \mathcal{V}, \ x \geqslant x \otimes y$;

3. $\forall x, y \in \mathcal{V}, \ [y, x] \geqslant x$;

4. $\forall x \in \mathcal{V}, \ [x, x] = \top$.

Proof. $2 \Rightarrow 1$; in particular, $I \geqslant I \otimes \top = \top$ and as $\top \geqslant I$ we get $I = \top$.

$2 \Leftrightarrow 3$; consider $x, y \in \mathcal{V}$. Because $[y, -]$ is a right adjoint to $- \otimes y$, $x \geqslant x \otimes y$ if and only if $[y, x] \geqslant x$.

$1 \Rightarrow 4$; suppose $I = \top$. Let us consider $x \in \mathcal{V}$. Because of the adjoint situation between $x \otimes -$ and $[x, -]$, and as $x \otimes I \leqslant x$, $[x, x] \geqslant I = \top$ so that by definition of $\top$, $[x, x] = \top$.

$4 \Rightarrow 2$; suppose that for all $x \in \mathcal{V}$, $[x, x] = \top$. Let us consider $x, y \in \mathcal{V}$. Still using the adjoint situation and because $[x, x] \geqslant \top$, $x \geqslant x \otimes \top$, and, as $\top \geqslant y$ and $x \otimes -$ is monotone, $x \geqslant x \otimes y$. $\qquad \square$

Example 2. In the quantales 2 and $\overline{\mathbb{R}}_+$, $I = \top$, as $1 = \top$ and $0 = \top$ respectively.

We can now consider predicates and relations defined on a quantale, and give a generalization of equivalence relations and pseudometrics for any quantale.

1.3 $\mathcal{V}$-predicates, $\mathcal{V}$-relations, and $\mathcal{V}$-pseudometrics

Now that a structure (quantales) has been introduced to represent truth values we want a generalization of notions used for behavior equivalences or behavioral metrics: predicates, relations, generalized metrics, pseudometrics, and metrics. This way of representing pseudometrics originates in Lawvere's work [10]. In this Section are defined $\mathcal{V}$-valued predicates and relations, particular properties of $\mathcal{V}$-valued relations such as reflexivity, transitivity, or symmetry, and finally $\mathcal{V}$-pseudometrics and metrics. This first part follows the presentation of $\mathcal{V}$-structures that is made in [5]. In the second part of this Section we define a particular $\mathcal{V}$-relation that will be used in the definition of the Kantorovich lifting: the euclidean relation; some of its properties are given. Finally, we introduce the Technical lemma that will be particularly important in the study of the Kantorovich-Rubinstein duality. Throughout this Section, $\mathcal{V}$ will denote an arbitrary quantale. When considering maps, the order on $\mathcal{V}$ will be extended pointwise:

$$\forall f, g \colon X \to \mathcal{V}, \ f \leqslant g \Leftrightarrow (\forall x \in X, \ f(x) \leqslant g(x))$$

Definition 6. A $\mathcal{V}$-valued predicate on a set X is a map $p \colon X \to \mathcal{V}$.

Proposition 3. *The data made of:*

- *objects: $\mathcal{V}$-valued predicates $p \colon X \to \mathcal{V}$;*

- *morphisms: maps $f \colon X \to Y$ such that $p \leqslant q \circ f$ from $p \colon X \to \mathcal{V}$ to $q \colon Y \to \mathcal{V}$,*

*is a well-defined category noted $\mathcal{V}$-**Pred** and called the category of $\mathcal{V}$-valued predicates.*

Definition 7. A $\mathcal{V}$-valued relation on a set X is a map $r \colon X \times X \to \mathcal{V}$.

Proposition 4. *The data made of:*

- *objects: $\mathcal{V}$-valued relations $r \colon X \times X \to \mathcal{V}$;*

- *morphisms: maps $f \colon X \to Y$ such that $r \leqslant s \circ (f \times f)$ from $r \colon X \times X \to \mathcal{V}$ to $s \colon Y \times Y \to \mathcal{V}$,*

*is a well-defined category noted $\mathcal{V}$-**Rel** and called the category of $\mathcal{V}$-valued relations.*

The condition defining morphisms of $\mathcal{V}$-relations can be seen as a generalization of inclusion for usual relations, or nonexpansiveness for functions between pseudometric spaces.

Usual properties of $\mathcal{V}$-relations are now defined:

Definition 8. Let X be a set and $r, s \colon X \times X \to \mathcal{V}$ be two $\mathcal{V}$-valued relations.

- The *composition* of r with s is defined by

$$r{\cdot}s \colon X \times X \to \mathcal{V}$$
$$(x, y) \mapsto \bigvee \{r(x, z) \otimes s(z, y) \mid z \in X\}$$

- The *diagonal $\mathcal{V}$-valued relation* on X is given by

$$\mathrm{diag}_X \colon X \times X \to \mathcal{V}$$
$$(x, y) \mapsto \begin{cases} I \text{ if } x = y \\ \bot \text{ else} \end{cases}$$

- The *symmetry morphism* is

$$\mathrm{sym}_X \colon X \times X \to X \times X$$
$$(x, y) \mapsto (y, x)$$

- The $\mathcal{V}$-relation r is said

  - *reflexive* if $r \geqslant \mathrm{diag}_X$;
  - *transitive* if $r{\cdot}r \leqslant r$;
  - *symmetric* if $r = r \circ \mathrm{sym}_X$.

Those properties allow generalizations of pseudometrics and metrics to be defined:

Definition 9. A reflexive, transitive, and symmetric $\mathcal{V}$-relation will be called a $\mathcal{V}$-*pseudometric*. When symmetry does not hold, the relation is called a *generalized $\mathcal{V}$-metric*. A $\mathcal{V}$-pseudometric $r \colon X \times X \to \mathcal{V}$ will be called a $\mathcal{V}$-*metric* whenever for all $x \neq y \in X$, $r(x, y) \neq \top$.

Example 3. With:

- $\mathbf{2}$, pseudometrics are equivalence relations;
- $\overline{\mathbb{R}}_+$, pseudometrics are usual pseudometrics, metrics usual metrics.

Proposition 5. *The data made of the part of $\mathcal{V}$-$\mathbf{Rel}$ with generalized $\mathcal{V}$-metric is a full subcategory of $\mathcal{V}$-$\mathbf{Rel}$ noted $\mathcal{V}$-$\mathbf{Cat}$. If the relations are $\mathcal{V}$-pseudometrics we also get a full subcategory noted $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sym}}$. If the relations are $\mathcal{V}$-metrics, we again get a full subcategory noted $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{met}}$.*

Proposition 6. *Equivalently $\mathcal{V}$-$\mathbf{Cat}$ is the category of small $\mathcal{V}$-enriched categories.*

The enriched character of this construction is not really of interest here. The interested reader may look at [8] for an introduction to enriched category theory.

The Kantorovich lifting that will be defined in Section 2 was originally introduced for the quantale $\overline{\mathbb{R}}_+$ only. This first definition (see [3, 4]) made use of the extended euclidean distance $\mathrm{d_e}\colon \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$. In order to define the Kantorovich lifting for any symmetric unital quantale, we introduce the euclidean relation that coincides with the extended euclidean distance for the quantale $\overline{\mathbb{R}}_+$.

In the rest of this Section $\mathcal{V}$ will always denote a symmetric unital quantale.

Definition 10. The *euclidean relation* on $\mathcal{V}$ is noted $\mathrm{d_e}\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and defined as:

$$\forall x, y \in \mathcal{V}, \ \mathrm{d_e}(x, y) = \bigwedge \{[x, y], [y, x]\}$$

Example 4. With

- 2, $\mathrm{d_e}(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ else} \end{cases}$;

- $\overline{\mathbb{R}}_+$, $\mathrm{d_e}$ is the extended euclidean distance.

Let us comment the intuitive meaning of $\mathrm{d_e}$. The operation $\otimes$ on $\mathcal{V}$ corresponds, intuitively, to adding x's and y's falseness. We would like $\mathrm{d_e}$ to measure the difference of x's and y's truthness. The adjoint of $x \otimes -$ given by the internal hom $[-, x]$ is the closest thing we can find that would define an inverse of $\otimes$, i.e. a difference. Because we do not know which one of x or y is the "most true", we take the meet over $[x, y]$ and $[y, x]$.

For 2 and $\overline{\mathbb{R}}_+$, $\mathrm{d_e}$ is a $\mathcal{V}$-metric. We are going to prove that this holds whenever $I = \top$.

Proposition 7. *The euclidean relation* $\mathrm{d_e}$ *is a* $\mathcal{V}$-*metric.*

Proof. Note that by definition $\mathrm{d_e}$ is always symmetric. The three following lemmas end the proof. □

Lemma 3. *The euclidean relation is reflexive.*

Proof. Let $x \in \mathcal{V}$. Then,
$$\mathrm{d_e}(x, x) = [x, x]$$
As $x \geqslant x = x \otimes I$, we get $[x, x] \geqslant I$, meaning $\mathrm{d_e}(x, x) \geqslant I$. □

Lemma 4. *If* $[-, -]$ *is transitive as a* $\mathcal{V}$-*relation, then so is* $\mathrm{d_e}$.

Proof. By definition of d_e we know that $[x, z] \geqslant d_e(x, z)$, and $[z, y] \geqslant d_e(z, y)$, so that $[x, z] \otimes [z, y] \geqslant d_e(x, z) \otimes d_e(z, y)$ and as $[-, -]$ is transitive, $[x, y] \geqslant d_e(x, z) \otimes d_e(z, y)$. Similarly $[y, x] \geqslant d_e(y, z) \otimes d_e(z, x)$ and as d_e and $\otimes$ are both symmetric,

$$d_e(y, z) \otimes d_e(z, x) = d_e(x, z) \otimes d_e(z, y)$$

proving that $d_e(x, y) \geqslant d_e(x, z) \otimes d_e(y, z)$ and that d_e is transitive. $\qquad\square$

Lemma 5. *The $\mathcal{V}$-relation $[-, -]$ is transitive.*

Proof. Let $x, y, z \in \mathcal{V}$. Then, using the fact $\otimes$ is monotone and the lemma 1 we get

$$x \otimes [x, z] \otimes [z, y] \leqslant z \otimes [z, y]$$
$$\leqslant y$$

Using the adjoint situation between $x \otimes -$ and $[x, -]$, we get

$$[x, y] \geqslant [x, z] \otimes [z, y]$$

This being being true for all z, we get that

$$[x, y] \geqslant \bigwedge \{[x, z] \otimes [z, y] \mid z \in \mathcal{V}\}$$

and thus,

$$[-, -] \geqslant [-, -] \cdot [-, -]$$

proving $[-, -]$ is transitive. $\qquad\square$

We give a last result on the euclidean relation that will be useful:

Lemma 6. *For all $x, y \in \mathcal{V}$,*

$$d_e(x, y) = \top \Rightarrow x = y$$

If furthermore $I = \top$ then

$$d_e(x, y) = \top \Leftrightarrow x = y$$

Proof. Whenever $I = \top$ we know using the lemma 2 that $[x, x] = \top$ meaning $d_e(x, x) = \top$. Suppose $d_e(x, y) = \top$. This means that $[x, y] = [y, x] = \top = I$. Using the fact $[z, -]$ and $z \otimes -$ are adjoints for any $z \in \mathcal{V}$, this implies $x \geqslant y$ and $y \geqslant x$ meaning $x = y$. $\qquad\square$

We know give a result we call the Technical lemma that will be used extensively in Sections 2 and 3 (see Propositions 22, 26, 27, 32). It is our main tool to study the Kantorovich-Rubinstein duality on particular functors (see Section 3.2). It depends on the following lemma stating that under the right hypotheses, we can always extend a morphism in $\mathcal{V}$-**Rel**.

Lemma 7. *Let $r \colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-pseudometric, and $i \colon Y \hookrightarrow X$ an injective map. We will note $i^*r = r \circ (i \times i)$ and $i^*f = f \circ i$ for $f \colon X \to \mathcal{V}$. Then for all morphisms $g \colon i^*(r) \to \mathrm{d_e}$ in $\mathcal{V}$-**Pred** there exists an "extension" of g to r. More precisely there exists $f \colon r \to \mathrm{d_e}$ such that*

$$g = i^*(f)$$

Remark 1. The notation i^* comes from a functor defined later (Definition 13).

Proof. We will identify Y with its image in X. We define $f \colon r \to \mathrm{d_e}$ by the following:

$$\forall x \in X, \ f(x) = \bigvee \{g(u) \otimes r(x, u) \mid u \in Y\}$$

We have to prove that f is a well-defined morphism from r to $\mathrm{d_e}$ in $\mathcal{V}$-**Rel** and that $i^*(f) = g$.

First, let us show f is well-defined, i.e. that

$$r \leqslant \mathrm{d_e} \circ (f \times f)$$

Let us consider $z, w \in X$,

$$f(z) = \bigvee \{g(u) \otimes r(z, u) \mid u \in Y\}$$
$$f(w) = \bigvee \{g(u) \otimes r(w, u) \mid u \in Y\}$$

Then, r being transitive, for all $u \in Y$,

$$r(u, z) \geqslant r(u, w) \otimes r(w, z)$$

which gives,

$$g(u) \otimes r(u, z) \geqslant r(u, w) \otimes r(w, z) \otimes g(u)$$

which, by taking the join gives,

$$f(z) \geqslant f(w) \otimes r(w, z)$$

which is equivalent to

$$[f(w), f(z)] \geqslant r(w, z)$$

By inverting z and w roles,

$$[f(z), f(w)] \geqslant r(z, w)$$

and finally by symmetry of r and definition of $\mathrm{d_e}$,

$$\mathrm{d_e} \circ (f \times f)(w, z) \geqslant r(w, z)$$

This being true in any case, f is a well-defined morphism from r to $\mathrm{d_e}$.

Now let us prove $i^*(f) = g$, i.e., identifying Y with $i[Y]$,

$$\forall y \in Y, \ f(y) = g(y)$$

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Consider $y \in Y$. By definition,

$$f(y) = \bigvee \{g(u) \otimes r(y, u) | u \in Y\}$$

Let us fix $u \in Y$. We know that g is a morphism from $i^*(r)$ to $\mathrm{d_e}$.

$$r(y, u) \leqslant \mathrm{d_e}(g(y), g(u))$$

By definition of $\mathrm{d_e}$ this implies that

$$r(y, u) \leqslant [g(u), g(y)]$$

and using the adjoint situation between $[-, -]$ and $\otimes$,

$$r(y, u) \otimes g(u) \leqslant g(y) \leqslant g(y) \otimes r(y, y)$$

as r is reflexive. Thus $f(y) = g(y) \otimes r(y, y) = g(y)$ and f is indeed an extension of g to r. □

Lemma 8 (*Technical lemma*). *Let* $r \colon X \times X \to \mathcal{V}$ *be a* $\mathcal{V}$-*pseudometric. Given a map* $\Delta \colon \mathrm{Hom}(r, \mathrm{d_e}) \to \mathcal{V}$, *if there exists an injective map* $i \colon Y \hookrightarrow X$ *and a morphism* $g \colon i^*(r) \to \mathrm{d_e}$ *in* $\mathcal{V}$-**Rel** *such that*

$$\forall f \colon r \to \mathrm{d_e}, \ (i^*(f) = g) \Rightarrow (\Delta(f) = s)$$

for some $s \in \mathcal{V}$, *then there exists* $f \colon r \to \mathrm{d_e}$ *such that* $\Delta(f) = s$.

Proof. This is a direct application of the lemma 7. □

Remark 2. Note that the condition $\Delta(f) = s$ could be replaced by any property on f. However, for our use, this is all we need.

1.4 Fibrations

When considering behavior equivalences, or behavioral metrics on the objects of a category, one can regard the structure expressing this behavior notion as above this category. Here, $\mathcal{V}$-relations above the category **Set**. This could be expressed through forgetful functors (as in [3]), but following [5] fibrations are more suited. For a full introduction to fibrations see [6]. We give the definitions of cartesian liftings, fibrations, and of some properties of the latter such as split fibrations and bifibrations. We mention the reindexing functor associated to a fibration, and the direct image functor associated to a split fibration. Finally we take a look at the bifibrations defined on the categories $\mathcal{V}$-**Pred** and $\mathcal{V}$-**Rel**. Thus, $\mathcal{V}$ will denote an arbitrary quantale in this Section. We give more or less the same presentation of fibrations than in [5]. At first, $p \colon \mathcal{E} \to \mathcal{B}$ will denote a functor from a category $\mathcal{E}$ to a category $\mathcal{B}$.

Cartesian liftings of morphisms are necessary to define fibrations:

Definition 11. Let f in $\mathcal{B}$ be a morphism. A *cartesian lifting* of $f\colon X \to Y$ with regard to p is a morphism $\widetilde{f}\colon S \to R$ in $\mathcal{E}$ such that $p(\widetilde{f}) = f$, which is universal in the following way: for any morphism $u\colon Q \to R$ in $\mathcal{E}$ with $p(u) = f \circ g$ for some $g\colon Z \to X$ in $\mathcal{B}$, there is a unique $v\colon Q \to S$ with $p(v) = g$ such that $u = \widetilde{f} \circ v$, i.e. we have the following diagram:
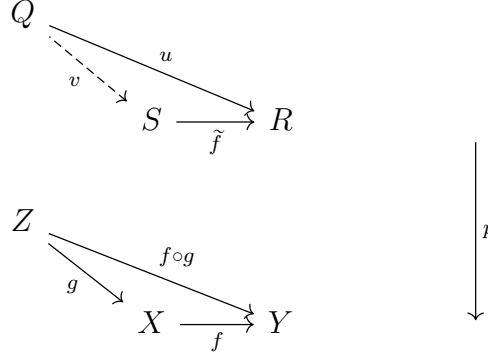
Definition 12. The functor $p\colon \mathcal{E} \to \mathcal{B}$ is called a *fibration* if for all $f\colon X \to Y$ in $\mathcal{B}$, and $R \in E$ with $p(R) = Y$, there is a cartesian lifting $\widetilde{f_R}\colon f^*(R) \to R$ of f. We say that:

- $Q \in \mathcal{E}$ is *above* $Z \in \mathcal{B}$ when $p(Q) = Z$, and similarly for morphisms;

- $\mathcal{E}$ is the *total category*;

- $\mathcal{B}$ is the *base category*;

- the subcategory $\mathcal{E}_X$ of $\mathcal{E}$ made of objects above X and morphisms above Id_X is the *fibre* above X.

From now on and until the end of this Section, $p\colon \mathcal{E} \to \mathcal{B}$ will always denote a fibration.

The idea is that the fibre $\mathcal{E}_X$ above X represents some predicates over X. The cartesian liftings then expresse a kind of precondition semantics, allowing one to make the predicates work with the transition structure of the system that will be defined in the base category $\mathcal{B}$. The operation sending R to $f^*(R)$ is actually more than a map of objects:

Proposition 8. *Given a fibration $p\colon \mathcal{E} \to \mathcal{B}$ and a morphism $f\colon X \to Y$ in $\mathcal{B}$, the data made of*

- *the map $f^*\colon R \in \mathcal{E}_Y \mapsto f^*(R) \in \mathcal{E}_X$ on objects;*

- *the map f^* on morphisms of $\mathcal{E}_Y$ sending $m\colon R \to S$ to v the unique morphism above Id_X such that by cartesian property of $\widetilde{f_S}$, $\widetilde{f_S} \circ v = \widetilde{f_R} \circ m$,*

is a functor $f^*\colon \mathcal{E}_Y \to \mathcal{E}_X$.

Definition 13. The previously defined functor $f^* \colon \mathcal{E}_Y \to \mathcal{E}_X$ is called the *reindexing functor* along f.

Remark 3. Note that cartesian liftings are unique up-to isomorphism, so that the definition of f^* actually requires a choice on morphisms. Thus a reindexing functor always exists if and only if the axiom of choice is supposed true. Sometimes this choice can be made in a way that is coherent. That is the meaning of the next definition.

Definition 14. The fibration $p \colon \mathcal{E} \to \mathcal{B}$ is called *split* when for all morphisms f and g compatible,
$$(gf)^* = f^* g^*$$

Fibrations that we will consider are all split and of a particular kind:

Definition 15. The fibration $p \colon \mathcal{E} \to \mathcal{B}$ is called a *bifibration* when both p and p^{op} are fibrations.

There is a nice and useful characterization of bifibrations (see [6, Lemma 9.1.2]).

Proposition 9. *The fibration p is a bifibration if and only if the reindexing functor f^* has a left-adjoint functor noted Σ_f and called the direct image functor along f.*

As said earlier, we want $\mathcal{V}$-valued predicates and relations to correspond to behavior structures. Note $\mathcal{V}\text{-}\mathbf{Rel} \overset{i}{\hookrightarrow} \mathcal{V}\text{-}\mathbf{Pred}$. We consider the following fibrations:

Proposition 10. *The forgetful functor $U \colon \mathcal{V}\text{-}\mathbf{Pred} \to \mathbf{Set}$ defines a bifibration. Furthermore, through the diagonal functor $\Delta \colon \mathbf{Set} \to \mathbf{Set}$ sending X to $X \times X$ and $f \colon X \to Y$ to $f \times f \colon X \times X \to Y \times Y$, we get the following situation in $\mathbf{Cat}$ the category of (small) categories, with a pullback corresponding to a change-of-base situation,*

$$
\begin{array}{ccc}
\mathcal{V}\text{-}\mathbf{Rel} & \overset{i}{\longhookrightarrow} & \mathcal{V}\text{-}\mathbf{Pred} \\
\downarrow & \lrcorner & \downarrow \\
\mathbf{Set} & \underset{\Delta}{\longrightarrow} & \mathbf{Set}
\end{array}
$$

wich yields another bifibration on $\mathcal{V}$-relations. For $p \colon X \to \mathcal{V}$, $q \colon Y \to \mathcal{V}$, $r \colon X \times X \to \mathcal{V}$, $s \colon Y \times Y \to \mathcal{V}$, and $f \colon X \to Y$,

$$f^*(q) = q \circ f \quad ; \quad \Sigma_f(p)(y) = \bigvee \left\{ p(x) \mid x \in f^{-1}(y) \right\}$$

and

$$f^*(s) = s \circ (f \times f) \quad ; \quad \Sigma_f(r)(y, y') = \bigvee \left\{ r(x, x') \mid (x, x') \in (f \times f)^{-1}(y, y') \right\}$$

Now let us see how to construct liftings, but before we introduce the last important notion necessary to our study: evaluation maps.

1.5 Data, modality, evaluation with algebras

In categorical computer science algebras have been widely used to represent finite data (see [7, Section 2.4] for a general presentation). Here they are used as evaluation maps for behavioral metrics. They are the dual (in a categorical sens) of coalgebras. We only give the one definition.

Definition 16. Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor. An *F-algebra* is a data made of:

- an object $X \in \mathcal{C}$;

- an arrow $a\colon FX \to X$ in $\mathcal{C}$.

Algebras are usually represented using their arrow $a\colon FX \to X$ only.

Given a fibration $p\colon \mathcal{E} \to \mathcal{B}$ and a functor $F\colon \mathcal{B} \to \mathcal{B}$, a lifting of F will be another functor $\widetilde{F}\colon \mathcal{E} \to \mathcal{E}$ that looks a lot like F, but in $\mathcal{E}$. Such liftings will be first defined for the fibration on $\mathcal{V}$-**Pred** and then for the one on $\mathcal{V}$-**Rel**. It so happens that some of those liftings are characterized by particular algebras of the form $a\colon F\mathcal{V} \to \mathcal{V}$ and are used to define the Wasserstein lifting. The idea is to define $\widetilde{F} = a \circ F$ on $\mathcal{V}$-predicates. Furthermore, algebras are used in [3] in the definition of the Kantorovich lifting, and our definition follows the same scheme. For all those reasons such algebras are quite important here.

We will name *evaluation map* or evaluation function an algebra of the form $F\mathcal{V} \to \mathcal{V}$, and note it ev$\colon F\mathcal{V} \to \mathcal{V}$ with a possible subscript. Evaluation maps are a way to "smash" the values in $F\mathcal{V}$ to simple truth values in $\mathcal{V}$ in a way that is coherent.

In the literature evaluation maps are also called modalities and noted $\tau\colon F\Omega \to \Omega$ for Ω an object of truth values (see [9] for example).

1.6 Liftings on $\mathcal{V}$-Pred

This Section gives a general presentation of liftings of functors on fibrations. Doing so will enable us to define the objects of our study in the next Section: the Wasserstein and the Kantorovich liftings. In this Section we define liftings in general, fibred liftings, and we give a characterization of fibred liftings on $\mathcal{V}$-**Pred** using evaluation maps. Then we introduce different classes of evaluation maps that are of importance: the canonical evaluation map, behaved, and well-behaved evaluation maps. This Section is very similar to the presentation of liftings made in [5]. Still we link it to [3]'s version through the lemma 9. Throughout this Section, $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{E}$ and $\mathcal{E}'$ will be categories, $F\colon \mathcal{B} \to \mathcal{B}'$ a functor, $p\colon \mathcal{E} \to \mathcal{B}$ and $p'\colon \mathcal{E}' \to \mathcal{B}'$ fibrations, and $\mathcal{V}$ a quantale.

Definition 17. A *lifting* of F for the fibrations p and p' is a functor $\widetilde{F}\colon \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\widetilde{F}} & \mathcal{E}' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\
\mathcal{B} & \xrightarrow[F]{} & \mathcal{B}'
\end{array}
$$

Remark 4. In general liftings will be noted $\widetilde{F}$. Liftings to $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}' = \mathcal{V}\text{-}\mathbf{Pred}$ will be noted $\widehat{F}$.

Particular liftings are of importance in this study:

Proposition 11. *Let* $\widetilde{F} \colon \boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{E}}'$ *be a lifting of* F *and* $f \colon X \to Y$ *a morphism in* $\boldsymbol{\mathcal{B}}$. *Then there are canonical morphisms*

$$\forall R \in \boldsymbol{\mathcal{E}}_Y, \ \widetilde{F} \circ f^*(R) \to (Ff)^* \circ \widetilde{F}(R)$$

which are the components of a natural transformation $\widetilde{F} \circ f^* \Rightarrow (Ff)^* \circ \widetilde{F}$.

Definition 18. The lifting $\widetilde{F}$ is called a *fibred lifting* when the natural transformation $\widetilde{F} \circ f^* \Rightarrow (Ff)^* \circ \widetilde{F}$ is a natural isomorphism.

From now on, $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}' = \mathbf{Set}$, F is a $\mathbf{Set}$ endofunctor, and $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}' = \mathcal{V}\text{-}\mathbf{Pred}$. We focus on $\mathcal{V}\text{-}\mathbf{Pred}$ liftings.

In order to define the Wasserstein lifting in Section 2, we want a way to construct liftings from $\mathbf{Set}$ to $\mathcal{V}\text{-}\mathbf{Pred}$. In this prospect we have the following result [5, Proposition 12] that makes use of monotone evaluation maps.

Definition 19. Let $\mathrm{ev} \colon F\mathcal{V} \to \mathcal{V}$ an evaluation map. We say that ev is *monotone* when it is monotone along the relation lifting (see [7, Chapters 3 and 4]) of the order $\leqslant$ of $\mathcal{V}$; more precisely the relation lifting $\ll$ of $\leqslant$ is defined by,

$$\forall x_1, x_2 \in F\mathcal{V}, \ (x_1 \ll x_2) \Leftrightarrow (\exists r \in F \leqslant, \ F\pi_1 r = x_1 \wedge F\pi_2 r = x_2)$$

and ev being monotone is expressed by,

$$\forall x_1, x_2 \in F\mathcal{V}, \ (x_1 \ll x_2) \Rightarrow (\mathrm{ev}(x_1) \leqslant \mathrm{ev}(x_2))$$

Remark 5. A priori the relation $\ll$ is not an order nor even a preorder. What we do know is that if F preserves weak pullbacks then $\ll$ is a preorder (see [1]).

Proposition 12. *There is a one-to-one correspondence between:*

- *fibred liftings* $\widehat{F}$ *of* $\mathcal{V}\text{-}\mathbf{Pred}$;

- *monotone evaluation maps* $\mathrm{ev} \colon F\mathcal{V} \to \mathcal{V}$.

In particular, when $\widehat{F}$ *is fibred,* $\widehat{F}(p) = \mathrm{ev} \circ F(p)$.

Remark 6. Note that whenever ev is not monotone, $\mathrm{ev} \circ F$ cannot define a $\mathcal{V}\text{-}\mathbf{Pred}$ lifting. The setting in which the Wasserstein and the Kantorovich liftings of Section 2 will be compared implies the use of a lifting to $\mathcal{V}\text{-}\mathbf{Pred}$ defined using an evaluation map. Thus we will always use a monotone evaluation map.

In [5] a particular evaluation map is considered and gives good properties (see [5, Proposition 22]). We give its definition here, but we will show (Proposition 33) that in general it does not yield good properties regarding the Kantorovich-Rubinstein duality.

Definition 20. The *canonical evaluation map* associated to F and $\mathcal{V}$ is given by

$$\mathrm{ev}_{\mathrm{can}}\colon F\mathcal{V} \to \mathcal{V}$$
$$u \mapsto \bigvee \{r \mid u \in F(\uparrow r)\}$$

Whenever this map is monotone it gives rise to a fibred lifting of F called the *canonical $\mathcal{V}$-**Pred** lifting* of F and noted $\widehat{F}_{\mathrm{can}}$; with $p\colon X \to \mathcal{V}$ and $u \in FX$,

$$\widehat{F}_{\mathrm{can}}(p)(u) = \bigvee \{r \mid F(p)(u) \in F(\uparrow r)\}$$

Following [5, Lemma 43] we give a condition for the monotonicity of the canonical evaluation map.

Proposition 13. *With the previous setting, if F is a weak pullback-preserving functor then $\mathrm{ev}_{\mathrm{can}}$ is monotone.*

Other more general classes of evaluation maps that we will use are the following (see [5, Theorem 21]).

Definition 21. An evaluation map $\mathrm{ev}\colon F\mathcal{V} \to \mathcal{V}$ is called *behaved* if F preserves weak pullbacks and if

- it is monotone;

- the associated fibred $\mathcal{V}$-**Pred** lifting $\widehat{F}$ is such that for all $\mathcal{V}$-predicates p and q, $\widehat{F}(p \otimes q) \geqslant \widehat{F}(p) \otimes \widehat{F}(q)$.

If, furthermore, the evaluation map is such that for all X, $\widehat{F}(\kappa_X) \geqslant \kappa_{FX}$, then it is called *well-behaved*.

Remark 7. This notion of well-behavness is different from the one that can be found in [3], even though there are clear similarities.

The second condition of behavness with $[0, \infty]$ as a quantale will translate to particular functional inequalities. For example, with the identity functor (see Section 3.2.2), it will be equivalent to

$$\forall x, y \in \overline{\mathbb{R}}_+, \ \mathrm{ev}(x + y) \leqslant_{\mathbb{R}} \mathrm{ev}(x) + \mathrm{ev}(y)$$

and well-behaved evaluation maps will exactly be monotone subadditive functions such that $\mathrm{ev}(0) = 0$.

Through the next result we link our notion of well-behavness with the one in [3, Definition 4.3] defined on $\mathrm{ev}\colon F[0, \top] \to [0, \top]$ for $\top \in (0, \infty]$ by:

- ev is monotone;

- for all $t \in F([0, \top] \times [0, \top])$ a coupling of t_1 and t_2,

$$d_e(ev(t_1), ev(t_2)) \geqslant \widehat{F}(d_e)(t);$$

- $ev^{-1}[\{0\}] = Fi[F\{0\}]$ for $i \hookrightarrow \{0\} \to [0, \top]$ the inclusion map.

Note that the element $\top$ in $[0, \top]$ corresponds, in our interpretation to the element $\perp$ of $\mathcal{V}$ as the order in reversed. The third condition implies our third condition of well-behavness. The following lemma links the second condition of the two definitions.

Lemma 9. *Suppose F is a weak pullback-preserving* **Set** *endofunctor. Let $t \in F(\mathcal{V} \times \mathcal{V})$ be a coupling of $t_1 = F\pi_1 t$ and $t_2 = F\pi_2 t$ and $ev\colon F\mathcal{V} \to \mathcal{V}$ be a behaved evaluation map. Then,*

$$d_e(ev(t_1), ev(t_2)) \geqslant \widehat{F}(d_e)(t)$$

where $\widehat{F}$ is the fibred lifting associated to the monotone evaluation map ev.

Proof. First, let us rewrite the first term of the inequality; using $t_i = F\pi_i t$ and $\widehat{F} = ev \circ F$, one gets,

$$
\begin{aligned}
d_e(ev(t_1), ev(t_2)) &= d_e(ev \circ F\pi_1(t), ev \circ F\pi_2(t)) \\
&= d_e(\widehat{F}\pi_1 t, \widehat{F}\pi_2 t) \\
&= \bigwedge \left\{ [\widehat{F}\pi_1 t, \widehat{F}\pi_2 t], [\widehat{F}\pi_2 t, \widehat{F}\pi_1 t] \right\}
\end{aligned}
$$

Note that by definition of meet,

$$\bigwedge \left\{ [\widehat{F}\pi_1 t, \widehat{F}\pi_2 t], [\widehat{F}\pi_2 t, \widehat{F}\pi_1 t] \right\} \geqslant \widehat{F}(d_e)(t)$$

$$\Leftrightarrow$$

$$\forall (i,j) \in \{(1,2),(2,1)\}, \; [\widehat{F}\pi_i t, \widehat{F}\pi_j t] \geqslant \widehat{F}(d_e)(t)$$

Using the adjoint situation $[x, -] \dashv x \otimes -$,

$$[\widehat{F}\pi_i t, \widehat{F}\pi_j t] \geqslant \widehat{F}(d_e)(t)$$

$$\Leftrightarrow$$

$$\widehat{F}(d_e)(t) \otimes \widehat{F}\pi_i(t) \leqslant \widehat{F}\pi_j(t)$$

Now, using lemma 1 it is immediate to prove that,

$$d_e \otimes \pi_i \leqslant \pi_j$$

Applying $\widehat{F}$,

$$\widehat{F}(d_e \otimes \pi_i) \leqslant \widehat{F}\pi_j$$

and ev being behaved,

$$\widehat{F}(d_e \otimes \pi_i) \geqslant \widehat{F}(d_e) \otimes \widehat{F}(\pi_i)$$

yielding, through all the equivalences and equalities,

$$d_e(ev(t_1), ev(t_2)) \geqslant \widehat{F}(d_e)(t) \qquad \square$$

2 The Kantorovich and Wasserstein liftings

This Section is devoted to the definitions and study of the Wasserstein (Section 2.1) and Kantorovich liftings (Section 2.2). Both are liftings from the category **Set** to the category $\mathcal{V}$-**Rel** of $\mathcal{V}$-relations for a given quantale $\mathcal{V}$. The first part defines the Wasserstein lifting and gives some of its properties. It mainly comes from [5] even though a property of [3] have been adapted to our setting. Aside from the definition of the Kantorovich lifting, the goal of the second part is to find conditions for it to have the same properties as its counterpart. This is a prerequisite to the study of the Kantorovich-Rubinstein duality in Section 3. In particular those conditions imply restricting both liftings to $\mathcal{V}$-pseudometrics. Throughout this section, $\mathcal{V}$ will denote an arbitrary symmetric unital quantale and $F\colon \textbf{Set} \to \textbf{Set}$ an arbitrary **Set** endofunctor. The Wasserstein lifting can be defined for arbitrary quantales, but the Kantorovich lifting uses the euclidean relation and thus requires a symmetric unital quantale.

2.1 The Wasserstein lifting

The first defined lifting is the Wasserstein lifting. It should be seen as a generalization of relation liftings as found in Chapter 4 of [7]. This Section starts with the construction of the Wasserstein lifting, and ends with some of its properties. Conditions for this lifting to be fibred and to restrict to the category $\mathcal{V}$-**Cat**$_{\text{sym}}$ of $\mathcal{V}$-pseudometrics and $\mathcal{V}$-**Cat**$_{\text{met}}$ of $\mathcal{V}$-metrics are given.

First, let us take a look at relation liftings. Consider the category **Rel**(**Set**) of usual relations as objects and relation preserving pairs of maps as morphisms. Note that the restriction of **Rel**(**Set**) to relations $r \subseteq X \times X$ for some set X and pairs of maps $f \times f$ is the category $\mathcal{V}$-**Rel** for the quantale $\mathbf{2}$ presented in Example 1. There is a fibration $p\colon \textbf{Rel}(\textbf{Set}) \to \textbf{Set} \times \textbf{Set}$. The goal is to define a functor $\widetilde{F}\colon \textbf{Rel} \to \textbf{Rel}$ lifting of F (more correctly of $F \times F$) to **Rel**(**Set**) along p.

Note our category **Rel**(**Set**) is different from the usual category **Rel** in which relations are morphisms, not objects.

To define the lifting of F for a relation $r \subseteq X \times Y$, the goal is to define a relation $\widetilde{F}r \subseteq FX \times FY$. Obviously, it should make use of F. The functor F applies to sets and maps only. The first step is thus to consider r as an injective map $r\colon R \hookrightarrow X \times Y$ for some set R. Then, Fr is a map from FR to $F(X \times Y)$. To end the definition of the relation lifting, it is sufficient to find a map from $F(X \times Y)$ to $FX \times FY$. The image of the projection maps by F are suited to do that. On relations, the lifting $\widetilde{F}r$ is defined by the monomorphism of the epi-mono factorization of $\langle F\pi_1, F\pi_2 \rangle \circ Fr$. A proof that this definition makes $\widetilde{F}$ a lifting of F can be found in [7, Proposition 4.4.2.].

The construction of the Wasserstein lifting is very similar to this definition of relation lifting. The goal is, starting on a $\mathcal{V}$-relation $r\colon X \times X \to \mathcal{V}$, to get a $\mathcal{V}$-relation $\widetilde{F}r\colon FX \times FX \to \mathcal{V}$. Note that $\mathcal{V}$-relations are not injective maps as usual relations were above, so that applying F to r only gives a map from $F(X \times X) \to F\mathcal{V}$. Post-composing by an evaluation map defines a map $F(X \times X) \to \mathcal{V}$. More generally it suffices to consider a $\mathcal{V}$-predicate lifting to get a map $\widehat{F}r\colon F(X \times X) \to \mathcal{V}$. Finally,

the transition from $F(X \times X) \to \mathcal{V}$ to a $\mathcal{V}$-relation $FX \times FX \to \mathcal{V}$ is done using the projection maps as for the lifting to $\mathbf{Rel}(\mathbf{Set})$ above.

Let us note that the change-of-base situation in Proposition 10 yields isomorphisms $i_X \colon \mathcal{V}\text{-}\mathbf{Rel}_X \to \mathcal{V}\text{-}\mathbf{Pred}_{X \times X}$ on fibres. The projection maps give morphisms in $\mathbf{Set}$ of the form $\lambda_X \colon F(X \times X) \to FX \times FX$ by $\langle F\pi_1, F\pi_2 \rangle$. They yield, through the direct image functor of the bifibration on $\mathcal{V}\text{-}\mathbf{Pred}$, functors $\Sigma_{\lambda_X} \colon \mathcal{V}\text{-}\mathbf{Pred}_{F(X \times X)} \to \mathcal{V}\text{-}\mathbf{Pred}_{FX \times FX}$. Now that notations are set, let us get to the actual definition (see [5, Section 5.2]).

Definition 22. Given $\widehat{F} \colon \mathcal{V}\text{-}\mathbf{Pred} \to \mathcal{V}\text{-}\mathbf{Pred}$ a $\mathcal{V}$-predicate lifting of F, the *Wasserstein lifting* of F along $\widehat{F}$ is the data $F^{\downarrow} \colon \mathcal{V}\text{-}\mathbf{Rel} \to \mathcal{V}\text{-}\mathbf{Rel}$ made of

- a map on objects; if $r \in \mathcal{V}\text{-}\mathbf{Rel}_X$, $F^{\downarrow}(r)$ is defined by

$$F^{\downarrow}(r) = i_{FX}^{-1} \circ \Sigma_{\lambda_X} \circ \widehat{F}_{X \times X} \circ i_X(r);$$

- a map on morphisms; if $f \colon r \to s$ is a morphism in $\mathcal{V}\text{-}\mathbf{Rel}$, then

$$F^{\downarrow} f = F f.$$

From now on, $\widehat{F}$ will denote an arbitrary $\mathcal{V}$-predicate lifting of F.

With this definition (see [5, Proposition 17]),

Proposition 14. *The Wasserstein lifting is a well-defined lifting of F to $\mathcal{V}\text{-}\mathbf{Rel}$. In particular, with $r \in \mathcal{V}\text{-}\mathbf{Rel}_X$ and $t_1, t_2 \in FX$,*

$$F^{\downarrow}(r)(t_1, t_2) = \bigvee \left\{ \widehat{F}(r)(t) \mid t \in F(X \times X), \ F\pi_i(t) = t_i \right\}$$

Now we give a few properties of the Wasserstein lifting (see [5, Proposition 17, Theorem 21]),

Proposition 15. *If F preserves weak-pullbacks and $\widehat{F}$ is fibred, then $F^{\downarrow}$ is fibred.*

Proposition 16.

- $F^{\downarrow}$ *preserves symmetric $\mathcal{V}$-relations;*
- *whenever F preserves weak-pullbacks and $\widehat{F}$ is fibred associated to a behaved evaluation map, $F^{\downarrow}$ preserves transitive $\mathcal{V}$-relations;*
- *when furthermore the evaluation map is well-behaved, $F^{\downarrow}$ preserves reflexive $\mathcal{V}$-relations.*

Thus when $\widehat{F}$ is fibred associated to a well-behaved evaluation map, $F^{\downarrow}$ restricts to both a $\mathcal{V}\text{-}\mathbf{Cat}$ and a $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$ lifting of F.

The two next properties will be used to find counterexamples to the Kantorovich-Rubinstein duality. The second one (see [3, Proposition 4.8]) describes situations in which the Wasserstein lifting preserves $\mathcal{V}$-metrics. The first one (see [5, Proposition 22]) helps finding such situations. Whenever the Kantorovich lifting does not preserve $\mathcal{V}$-metrics the liftings cannot be equal and duality cannot hold.

Proposition 17. *If F preserves weak-pullbacks, then* $\mathrm{ev_{can}}$ *is well-behaved.*

The next proposition is slightly different from the one found in [3]; a proof is provided here, even though it is the exact same than in [3], replacing $\overline{\mathbb{R}}_+$ by $\mathcal{V}$.

Proposition 18. *Let* ev *be an associated well-behaved evaluation map, and r be a $\mathcal{V}$-metric. If F is weak pullback preserving, if ev is such that*

$$\mathrm{ev}^{-1}[\{\top\}] = F\kappa_{\{\top\}}[F(\{\top\})] \tag{1}$$

and if every join in the formula of $F^{\downarrow}r$ is a maximum, then $F^{\downarrow}r$ is a $\mathcal{V}$-metric.

Proof. Let us note $!_X \colon X \to \{\top\}$ the uniquely defined map, and $\Delta_X = \Delta X = \{(x,x) | x \in X\}$.
Because r is a $\mathcal{V}$-metric (i.e. its kernel is exactly Δ_X), and using condition 1, one gets the two following weak pullbacks:

$$
\begin{array}{ccc}
\Delta_X & \xrightarrow{!_{\Delta_X}} & \{\top\} \\
\downarrow{\scriptstyle e} & \lrcorner & \downarrow{\scriptstyle i} \\
X \xleftarrow[\pi_2]{\pi_1} X \times X \xrightarrow{r} & & \mathcal{V}
\end{array}
\qquad
\begin{array}{ccc}
F\{\top\} & \xrightarrow{!_{F\{\top\}}} & \{\top\} \\
\downarrow{\scriptstyle Fi} & & \downarrow{\scriptstyle i} \\
F\mathcal{V} & \xrightarrow{\mathrm{ev}} & \mathcal{V}
\end{array}
$$

Because F preserves weak-pullbacks, we get the following situation where all three squares are weak-pullbacks

$$
\begin{array}{ccccc}
F\Delta_X & \xrightarrow{F!_{\Delta_X}} & F\{\top\} & \xrightarrow{!_{F\{\top\}}} & \{\top\} \\
& \downarrow{\scriptstyle Fe} & \downarrow{\scriptstyle Fi} & & \downarrow{\scriptstyle i} \\
FX \xleftarrow[F\pi_2]{F\pi_1} & FX \times X & \xrightarrow{Fr} F\mathcal{V} & \xrightarrow{\mathrm{ev}} & \mathcal{V}
\end{array}
$$

Let $x, y \in FX \times FX$ such that $F^{\downarrow}(x,y) = \top$. Then by hypothesis the join in the definition of $F^{\downarrow}$ is a maximum and there exists a coupling t of x and y such that

$$\mathrm{ev} \circ Fr(t) = F^{\downarrow}r(x,y) = \top$$

Because of the weak-pullback situation, we get that there exists $t' \in F\Delta_X$ such that $Fe(t') = t$. But as Fe is a regular monomorphism (as limit of an equalizer[1]), $t = t'$ so that $t_1 = F\pi_1 t = F\pi_1 \circ Fe(t') = F\pi_2 \circ Fe(t') = F\pi_2 t = t_2$ and $F^{\downarrow}r$ is a $\mathcal{V}$-metric. $\square$

Remark 8. This last proposition can be generalized (see [4, Theorem 5.24]) but is sufficient here.

---

[1]`https://ncatlab.org/nlab/show/regular+monomorphism`

The definition of the Wasserstein lifting makes use of coupling $t \in F(X \times X)$ that project to t_1 and t_2 through $F\pi_1$ and $F\pi_2$. Some functors have optimal couplings in the sens that the Wasserstein lifting exactly equals $\hat{F}r(t)$ for t an optimal coupling (see polynomial functors in Section **??** for example).

Definition 23. Let $r: X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation, and $t_1, t_2 \in FX$. An *optimal coupling* for F, r, and (t_1, t_2) is a coupling $t \in F(X \times X)$ of t_1 and t_2 such that:

$$F^{\downarrow}r(t_1, t_2) = \hat{F}r(t)$$

When an optimal coupling always exists we say that F have all the optimal couplings.

2.2 The Kantorovich lifting

This Section introduces the Kantorovich lifting. Its definition is an adaptation of [3, Definition 3.1] to the setting of Section 1, that is the setting of [5]. After giving its definition some properties of the Kantorovich lifting are given and proven. In particular we prove that it restricts to $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$ and that the restriction is fibred. Recall that $F: \mathbf{Set} \to \mathbf{Set}$ denotes an arbitrary functor and that $\mathcal{V}$ is an arbitrary symmetric unital quantale. Note that given an evaluation map $\mathrm{ev}: F\mathcal{V} \to \mathcal{V}$, $\mathrm{ev} \circ F$ is often noted $\hat{F}$ even when this definition is not functorial.

In [3] the Kantorovich lifting was defined for arbitrary pseudometrics on reals, i.e. for quantales of the form $[0, \top]$ for $\top \in (0, \infty]$. Let us consider the case of $\top = \infty$, i.e. of the quantale $\overline{\mathbb{R}}_+$. In this context, given $d: X \times X \to \overline{\mathbb{R}}_+$ a pseudometric on X and $\mathrm{ev}: F\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ an evaluation map, the Kantorovich lifting of d is defined in [3] by:

$$\forall x, y \in FX, \ F^{\uparrow}d(x, y) = \sup \left\{ \mathrm{d_e}(\hat{F}f(x), \hat{F}f(y)) \mid f: (X, d) \to (\overline{\mathbb{R}}_+, \mathrm{d_e}) \right\}$$

where $\hat{F}f$ is equal to $\mathrm{ev} \circ Ff$, $\mathrm{d_e}$ is the extended euclidean distance on $\overline{\mathbb{R}}_+$, and functions $f: (X, d) \to (\overline{\mathbb{R}}_+, \mathrm{d_e})$ are non-expansive maps of metric spaces. The goal is to translate this formula in our setting. Using tools introduced in Section 1 we obtain the following definition:

Definition 24. Given $\mathrm{ev}: F\mathcal{V} \to \mathcal{V}$ an evaluation map, the *Kantorovich lifting* of F along ev is the data $F^{\uparrow}: \mathcal{V}\text{-}\mathbf{Rel} \to \mathcal{V}\text{-}\mathbf{Rel}$ defined by:

- a map on objects; on $r \in \mathcal{V}\text{-}\mathbf{Rel}$,

$$\forall t_1, t_2 \in FX, \ F^{\uparrow}r(t_1, t_2) = \bigwedge \left\{ \mathrm{d_e}(\hat{F}f(t_1), \hat{F}f(t_2)) \mid f: r \to \mathrm{d_e} \text{ in } \mathcal{V}\text{-}\mathbf{Rel} \right\}$$

   where $\hat{F}f = \mathrm{ev} \circ Ff$;

- a map on morphisms; on $f: X \to Y$ in $\mathcal{V}\text{-}\mathbf{Rel}$, $F^{\uparrow}f = Ff$.

Before looking at properties of the Kantorovich lifting, it must be proven that this data is indeed a lifting.

Proposition 19. *The Kantorovich lifting is a well-defined lifting of F to the category of $\mathcal{V}$-relations.*

Proof. Two things to show:

- $F^\uparrow$ is a functor; Note that if $f\colon r \to s$ is a morphism in $\mathcal{V}\text{-}\mathbf{Rel}$ from $r\colon X \times X \to \mathcal{V}$ to $s\colon Y \times Y \to \mathcal{V}$, i.e. it is a map $f\colon X \to Y$ such that $r \leqslant s \circ (f \times f)$, then $F^\uparrow f = Ff$ is a map from FX to FY. As $F^\uparrow r$ is a $\mathcal{V}$-relation of the form $F^\uparrow r\colon FX \times FX \to \mathcal{V}$, and similarly $F^\uparrow s\colon FY \times FY \to \mathcal{V}$, f can be considered as a $\mathcal{V}\text{-}\mathbf{Rel}$ map from $F^\uparrow r$ to $F^\uparrow s$ if and only if the following holds:

$$F^\uparrow r \leqslant F^\uparrow s \circ (Ff \times Ff)$$

  By definition,

$$F^\uparrow(s) \circ (Ff \times Ff)(x,y) = \bigwedge \left\{ d_e(\widehat{F}g(Ff(x)), \widehat{F}g(Ff(y))) \mid g\colon s \to d_e \right\}$$

$$F^\uparrow r(x,y) = \bigwedge \left\{ d_e(\widehat{F}h(x), \widehat{F}h(y)) \mid h\colon r \to d_e \right\}$$

  Taking $h = g \circ f$, one gets,

$$\widehat{F}g \circ Ff = \widehat{F}(g \circ f)$$
$$= \widehat{F}h$$

  so that by composition of $f\colon r \to s$ and $g\colon s \to d_e$, h is of the form $r \to d_e$ and,

$$\left\{ d_e(\widehat{F}g(Ff(x)), \widehat{F}g(Ff(y))) \mid g\colon s \to d_e \right\} \subseteq \left\{ d_e(\widehat{F}h(x), \widehat{F}h(y)) \mid h\colon r \to d_e \right\}$$

  meaning $F^\uparrow r \leqslant F^\uparrow s \circ (Ff \times Ff)$. So $F^\uparrow f$ is indeed a map of $\mathcal{V}\text{-}\mathbf{Rel}$ from $F^\uparrow r$ to $F^\uparrow s$. Thus, $F^\uparrow$ is well-defined as a pair of maps on objects and on morphisms. It is the right data to be a functor. Remains to prove the relations functoriality requires hold.

  Consider $\mathrm{Id}_r\colon r \to r$ the identity morphism on $r\colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation. Then as a map on sets $\mathrm{Id}_r = \mathrm{Id}_X$. Thus $F^\uparrow(\mathrm{Id}_r) = F(\mathrm{Id}_X) = \mathrm{Id}_{FX} = \mathrm{Id}_{F^\uparrow(r)}$ so that $F^\uparrow$ preserves identity morphisms. Furthermore, given two compatible $\mathcal{V}\text{-}\mathbf{Rel}$ arrows f and g,

$$F^\uparrow(g \circ f) = F(g \circ f)$$
$$= Fg \circ Ff$$
$$= F^\uparrow f \circ F^\uparrow g$$

  and $F^\uparrow$ is indeed a $\mathcal{V}\text{-}\mathbf{Rel}$ endofunctor.

- $F^\uparrow$ is a lifting; let us note $p\colon \mathcal{V}\text{-}\mathbf{Rel} \to \mathbf{Set}$ the bifibration introduced above (see Proposition 10). Take $r\colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation.

$$p \circ F^\uparrow(r) = p(F^\uparrow r)$$
$$= FX$$
$$= F \circ p(r)$$

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With $f\colon r \to s$ a morphism that is a map $f\colon X \to Y$ on sets,

$$
\begin{aligned}
p \circ F^\uparrow(f) &= p(Ff) \\
&= Ff \\
&= F \circ p(f)
\end{aligned}
$$

so that $p \circ F^\uparrow = F \circ p$ and $F^\uparrow$ is indeed a well-defined lifting. $\qquad\square$

Remark 9. In opposition to the Wasserstein one, the Kantorovich lifting is defined with $\widehat{F} = \mathrm{ev} \circ F$ for any ev, even when $\mathrm{ev} \circ F$ is not a $\mathcal{V}$-**Pred** lifting. That is because the previous proof requires the following equality to hold: $\widehat{F}g \circ Ff = \widehat{F}(g \circ f)$. In return this equality defines an evaluation map by $\mathrm{ev} = \widehat{F}\mathrm{Id}_\mathcal{V}$ but does not imply that $\widehat{F}$ defines a $\mathcal{V}$-**Pred** lifting.

Note that even though in the definition of Kantorovich liftings we have $f\colon r \to \mathrm{d_e}$ in $\mathcal{V}$-**Rel** a morphism, it is considered an object $f\colon X \to \mathcal{V} \in \mathcal{V}$-**Pred** when $\widehat{F}$ is applied to f in $\widehat{F}f$. This could have been expressed by stating that here, $\widehat{F}$ is a functor on the slice category **Set**$/\mathcal{V}$.

The goal is to know when both liftings coincide. As the Wasserstein lifting is fibred under some conditions, knowing when the Kantorovich lifting is fibred is mandatory. This is not true in general. Furthermore for some functors, the Kantorovich lifting is not fibred except for trivial cases.

Proposition 20. *Let* $\mathrm{ev}\colon F\mathcal{V} \to \mathcal{V}$ *be an evaluation map. If F maps constant maps to constant maps and if $I = \top$ in $\mathcal{V}$ then $F^\uparrow$ is fibred if and only if ev is constant.*

Proof. Let us suppose that F maps constant maps to constant maps and that $I = \top$ in $\mathcal{V}$. Consider an evaluation map defining a Kantorovich lifting $F^\uparrow$ of F to $\mathcal{V}$-**Rel**. We will note $\widehat{F}f = \mathrm{ev} \circ Ff$. By definition, this lifting is fibred when

$$
(Ff)^* \circ F^\uparrow = F^\uparrow \circ f^*
$$

This should be true for any $\mathcal{V}$-relation. Take $r\colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation. Then for all $x, y \in FX$,

$$
\begin{aligned}
(Ff)^* \circ F^\uparrow(r)(x,y) &= \bigwedge \left\{ \mathrm{d_e}\left(\widehat{F}g(Ff(x)), \widehat{F}g(Ff(y))\right) ;\ g\colon r \to \mathrm{d_e} \right\} \\
F^\uparrow \circ f^*(r)(x,y) &= \bigwedge \left\{ \mathrm{d_e}\left(\widehat{F}h(x), \widehat{F}h(y)\right) ;\ h\colon f^*r \to \mathrm{d_e} \right\}
\end{aligned}
$$

Take r to be the following:

$$
\forall (x,y) \in X \times X,\ r(x,y) = \begin{cases} \bot \text{ if } x = y \\ \top \text{ else} \end{cases}
$$

with X arbitrary, and let $f\colon X \to \mathcal{V}$ be a constant function. The set

$$
\left\{ \mathrm{d_e}\left(\widehat{F}g(Ff(x)), \widehat{F}g(Ff(y))\right) ;\ g\colon r \to \mathrm{d_e} \right\}
$$

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is either empty, in which case the meet of the empty set being $\top$, $(Ff)^* \circ F^{\uparrow}(r)(x,y) = \top$, either it is not empty, but then, as f is constant and F maps constant maps to constant maps, Ff is again constant and the previous set is reduced to the singleton $\{\top\}$ as $I = \top$. In any case, $(Ff)^* \circ F^{\uparrow}(r)(x,y) = \top$.

The lifting being fibred implies $F^{\uparrow} \circ f^*(r)(x,y) = \top$, meaning that

$$\forall h \colon f^* r \to \mathrm{d_e}, \ \mathrm{d_e}(\widehat{F}h(x), \widehat{F}h(y)) = \top$$

Having $h \colon f^* r \to \mathrm{d_e}$ means $r \circ (f \times f) \leqslant \mathrm{d_e} \circ (h \times h)$. As f is a constant function and by definition of r, $r \circ (f \times f)$ is constant and equal to $\bot$, the equation characterizing h is always true: h is any function from X to $\mathcal{V}$.

$$F^{\uparrow} \circ f^*(r)(x,y) = \top \Leftrightarrow \forall h \colon X \to \mathcal{V} \text{ in } \mathbf{Set}, \ \mathrm{d_e}(\widehat{F}h(x), \widehat{F}h(y)) = \top$$

Using the Lemma 6, this holds if and only if for all $h \colon X \to \mathcal{V}$, $\widehat{F}h(x) = \widehat{F}h(y)$. Taking $h \colon \mathcal{V} \to \mathcal{V}$ the identity, by functoriality, this implies that ev is constant.

Conversely, the Kantorovich lifting is obviously fibred for any constant evaluation map as when $I = \top$, for all $x \in \mathcal{V}$, $[x,x] = \top$. $\qquad\square$

The Proposition 20 was first obtained in the following form:

Corollary 1. *Let* ev $\colon F\mathcal{V} \to \mathcal{V}$ *an arbitrary evaluation map. If* $I = \top$ *in* $\mathcal{V}$ *and if* F *is full, then* $F^{\uparrow}$ *is fibred if and only if* ev *is constant.*

Example 5. The first three examples use $\overline{\mathbb{R}}_+$-relations, $(\overline{\mathbb{R}}_+, +, 0)$ with reversed order being the quantale structure. Note $0 = I = \top$.

- The following example was used in [3] to prove the Kantorovich and Wasserstein liftings differ in general. Let $\Delta \colon \mathbf{Set} \to \mathbf{Set}$ be the diagonal functor mapping $X \in \mathbf{Set}$ to $X \times X$ and f in $\mathbf{Set}$ to $f \times f$. Then, as $I = \top$ and Δ is full, the associated Kantorovich lifting is fibred if and only if ev is constant.

- Consider $F = \mathrm{Id}_{\mathbf{Set}}$ the identity functor. It is also full, so that an associated Kantorovich lifting is fibred if and only if ev is constant.

- The finite powerset functor $\mathcal{P}$ maps a constant map to a constant map, thus an associated Kantorovich lifting is fibred if and only if ev is constant.

- The finite distribution functor $\mathcal{D}$ maps a constant map $f \colon x \in X \mapsto y \in Y$ to $\mathcal{D}(f) \colon P \in \mathcal{D}(X) \to \delta_y \in \mathcal{D}(Y)$ a constant map $(\mathbb{P}(\delta_y = y) = 1$ and y is entirely determined by f). This particular example uses $([0,1], \min\{+, 1\}, 0)$ with reversed order as a quantale. As $I = \top = 0$ any associated Kantorovich lifting is fibred if and only if the associated evaluation map is constant.

Note however that on some functors of interest, such as the ones modeling finite deterministic automata $X^{\Sigma} \times \{0,1\}$, or functors build with the coproduct such as $X + X^2$, this theorem is useless.

In [3] the last three examples are used to illustrate cases where the Kantorovich-Rubinstein duality holds, i.e. the Wasserstein and Kantorovich liftings are equal. Here it clearly does not as one lifting may be fibred but not the other whenever the evaluation map is not constant; and even when it is, the Theorem 1 will prove that for the two liftings to coincide ev must be constant and equal to $\top$. It may look contradictory, but [3] considered liftings on pseudometrics, i.e. restrictions to $\mathcal{V}\text{-}\mathbf{Cat}_{\text{sym}}$ and on particular quantales, quantales on reals. Interestingly enough, duality can hold on $\mathcal{V}\text{-}\mathbf{Cat}_{\text{sym}}$ when it does not on $\mathcal{V}\text{-}\mathbf{Rel}$. The next step is to give reasonable conditions for the Kantorovich lifting to restrict to $\mathcal{V}$-pseudometrics. In fact this is always true.

Proposition 21. *Let* ev *be an* *evaluation map* *yielding the* *Kantorovich lifting* $F^{\uparrow}$ *of* F *along* $\widehat{F} = \text{ev} \circ F$*. Then,* $F^{\uparrow}$ *restricts to a* *lifting on* $\mathcal{V}\text{-}\mathbf{Cat}$ *and* $\mathcal{V}\text{-}\mathbf{Cat}_{\text{sym}}$*.*

Proof. Let us prove that $F^{\uparrow}$ preserves symmetric, reflexive, and transitive $\mathcal{V}$-relations:

- $F^{\uparrow}$ preserves symmetric $\mathcal{V}$-relations; take $r \colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation. Note that d_e is a symmetric $\mathcal{V}$-relation. Thus,

$$\forall (x,y) \in FX \times FX, \ F^{\uparrow}r(x,y) = \bigwedge \left\{ d_e(\widehat{F}f(x), \widehat{F}f(y)) \mid f \colon r \to d_e \right\}$$
$$= \bigwedge \left\{ d_e(\widehat{F}f(y), \widehat{F}f(x)) \mid f \colon r \to d_e \right\}$$
$$= F^{\uparrow}r(y,x)$$
$$= F^{\uparrow}r \circ \text{sym}_{FX}(x,y)$$

and in any case $F^{\uparrow}r$ is symmetric. In particular, $F^{\uparrow}$ preserves symmetric $\mathcal{V}$-relations.

- $F^{\uparrow}$ preserves reflexive $\mathcal{V}$-relations; let $r \colon X \times X \to \mathcal{V}$ be a $\mathcal{V}$-relation. We know that for all $v \in \mathcal{V}$ and $x, y \in X \times X$ such that $x \neq y$,

$$[v, v] \geqslant I = \text{diag}_X(x, x) \text{ and } v \geqslant \bot = \text{diag}_X(x, y)$$

Then,

$$\forall x \in FX, \ F^{\uparrow}r(x,x) = \bigwedge \left\{ d_e(\widehat{F}f(x), \widehat{F}f(x)) \mid f \colon r \to d_e \right\}$$
$$= \bigwedge \left\{ [\widehat{F}f(x), \widehat{F}f(x)] \mid f \colon r \to d_e \right\}$$
$$\geqslant \bigwedge \{ I \mid f \colon \text{diag}_X \to d_e \}$$
$$= I$$
$$= \text{diag}_X(x,x)$$
$$\forall x, y \in FX \times FX, \ x \neq y \Rightarrow F^{\uparrow}r(x,y) \geqslant \bot$$
$$= \text{diag}_X(x,y)$$

Thus, $F^{\uparrow}r \geqslant \text{diag}_{FX}$ meaning $F^{\uparrow}r$ is reflexive. In particular $F^{\uparrow}$ preserves reflexive $\mathcal{V}$-relations.

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- Suppose d_e is transitive. Then for any relation $r \colon X \times X \to \mathcal{V}$,

$$\forall t_1, t_2, t_3 \in FX,$$
$$F^\uparrow r(t_1, t_2) \otimes F^\uparrow r(t_2, t_3)$$
$$= \bigwedge \left\{ d_e(\widehat{F}f(t_1), \widehat{F}f(t_2)); \ f \colon r \to d_e \right\} \otimes \bigwedge \left\{ d_e(\widehat{F}f(t_2), \widehat{F}f(t_3)); \ f \colon r \to d_e \right\}$$
$$\leqslant \bigwedge \left\{ d_e(\widehat{F}f(t_1), \widehat{F}f(t_2)) \otimes d_e(\widehat{F}f(t_2), \widehat{F}f(t_3)); \ f \colon r \to d_e \right\}$$
$$\leqslant \bigwedge \left\{ d_e(\widehat{F}f(t_1), \widehat{F}f(t_3)); \ f \colon r \to d_e \right\}$$
$$= F^\uparrow r(t_1, t_3)$$

and in particular $F^\uparrow$ preserves transitive $\mathcal{V}$-relations. Using the Lemmas 4 and 5, $F^\uparrow$ preserves transitive $\mathcal{V}$-relations in any case. $\qquad\square$

Remark 10. Note that the proof gives a stronger result: $F^\uparrow \colon \mathcal{V}\text{-}\mathbf{Rel} \to \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$

Now we know the Kantorovich lifting restricts to $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$, the next step is to know wether it is fibred or not.

Proposition 22. *Let* ev *be an evaluation map defining a Kantorovich lifting* $F^\uparrow \colon \mathcal{V}\text{-}\mathbf{Rel} \to \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$. *Then* $F^\uparrow$ *is fibred on* $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$.

Proof. By definition, stating that $F^\uparrow$ is fibred amounts to

$$\forall f \colon Y \to X, \ (Ff)^* \circ F^\uparrow = F^\uparrow \circ f^*$$

Instanciating this situation, the goal is to prove that for all $f \colon Y \to X$, $r \colon X \times X \to \mathcal{V}$, and $(x, y) \in FY \times FY$,

$$r \in \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}} \Rightarrow (Ff)^* \circ F^\uparrow r(x, y) = F^\uparrow \circ f^* r(x, y)$$

By definition of $(-)^*$ and of $F^\uparrow$,

$$(Ff)^* \circ F^\uparrow r(x, y) = \bigwedge \left\{ d_e(\widehat{F}(g \circ f)(x), \widehat{F}(g \circ f)(y)) \mid g \colon r \to d_e \right\}$$
$$F^\uparrow \circ f^* r(x, y) = \bigwedge \left\{ d_e(\widehat{F}h(x), \widehat{F}h(y)) \mid h \colon f^* r \to d_e \right\}$$

Note that $(Ff)^* \circ F^\uparrow r$ being a cartesian lifting of $F^\uparrow r$, we already have a morphism from $F^\uparrow \circ f^*$ to $(Ff)^* \circ F^\uparrow$. Because both are above Ff, this morphism is the identity, and,

$$F^\uparrow \circ f^* \leqslant (Ff)^* \circ F^\uparrow$$

Here we are going to prove that

$$F^\uparrow \circ f^* = (Ff)^* \circ F^\uparrow$$

To do so we will use that given $\bigwedge A$ and $\bigwedge B$,

$$(\forall x \in A, \ \exists y \in B, \ y \leqslant x) \Rightarrow \bigwedge A \geqslant \bigwedge B$$

To prove the equality, we only have to prove that

$$\forall h \colon f^*r \to \mathrm{d_e}, \ \exists g \colon r \to \mathrm{d_e}, \ \mathrm{d_e}(\widehat{F}(g \circ f)(x), \widehat{F}(g \circ f)(y)) \leqslant \mathrm{d_e}(\widehat{F}h(x), \widehat{F}h(y))$$

Given $h \colon f^*r \to \mathrm{d_e}$, we want to construct $g \colon X \to \mathcal{V}$ such that this holds. First we define $g \colon f(Y) \to \mathcal{V}$ on $f(Y)$ by

$$\forall z \in Y, \ g(f(z)) = h(z)$$

This is well-defined as whenever $f(z) = f(z')$, then $r \circ (f \times f)(z, z') = \top \leqslant \mathrm{d_e} \circ (h \times h)(z, z')$ meaning $h(z) = h(z')$ using the Lemma 6. Then, with $i \colon f(Y) \hookrightarrow X$,

$$\forall u \colon r \to \mathrm{d_e}, \ (i^*(u) = g) \Rightarrow (\mathrm{d_e}(\widehat{F}(u \circ f)(x), \widehat{F}(u \circ f)(y)) = \mathrm{d_e}(\widehat{F}h(x), \widehat{F}h(y)))$$

Because r is a $\mathcal{V}$-pseudometric and using the Technical lemma, g can be extended to a map $X \to \mathcal{V}$ giving the result. $\qquad \square$

3 The Kantorovich-Rubinstein duality

Here the goal is to study the link between the Kantorovich and the Wasserstein liftings. It has been found that in some cases, the liftings coincide (see [3]; the examples given in this article are also treated here). When they do the *Kantorovich-Rubinstein duality* is said to hold. It has also been found that they can differ (see [3, 4]). Our main goal is to find some conditions so that the Kantorovich-Rubinstein duality holds. Section 3.1 concentrates on duality for general $\mathcal{V}$-relations, and Section 3.2 on duality for $\mathcal{V}$-pseudometrics.

Considering both the Wasserstein and the Kantorovich liftings in the same setting means constructing them using the same evaluation map. For that to be possible the latter must be monotone, yielding a $\mathcal{V}$-**Pred** lifting defining the Wasserstein lifting. Throughout this section, $\mathcal{V}$ will be an arbitrary symmetric unital quantale and $F \colon \mathbf{Set} \to \mathbf{Set}$ an arbitrary **Set** endofunctor. Finally, $\mathrm{ev} \colon F\mathcal{V} \to \mathcal{V}$ will denote an arbitrary monotone evaluation map.

3.1 Duality for $\mathcal{V}$-relations

Some general results are given in Section 3.1. In particular, Theorem 1 will prove that for some functors duality only holds for the constant and equal to $\top$ evaluation map. The remaining of Section 3.1 is devoted to extending this result for polynomial functors. This is done by first looking at duality for constant functors in Section **??**, and for coproducts in Section **??**. The general case for polynomial functors is treated in Section **??**.

3.1.1 Duality, general results

We first give a generalization to arbitrary quantales of a "weak" version of duality (see [3, Proposition 4.9].

Proposition 23. *Suppose that F is a weak-pullback preserving functor and that ev is a behaved evaluation map $\mathrm{ev}\colon F\mathcal{V} \to \mathcal{V}$. Then,*

$$F^{\uparrow} \geqslant F^{\downarrow}$$

Proof. By definition,

$$F^{\uparrow} \geqslant F^{\downarrow}$$
$$\Leftrightarrow \forall r \in \mathcal{V}\text{-}\mathbf{Rel}\colon X \times X \to \mathcal{V},\ \forall t_1, t_2 \in FX,\ F^{\uparrow}r(t_1, t_2) \geqslant F^{\downarrow}r(t_1, t_2)$$

Again by definition,

$$F^{\uparrow}r(t_1, t_2) = \bigwedge \left\{ \mathrm{d_e}(\widehat{F}f(t_1), \widehat{F}f(t_2)) \mid f\colon r \to \mathrm{d_e} \text{ in } \mathcal{V}\text{-}\mathbf{Rel} \right\}$$
$$F^{\downarrow}r(t_1, t_2) = \bigvee \left\{ \widehat{F}(r)(t) \mid t \in F(X \times X),\ F\pi_i(t) = t_i \right\}$$

A meet is greater than or equal to a join if and only if the inequality holds for all the elements on which they are defined. Thus we have to prove that

$$\forall f\colon r \to \mathrm{d_e} \text{ in } \mathcal{V}\text{-}\mathbf{Rel},\ \forall t \in F(X \times X),\ F\pi_i(t) = t_i \Rightarrow \mathrm{d_e}(\widehat{F}f(t_1), \widehat{F}f(t_2)) \geqslant \widehat{F}(r)(t)$$

We have:

$$
\begin{aligned}
\mathrm{d_e}(\widehat{F}f(t_1), \widehat{F}f(t_2)) &= \mathrm{d_e}(\widehat{F}f(F\pi_1(t)), \widehat{F}f(F\pi_2(t))) \\
&= \mathrm{d_e}(\widehat{F}(f \circ \pi_1)(t), \widehat{F}(f \circ \pi_2)(t)) \\
&= \mathrm{d_e}(\widehat{F}(\pi_1 \circ (f \times f))(t), \widehat{F}(\pi_2 \circ (f \times f))(t)) \\
&= \mathrm{d_e}(\widehat{F}\pi_1(F(f \times f))(t), \widehat{F}\pi_2(F(f \times f))(t)) \\
&\geqslant \widehat{F}(\mathrm{d_e})(F(f \times f)(t)) \\
&\quad \text{(using the lemma 9)} \\
&= \widehat{F}(\mathrm{d_e} \circ (f \times f))(t) \\
&\geqslant \widehat{F}(r)(t) \\
&\quad \text{(because } f\colon r \to \mathrm{d_e} \text{ is a morphism in } \mathcal{V}\text{-}\mathbf{Rel})
\end{aligned}
$$

ending the proof. $\qquad\square$

In some cases the duality is simple to treat:

Theorem 1. *Suppose that $I = \top$ in $\mathcal{V}$ and that F maps constant maps to constant maps and is weak-pullback preserving. Then $F^{\uparrow} = F^{\downarrow}$ if and only if all the couplings on F exists and ev is constant and equal to $\top$.*

Proof. Obviously if ev is constant and equal to $\top$ it is behaved. In particular $F^{\downarrow}$ is defined, and if all the couplings exist the liftings are both constant and equal to $\top$.

Conversely, suppose duality holds. If $\mathcal{V}$ is a singleton, then obvioulsy ev is constant and equal to $\top$ and duality holds. Let us suppose $\bot \neq \top$.

Using Proposition 15, $F^{\downarrow}$ is fibred here. By Proposition 20 ev must be constant. As $I = \top$ using Lemma 6, for all $x \in \mathcal{V}$, $d_e(x, x) = \top$. The Kantorovich lifting is constant and equal to $\top$. Similarly, $F^{\downarrow}$ must be constant and equal to $\top$:

$$\forall t_1, t_2 \in FX, \ F^{\downarrow}(r)(t_1, t_2) = \top = \bigvee \left\{ \widehat{F}(r)(t) \mid t \in F(X \times X), \ F\pi_i t = t_i \right\}$$

The set defining this join cannot be empty. Else it would mean $F^{\downarrow}(r)(t_1, t_2)$ is equal to $\bot \neq \top$. All couplings exist.

Let us note v such that ev is constant and equal to v. Then as all couplings exists, $F^{\downarrow}r$ is constant and equal to v. Thus $v = \top$ and:

- all couplings along F exist;

- ev is constant and equal to $\top$. □

The remaining of Section 3.1's goal is to prove that Theorem 1 extends to polynomial functors. The proof of this results will use the two following lemmas, linking duality for different functors subject to natural transformations.

Lemma 10. *Let $G: \mathbf{Set} \to \mathbf{Set}$ be a $\mathbf{Set}$ endofunctor, and $\sigma: G \Rightarrow F$ a natural isomorphism. Then, $\mathrm{ev}: F\mathcal{V} \to \mathcal{V}$ gives duality for F if and only if $\mathrm{ev} \circ \sigma_{\mathcal{V}}$ gives duality for G.*

Proof. We will note ev_F for the evaluation map $\mathrm{ev}: F\mathcal{V} \to \mathcal{V}$ and $\mathrm{ev}_G = \mathrm{ev}_F \circ \sigma_{\mathcal{V}}$. For all maps $f: X \to Y$ the following commutes,

$$
\begin{array}{ccc}
GX & \xrightarrow{\ \sigma_X\ } & FX \\
{\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\
GY & \xrightarrow[\ \sigma_Y\ ]{} & FY
\end{array}
$$

Thus, if $f: X \to \mathcal{V}$, $Ff = \sigma_{\mathcal{V}} \circ Gf \circ \sigma_X^{-1}$, and

$$
\begin{aligned}
\widehat{F}f &= \mathrm{ev}_F \circ Ff \\
&= \mathrm{ev}_F \circ \sigma_{\mathcal{V}} \circ Gf \circ \sigma_X^{-1} \\
&= \mathrm{ev}_G \circ Gf \circ \sigma_X^{-1} \\
&= \widehat{G}f \circ \sigma_X^{-1} \\
\widehat{G}f &= \widehat{F}f \circ \sigma_X
\end{aligned}
$$

Suppose that duality holds for G with ev_G. Then, for all $\mathcal{V}$-relation $r\colon X \times X \to \mathcal{V}$, and $t_1, t_2 \in FX$,

$$
\begin{aligned}
F^\uparrow r(t_1, t_2) &= \bigvee \left\{ \mathrm{d_e}(\widehat{F}f(t_1), \widehat{F}f(t_2)) \mid f\colon r \to \mathrm{d_e} \right\} \\
&= \bigvee \left\{ \mathrm{d_e}(\widehat{G}f(\sigma_X^{-1}(t_1)), \widehat{G}f(\sigma_X^{-1}(t_2))) \mid f\colon r \to \mathrm{d_e} \right\} \\
&= G^\uparrow r(\sigma_X^{-1}t_1, \sigma_X^{-1}t_2) \\
&= G^\downarrow r(\sigma_X^{-1}t_1, \sigma_X^{-1}t_2) \\
&= \bigwedge \left\{ \widehat{G}r(t) \mid t \in G(X \times X) \text{ a coupling of } \sigma_X^{-1}t_1 \text{ and } \sigma_X^{-1}t_2 \right\} \\
&= \bigwedge \left\{ \widehat{F}f(\sigma_X t) \mid t \in G(X \times X) \text{ a coupling of } \sigma_X^{-1}t_1 \text{ and } \sigma_X^{-1}t_2 \right\}
\end{aligned}
$$

Note that given a coupling $t \in G(X \times X)$ of $\sigma_X^{-1}t_1$ and $\sigma_X^{-1}t_2$, $\sigma_X t$ is a coupling of t_1 and t_2 as $F\pi_i(\sigma_X t) = \sigma_X G\pi_i t = t_i$. The converse being true, couplings of t_1 and t_2 are in one-to-one correspondance with couplings of $\sigma_X^{-1}t_1$ and $\sigma_X^{-1}t_2$ by σ_X. Thus,

$$
\begin{aligned}
F^\uparrow r(x, y) &= \bigwedge \left\{ \widehat{F}f(\sigma_X t) \mid t \in G(X \times X) \text{ a coupling of } \sigma_X^{-1}t_1 \text{ and } \sigma_X^{-1}t_2 \right\} \\
&= \bigwedge \left\{ \widehat{F}f(t) \mid t \in F(X \times X) \text{ a coupling of } t_1 \text{ and } t_2 \right\} \\
&= F^\downarrow r(t_1, t_2)
\end{aligned}
$$

proving that duality holds. The converse situation being symmetric, we leave it to the reader. $\qquad\square$

Often we will not have a natural monomorphism instead of a natural isomorphism. In this case duality for F do not necessarily imply duality for G, further conditions are required.

Lemma 11. *Let $G\colon \mathbf{Set} \to \mathbf{Set}$ be a $\mathbf{Set}$ endofunctor, and $\sigma\colon G \Rightarrow F$ be a natural monomorphism. Suppose that F has all optimal couplings, and that with $t \in F(X \times X)$ an optimal coupling of $t_1, t_2 \in FX$, $\sigma_{X \times} t$ is an optimal coupling of $\sigma_X t_1$ and $\sigma_X t_2$. Then if duality holds for F with ev, duality holds for G with $\mathrm{ev} \circ \sigma_\mathcal{V}$.*

Proof. We will note ev_F for $\mathrm{ev}\colon F\mathcal{V} \to \mathcal{V}$ and $\mathrm{ev}_G = \mathrm{ev}_F \circ \sigma_\mathcal{V}$.

First, note that as $\mathbf{Set}$ has all pullbacks, a monic natural transformation is monic pointwise. Thus, σ_X is left-invertible for any X. We will note τ_X be such a left-inverse.

By naturality of σ, for all $f\colon X \to Y$,

$$
Gf = \tau_Y \circ Gf \circ \sigma_X
$$

In particular, with $f\colon X \to \mathcal{V}$,

$$\begin{aligned}
\widehat{G}f &= \mathrm{ev}_G \circ Gf \\
&= \mathrm{ev}_F \circ \sigma_{\mathcal{V}} \circ \tau_Y \circ Ff \circ \sigma_X \\
&= \mathrm{ev}_F \circ \sigma_{\mathcal{V}} \circ \tau_Y \circ \sigma_Y \circ Gf \\
&= \mathrm{ev}_F \circ \sigma_{\mathcal{V}} \circ Gf \\
&= \mathrm{ev}_F \circ Ff \circ \sigma_X \\
&= \widehat{F}f \circ \sigma_X
\end{aligned}$$

Suppose that duality holds for F with ev_F. Then, for all $\mathcal{V}$-relation $r\colon X \times X \to \mathcal{V}$ and $t_1, t_2 \in GX$,

$$\begin{aligned}
G^{\uparrow}r(t_1, t_2) &= \bigvee \left\{ \mathrm{d_e}(\widehat{G}f(t_1), \widehat{G}f(t_2)) \mid f\colon r \to \mathrm{d_e} \right\} \\
&= \bigvee \left\{ \mathrm{d_e}(\widehat{F}f(\sigma_X t_1), \widehat{F}f(\sigma_X(t_2))) \mid f\colon r \to \mathrm{d_e} \right\} \\
&= F^{\uparrow}r(\sigma_X t_1, \sigma_X t_2) \\
&= F^{\downarrow}r(\sigma_X t_1, \sigma_X t_2) \\
&= \bigwedge \left\{ \widehat{F}r(t) \mid t \text{ a coupling of } \sigma_X t_1 \text{ and } \sigma_X t_2 \right\}
\end{aligned}$$

Now, as by hypothesis t_1 and t_2 have an optimal coupling t_{opt} such that $\sigma_{X \times X} t_{\mathrm{opt}}$ is an optimal coupling of $\sigma_X t_1$ and $\sigma_X t_2$,

$$\begin{aligned}
G^{\uparrow}r(t_1, t_2) &= \bigwedge \left\{ \widehat{F}r(t) \mid t \text{ a coupling of } \sigma_X t_1 \text{ and } \sigma_X t_2 \right\} \\
&= \widehat{F}r(\sigma_{X \times X} t_{\mathrm{opt}}) \\
&= \widehat{G}r(t_{\mathrm{opt}}) \\
&= G^{\downarrow}r(t_1, t_2)
\end{aligned}$$

and duality holds for G with $\mathrm{ev}_G = \mathrm{ev}_F \circ \sigma_{\mathcal{V}}$. $\qquad\square$

3.1.2 Constant functors F_A

The simplest of all functors are probably *constant functors* F_A sending any set X to a fixed set A and any morphism to the identity morphism Id_A. With these functors duality never holds whenever $\mathcal{V}$ and A have more than one element,

Proposition 24. *Let A be a set. Consider the constant functor F_A along with a well-behaved evaluation map $\mathrm{ev}\colon A \to \mathcal{V}$. Suppose that $I = \top$ in $\mathcal{V}$. Then duality holds if and only if one of the following two conditions hold:*

- *$\mathcal{V}$ has only one element;*

- *A has only one element and ev is constant and equal to $\top$,*

In any case, ev *must be constant and equal to* $\top$.

Proof. Suppose that duality holds.

Note that given $(a_1, a_2) \in A^2$, a coupling $a \in F_A(A \times A) = A$ is such that $\mathrm{Id}_A a_1 = a$ and $\mathrm{Id}_A a_2 = a$, i.e. it exists if and only if $a_1 = a_2$. When it does the formula for the Wasserstein lifting gives $F_A^{\downarrow} r(a, a) = \mathrm{ev}(a)$ independently from r. The definition of the Kantorovich lifting gives $F_A^{\uparrow} r(a, a) = [a, a]$. As $I = \top$ and using Lemma 6, $F_A^{\uparrow} r(a, a) = \top$ so that duality implies either that $\mathcal{V}$ is a singleton, or that ev is constant and equal to $\top$. When $\mathcal{V}$ is a singleton, duality obviously holds.

Suppose that $\bot \neq \top$ so that ev is constant and equal to $\top$. Whenever $a_1 \neq a_2$, because a coupling of a_1 and a_2 does not exist, $F_A^{\downarrow} r(a_1, a_2) = \bigvee \varnothing = \bot$. But $F_A^{\uparrow} r(a_1, a_2) = \top$ as ev is constant and equal to $\top$. Thus, whenever $\bot \neq \top$ and there exists $a_1 \neq a_2$ in A, duality does not hold. Duality implies that A is a singleton. Then duality obviously holds. $\square$

3.1.3 Coproducts

Let us generalize what we did on constant functors. This will give us an inductive condition of duality for functors constructed using coproducts.

Proposition 25. *Let* $F, G \colon \mathbf{Set} \to \mathbf{Set}$ *be two* $\mathbf{Set}$ *endofunctors. They induce the functor* $F(-) + G(-) \colon \mathbf{Set} \to \mathbf{Set}$ *defined by* $X \mapsto F(X) + G(X)$ *and* $f \mapsto F(f) + G(f)$. *Let* ev *be a monotone evaluation map for* $F(-) + G(-)$. *Then, duality holds if and only if:*

- *the restriction of* ev *to* F *(resp.* G*) gives duality;*

- *for all* $r \colon X \times X \to \mathcal{V}$ *a* $\mathcal{V}$*-relation,* $(x, 1), (y, 2) \in F(X) + G(X)$*, with* $x \in F(X)$ *and* $y \in G(X)$,

$$\bigwedge \{\mathrm{d_e}(\mathrm{ev}(Ff(x), 1), \mathrm{ev}(Gf(y), 2)) | f \colon r \to \mathrm{d_e}\} = \bot$$

Proof. Suppose that duality holds.

Obviously, duality holds for the liftings of F and G when ev is restricted to them using the coproduct structure (this is a generalization of Lemma 11 not using optimal couplings).

Consider $r \colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation and $x \in F(X)$, $y \in G(X)$. Couplings of $(x, 1) \in F(X) + G(X)$ with $(y, 2)$ do not exist. Indeed such a coupling $t \in F(X \times X) + G(X \times X)$ would have to be of the form (t, i) for $i \in \{1, 2\}$ and such that after projections $F\pi_1 + G\pi_1$ and $F\pi_2 + G\pi_2$ one gets back $(x, 1)$ and $(y, 2)$, implying $1 = i = 2$. Thus, $(F(-) + G(-))^{\downarrow} r((x, 1), (y, 2)) = \bot$. Necessarily, $(F(-) + G(-))^{\uparrow} r((x, 1), (y, 2)) = \bot$ as duality holds. Thus,

$$\bigwedge \{\mathrm{d_e}(\mathrm{ev}(Ff(x), 1), \mathrm{ev}(Gf(y), 2)) | f \colon r \to \mathrm{d_e}\} = \bot$$

giving the other condition in the theorem statement.

Conversely, suppose both conditions are true. Then, the liftings coincide for elements of the form $(x, 1)$ and $(x', 1)$ (resp. $(y, 2)$ and $(y', 2)$). On elements $(x, 1)$ and $(y, 2)$, the liftings are both equal to $\perp$; the Wasserstein lifting because there is no coupling of $(x, 1)$ with $(y, 2)$ and the Kantorovich lifting using the second condition in the theorem statement. Overall, duality holds. $\square$

Remark 11. This result can be adapted to arbitrary coproducts simply by replacing the second condition by similar conditions applying for each pair of functors defining the coproduct.

This proposition states that an evaluation map giving duality for $F + G$ is composed of two evaluation maps, one for F and one for G, that are incompatible.

This result's proof is quite simple. However its implications are very interesting.

Example 6. In Example 5, two examples are used to illustrate that Proposition 20 do not apply to every functor; in particular the functor defined by $X \mapsto X + X^2$ on sets. This functor does not map constant maps to constant maps and thus Proposition 20 does not apply. For the same reason Theorem 1 does not apply on $X + X^2$. Still, using Proposition 25, one knows that if $ev\colon \mathcal{V} + \mathcal{V}^2 \to \mathcal{V}$ gives duality for $X + X^2$ then it gives duality on both its restrictions to $\mathcal{V}$ and $\mathcal{V}^2$. Then, using Theorem 1, ev is constant and equal to $\top$ on both restrictions, meaning that overall, it is constant and equal to $\top$, and the second condition of Proposition 25 does not apply: duality cannot hold for $X + X^2$.

More generally the next section will show that similar reasonings apply to every polynomial functor, extending Theorem 1 and Proposition 24 to all of them.

3.1.4 Polynomial functors

Here we study duality for *Kripke polynomial functors* as defined in [7, Section 2.2]; that is functors built using:

- Constant functors;

- Identity functors;

- Finite products;

- Arbitrary coproducts;

- Composition with exponent functor $(-)^B$ for arbitrary set B;

- Composition with the finite powerset functor.

Therafter we simply talk of polynomial functors or polynomial **Set** endofunctors.

First we give a lemma that will be used to prove a general result on polynomial functors:

Lemma 12. *Let F be a polynomial functor, $r: X \times X \to \mathcal{V}$ a $\mathcal{V}$-relation, and $x, y \in FX$. Then, either there are no couplings of x and y, either there exists an optimal coupling $t \in F(X \times X)$ of x and y, i.e. a coupling such that for all monotone evaluation map $\mathrm{ev}: F(\mathcal{V}) \to \mathcal{V}$,*

$$F^{\downarrow} r(x, y) = \mathrm{ev} \circ Fr(t)$$

Proof. The proof will be done by induction on the structure of polynomial functors. If $F = A$, $x, y \in A$, either $x \neq y$ and there are no couplings of x and y, either $x = y$ and $t = x = y$ is the only and thus optimal coupling of x and y (see the proof of Proposition 24).

If $F = \mathrm{Id}_{\mathbf{Set}}$, then x and y have a unique and thus optimal coupling $(x, y) \in \mathrm{Id}_{\mathbf{Set}}(X \times X)$.

Suppose that the lemma holds for F and G two polynomial functors. Let us consider the different construction possible for polynomial functors:

- $F \times G$. Let $(x, y) = ((x_1, y_1), (x_2, y_2)) \in (F(X) \times G(X))^2$. Then, a coupling for x and y is a pair made of a coupling for x_1 and y_1, and a coupling for x_2 and y_2. By induction hypothesis, either one of these does not exist and there are no couplings of x and y, either there exists couplings of both x_1 and y_1 and x_2 and y_2, and by induction hypothesis there are optimal couplings t_1 of x_1 and y_1 and t_2 of x_2 and y_2. Then, $t = (t_1, t_2)$ is an optimal coupling for x and y. This is proven using the definition of monotonicity for evaluation maps on products of functors.

- $F + G$. Let $(x, i), (y, j) \in F(X) + G(X)$. Then, if $i \neq j$ there is no coupling of (x, i) and (y, j). If $i = j$, then a coupling (t, i) of (x, i) and (y, i) is a coupling of x and y. By induction hypotheses, we can suppose t to be optimal for x and y. Then (t, i) is optimal for (x, i) and (y, i). Thus, there is an optimal coupling of (x, i) and (y, i).

- $F(-)^B$ for a fixed set B. Let $x, y \in F(X)^B$. Then, we can write $x = (x_b)_{b \in B}$, and $y = (y_b)_{b \in B}$ with $x_b, y_b \in F(X)$. Then, a coupling of x and y is $t = (t_b)_{b \in B} \in F(X \times X)^B$ such that for $b \in B$, t_b is a coupling of x_b and y_b. By induction hypotheses, x_b and y_b have an optimal coupling. We will note it t_b despite the above notations. Then, $t = (t_b)_{b \in B}$ is optimal for x and y.

- $\mathcal{P}(F(-))$. Let $X, Y \in \mathcal{P}(F(Z))$. Then, X and Y are subsets of $F(Z)$. A coupling of X and Y is a subset T of $\mathcal{P}(F(X \times X))$ such that for all $t \in T$, there are $x \in X$ and $y \in Y$, and t is a coupling of x and y and such that for all $x \in X$, there is $t \in T$ such that t is a coupling of x with some $y \in Y$, and similarly for all $y \in Y$. Consider T the coupling made of, for all $x \in X$, the coupling t defined as the join of all the optimal couplings of x with some $y \in Y$, and similarly for y. Then, this coupling is optimal for X and Y. (not clear TODO: rewrite this; still optimal couplings for $\mathcal{P}$ were already defined in [3] so not a big deal for now; however here optimal couplings depend on the considered pseudometric, so that this must be added in the definition of optimal couplings). □

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The following result solves duality for polynomial functors on $\mathcal{V}$-**Rel**. The following hypothesis will be made: any occurence of $\varnothing \times F(-)$ will simply be replaced by $\varnothing$. When a construction such as the product bifunctor, the coproduct, or the powerset functor are composed with constant functors only, the whole thing will be considered a constant functor. Thus if we consider $F(-) \times G(-)$ one of F or G at least is not a constant functor.

Theorem 2. *Let $F \colon$ **Set** $\to$ **Set** a polynomial **Set** endofunctor. Suppose that $I = \top$ in $\mathcal{V}$. Then for any monotone evaluation map $\mathrm{ev} \colon F\mathcal{V} \to \mathcal{V}$, duality holds if and only if $\mathcal{V}$ is a singleton or if*

- *there is not constant functors in the recursive definition of F that is not $\varnothing$ or a singleton;*

- *there is not coproducts in the recursive definition of F that is not with the constant functor $\mathrm{F}_\varnothing$;*

- *ev is constant and equal to $\top$.*

Proof. If $\mathcal{V}$ is a singleton, obviously duality holds. From now on suppose $\bot \neq \top$.

Suppose there is no constant functors apart from singletons or $\varnothing$, and no coproducts except when made on $\mathrm{F}_\varnothing$ in the recursive definition of F. Let us prove by induction that F maps constant maps to constant maps. If $F = \mathrm{F}_A$, then either A is a singleton, either $A = \varnothing$. In any case, F maps constant maps to Id_A which is constant. If $F = \mathrm{Id}_{\mathbf{Set}}$, F obviously maps constant maps to constant maps. Suppose that G and H are **Set** endofunctor mapping constant maps to constant maps. Then:

- if $F = H \times G$, then given a constant map f, $Ff = Gf \times Hf$ where Gf and Hf are constant maps by induction hypotheses. Thus, Ff is a constant map.

- if $F = H + G$, then without loss of generality, $G = \mathrm{F}_\varnothing$. In particular there is a natural isomorphism $F \cong H$, so that F maps constant maps to constant maps.

- if $F = H^B$ for B an arbitrary set, then with $f \colon X \to Y$ a map, $Ff = H(f)^B$ maps function $g \in B \to FX$ to $g \circ H(f)$ constant as $H(f)$ is constant by induction hypothesis.

- if $F = \mathcal{P}(H)$, then with $f \colon X \to Y$ a map, $Ff = \mathcal{P}(H(f))$ mapping a subset $S \subseteq \mathcal{P}(HX)$ to its image by $H(f)$. The latter being constant and equal to $y \in H(Y)$ by induction hypotheses, Ff maps any $S \subseteq \mathcal{P}(HX)$ to the singleton $\{y\}$, and Ff is a constant map.

By induction, any such polynomial functor F maps constant maps to constant maps. As $I = \top$ in $\mathcal{V}$ by hypothesis, using Theorem 1, duality holds for any such F if and only if ev is constant and equal to $\top$.

Conversely, suppose that a constant functor on a set with at least two elements or a coproduct with functors that are not $\mathrm{F}_\varnothing$ appear in F's inductive definition. Then, let

us prove by induction that duality cannot hold. We cannot have $F = \mathrm{Id}_{\mathbf{Set}}$. If $F = \mathrm{F}_A$ for A with at least two elements, then by Proposition 24, duality cannot hold as $\mathcal{V}$ is not a singleton. Suppose that the Theorem statement is true for polynomial functors built using n inductive step. Suppose F is built using $n + 1$ inductive steps. Then:

- if $F = H \times G$ then we cannot treat F's duality right away. Let us look at H inductive definition. Without loss of generality, we suppose that $H \neq K_1 \times K_2$. If $H = \mathrm{F}_A$ for some set A, then by hypothesis $A \neq \varnothing$ and either A is a singleton and $F \cong G$, and by induction duality cannot hold on F, either A has at leat two elements, and then we note that $F \cong \coprod_{a \in A} G$, and we treat this case using the next point. If $H = K_1 + K_2$, then $F \cong (K_1 \times G) + (K_2 \times G)$, and we treat this case using the next point. If $H = K^B$, then there is a monic natural transformation $K \times G \Rightarrow F$ sending optimal couplings to optimal couplings, so that by induction hypothesis on $K \times G$, and by the contraposis of Lemma 11, duality cannot hold for F. Similarly for $H = \mathcal{P}(K)$, there is a monic natural transformation $K \times G \Rightarrow F$ that implies that duality cannot hold for F.

  The final case is when $H = X$. Then, we treat the different inductive case in G's definition by the same procedue, leaving one case, $F = X \times X$. This case cannot happen by hypothesis on F.

- if $F = H + G$, then ev giving duality for F must, by restriction, give duality for H and for G. By induction hypotheses, that implies that ev is constant and equal to $\top$. As neither H nor G are the constant functor equal to $\varnothing$, and as $\mathcal{V}$ is not a singleton, there exists $v_H \in H(\mathcal{V})$ and $v_G \in G(\mathcal{V})$. Then, because we have a coproduct, there are no coupling of $(v_H, 1)$ and $(v_G, 2)$. Thus, choosing the right $\mathcal{V}$-relation, the Wasserstein lifting can be equal to $\bot$ when the Kantorovich lifting is constant and equal to $\top$, and duality cannot hold for F.

- if $F = H^B$, then we know that duality cannot hold for H. There is a monic natural transformation $H \Rightarrow F$ that associates to an elements of HX the function constant and equalt to it. Using the contraposis of Lemma 11 and the fact duality cannot holds for H, duality cannot hold for F neither.

- if $F = \mathcal{P}(H)$, then once again there is a monic natural transformation $H \Rightarrow F$ that associates to each element in HX the corresponding singleton in $\mathcal{P}(H(X))$. By the contraposis of Lemma 11 and because by induction hypothesis, duality cannot hold for H, duality cannot hold for F. $\qquad \square$

3.2 Duality for $\mathcal{V}$-pseudometrics

As shown already in the last Section, the Wasserstein and the Kantorovich liftings are better considered on $\mathcal{V}$-pseudometrics. Section 3.2 studies duality for such $\mathcal{V}$-relations.

Even though the methods are essentially the same for every functor, finding a general duality result seems a complicated task. Thus duality is considered one functor at a

time. Some results on general $\mathcal{V}$-relation are also true for $\mathcal{V}$-pseudometrics as shown in Section **??** and **??**.

Our approach rests on two things. First our interpretation of duality. It will be constructed in Section **??** and consolidated throughout Section 3.2. It gives intuitions as to what matters for duality to hold. The second tool is the almost permanent use of the Technical Lemma to actually prove liftings coincide in Section 3.2. Section 2 and Theorem 1 argue that the right place to look for duality is by restricting the Kantorovich and the Wasserstein liftings to $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sym}}$. To this end only well-behaved evaluation maps will be considered from now on. Often it will be supposed that $I = \top$ in $\mathcal{V}$.

Here we will answer several questions about duality: can duality hold for every functor ? No, in general constant functors do not give duality. Is there a generic evaluation map giving duality under mild conditions ? A good candidate would have been the canonical evaluation map. Still it does not give duality on the diagonal and finite probability distribution functors. Is it possible to solve duality, at least for a given functor ? Yes, here we solve this problem for constant functors and the likes, the identity functor, the finite powerset functor, and the diagonal functor. The proofs are very similar but difficult to generalize.

3.2.1 Constant functors, coproducts, and Observables

More generally, it is easy to see that duality holds whenever $\mathcal{V}$ is a singleton: for any set X there is only one $\mathcal{V}$-relation on X.

In a computational interpretation, constant functors are associated to those systems that are in a constant state, with one observable, an element of A. That being said, Proposition 24 is the starting point of our interpretation of duality: whenever an observable is explicitly given in the structure of the functor, then duality cannot hold except for those ev that are able to take observables into account. By explicit observables, we mean constant sets in the structure of the lifted functor. Explicit observables are highlighted by the Wasserstein lifting; they imply non-existence of some couplings making the Wasserstein lifting equal to $\bot$. This notion of explicit observable and its interpretation with duality is made formal in Corollary 2.

Furthermore, examples in the sequel will show that even "implicit" observables prevent duality to hold whenever they are not taken into account by ev. Let us explain what is meant by "implicit" observables. They are natural transformation $\sigma\colon F \Rightarrow G$. Then, given a $\mathcal{V}$-pseudometric $r\colon X \times X \to \mathcal{V}$, elements $t_1, t_2 \in FX$, whenever $\sigma_X t_1 = \sigma_X t_2$, that is with $!_{\mathbf{Set}}$ final in $\mathbf{Set}$, the following commutes for all $f\colon r \to \mathrm{d_e}$:

$$
\begin{array}{ccccc}
 & & ! & & \\
 & {}^{t_1}\swarrow & \downarrow & \searrow^{t_2} & \\
FX & \xrightarrow{\sigma_X} & GX & \xleftarrow{\sigma_X} & FX \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} & & \downarrow{\scriptstyle Ff} \\
F\mathcal{V} & \xrightarrow{\sigma_\mathcal{V}} & G\mathcal{V} & \xleftarrow{\sigma_\mathcal{V}} & F\mathcal{V}
\end{array}
$$

Thus, whenever the evaluation map ev only depends on $\sigma_{\mathcal{V}}$, two elements $t_1, t_2 \in FX$ with the same implicit observable $\sigma_X t_1 = \sigma_X t_2$ will give $\text{ev} \circ Ff(t_1) = \text{ev} \circ Ff(t_2)$ for all $f \colon r \to \text{d}_\text{e}$. Using Lemma 6, if $I = \top$ that implies $F^\uparrow r(t_1, t_2) = \top$. Thus implicit observables are highlighted by the Kantorovich lifting. This notion of implicit observables and its interpretation with duality is very difficult to translate in a formal general way. It will be recalled each time duality is solved using it: for the identity, powerset, diagonal, and finite probability distribution functors.

In this way, duality is a way of looking at the observables present in the system type represented by the functor that is lifted.

Let us generalize the previous result.

Corollary 2. *Let A a set, F an endofunctor on* **Set**. *Suppose $\mathcal{V}$ has at least two elements. Consider the functor $F(-) \times A$ which associates $F(X) \times A$ to a set X and $F(f) \times Id_A$ to a function f. Then, given a well-behaved evaluation map ev for $F(-) \times A$, duality holds if and only,*

- *for all $a_1 \neq a_2 \in A$, $v \in \mathcal{V}$, $\bigwedge \{[\text{ev}(v, a_1), \text{ev}(v, a_2)], [\text{ev}(v, a_2), \text{ev}(v, a_1)]\} = \bot$;*

- *for all $a \in A$, $\text{ev}_{|F(\mathcal{V}) \times \{a\}}$ would give duality as a well-behaved evaluation map for F.*

Proof. Suppose duality holds. Let $r \colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-pseudometrics. Consider $x_1 = (x, a_1)$ and $x_2 = (x, a_2)$ in $F(X) \times A$ with $a_1 \neq a_2$. Then, there exists no coupling of x_1 and x_2, so that $F(-) \times A^\downarrow r(x_1, x_2) = \bot$. Because duality holds, $F(-) \times A^\uparrow r(x_1, x_2) = \bot$. We know that

$$F(-) \times A^\uparrow r(x_1, x_2) = \bigwedge \{\text{d}_\text{e}(\text{ev}(Ff(x), a_1), \text{ev}(Ff(x), a_2)) \mid f \colon r \to \text{d}_\text{e}\}$$

This being equal to $\bot$ gives the first condition. Now consider $x = (x_1, a)$ and $y = (y_1, a)$ and $\text{ev}_{|F(\mathcal{V}) \times \{a\}}$. Then, as duality holds $F(-) \times A^\uparrow r(x, y) = F(-) \times A^\downarrow r(x, y)$, i.e.,

$$\bigwedge \{\text{d}_\text{e}(\text{ev}(Ff(x)), \text{ev}(Ff(y))) \mid f \colon r \to \text{d}_\text{e}\} = \bigvee \{\text{ev}(Ff(t)) \mid t \text{ a coupling of } x \text{ and } y\}$$

Now, as this is true for all r, x_1, and y_1, as a coupling is of the form (t_1, a) for t_1 a coupling of x_1 and y_1 for the functor F, and as in the previous equation we always apply $\text{ev}_{F(-) \times \{a\}}$, then it means $\text{ev}_{F(-) \times \{a\}}$ gives duality for $F(-)$ giving the second condition.

Conversely, suppose both conditions hold. Let $r \colon X \times X \to \mathcal{V}$ a pseudometric. Because on the second condition, whenever we look at the liftings on elements with the same observable in A, we get duality. Let $x = (x_1, a_1)$ and $y = (y_1, a_2)$ with $a_1 \neq a_2$. Then $F(-) \times A^\downarrow r(x, y) = \bot$ because there are no couplings of x and y. Let $v \in \mathcal{V}$. Using the technical lemma we can construct $f \colon r \to \mathcal{V}$ such that $Ff(x_1) = Ff(y_1) = v$. By inclusion,

$$F(-) \times A^\uparrow r(x, y) \leqslant \bigwedge \{\text{d}_\text{e}(\text{ev}(v, a_1), \text{ev}(v, a_2))\} = \bot$$

and thus $F(-) \times A^\uparrow r(x, y) = \bot$ ending the proof. $\qquad\square$

This proposition states that when an observable is made explicit through a constant set, then ev must take it into account so that the Kantorovich lifting can be equal to $\bot$ whenever the Wasserstein lifting is, that is whenever observables do not coincide. For example, when taking $F = \mathrm{Id}_{\mathbf{Set}}$ the identity functor and $\mathcal{V} = \overline{\mathbb{R}}_+$ the usual quantale, using propositions 26 and ??, with $A = [0, \infty]$, we know that $\mathrm{ev}(x, y) = x + y$ will not give duality, because with two different observable y and z, the Kantorovich lifting will give $|y - z| \neq \infty = (\mathrm{Id} \times A^\downarrow)r((x, y), (x, z))$ on (x, y) and (x, z). Still, duality holds takin $\mathrm{ev}(x, y) = x.y$.

3.2.2 The identity functor $\mathrm{Id}_{\mathbf{Set}}$

The simpler functor treated in [3] is $\mathrm{Id}_{\mathbf{Set}}$ the *identity functor*. The evaluation maps that are considered in [3] are of the form $\mathrm{ev}(v) = c.v$ for some constant $0 < c \leqslant 1$. Here we generalises this result.

Proposition 26. *Let us consider* $(\overline{\mathbb{R}}_+, +, 0)$ *the usual symmetric unital quantale where* $I = \top$ *and* ev *a well-behaved evaluation map for the identity functor. Then duality holds on $\mathcal{V}$-pseudometrics:* $\mathrm{Id}_{\mathbf{Set}}^{\uparrow} = \mathrm{Id}_{\mathbf{Set}}^{\downarrow}$.

Proof. Consider $r \colon X \times X \to \mathcal{V}$ a $\overline{\mathbb{R}}_+$-pseudometric. Note that for all $x, y \in X$ there is a unique coupling $(x, y) \in \mathrm{Id}_{\mathbf{Set}}(X \times X)$ that projects to both x and y. Thus,

$$\mathrm{Id}^\downarrow r(x, y) = \mathrm{ev} \circ r(x, y)$$

On the other hand,

$$\mathrm{Id}^\uparrow r(x, y) = \sup\left\{|\mathrm{ev}(f(x)) - \mathrm{ev}(f(y))|; f \colon r \to \mathrm{d}_e\right\}$$

Note that

$$|\mathrm{ev}(r(x, y)) - \mathrm{ev}(0)| = \mathrm{ev} \circ r(x, y)$$

so that we are in the situation of proposition ??. Thus there exists an $f \colon r \to \mathrm{d}_e$ such that $|\mathrm{ev}(f(x)) - \mathrm{ev}(f(y))| = \mathrm{Id}^\downarrow r(x, y)$ proving that duality holds. $\qquad\square$

Remark 12. With $\mathcal{V} = \overline{\mathbb{R}}_+$, well-behaved evaluation map for the identity functor are exactly monotone subadditive maps that are equal to 0 in 0. Thus, any additive maps $f(x) = c.x$ for $c \in [0, \infty)$ works. Another example is $f(x) = \sqrt{x}$. More generally, any increasing concave map that is equal to 0 in 0 works.

Following the previous interpretation we would say that the identity functor presents no observables, being explicit or implicit, that would prevent duality from holding. This will also be the case of the next functor. Still cases on Δ or $\mathcal{D}$ will be more difficult to understand using our interpretation.

3.2.3 The finite powerset functor $\mathcal{P}$

The second more eleborate functor treated in [3] is the *finite powerset functor* $\mathcal{P}$ which send a set X to its set of finite subset $\mathcal{P}(X)$ and a map f to the correspond direct image map $\mathcal{P}(f)$. In [3] it is noted that with $\mathrm{ev}(X) = \max X$ duality holds. We also give a generalization:

Proposition 27. *Let us consider $(\overline{\mathbb{R}}_+, +, 0)$ the usual unital symmetric quantale with $I = \top$ and ev an evaluation map. If ev is well-behaved for $\mathcal{P}$, then duality holds on pseudometrics.*

Proof. Let $\mathrm{ev} \colon \mathcal{P}\mathcal{V} \to \mathcal{V}$ be a well-behaved evaluation map for $\mathcal{P}$. First, let us look at what it means for ev to be monotone. Let $(X_1, X_2) \in (\mathcal{P}\mathcal{V})^2$ be two subsets of $\mathcal{V}$. The usual lifting of relation of $\leqslant$ gives a preorder $\ll$ on $\mathcal{P}\mathcal{V}$ along which ev should be monotone. In particular $\ll$ is defined by the following:

$$X_1 \ll X_2 \Leftrightarrow \exists r \in \mathcal{P} \leqslant, \ \forall i, \ \mathcal{P}\pi_i r = X_i$$

this means that there should exists a subset r of $\mathcal{P}(\mathcal{V} \times \mathcal{V})$ such that if $(x, y) \in r$ then $x \leqslant y$, and such that left (resp. right) projection of elements of r give X_1 (resp. X_2). I claim that $X_1 \ll X_2$ is equivalent to $\min X_1 \leqslant \min X_2$ and $\max X_1 \leqslant \max X_2$. Indeed if this holds then $r = \{(x, \max X_2) \mid x \in X_1\} \cup \{(\min X_1, y) \mid y \in X_2\}$ is a witness giving the inquality. Conversely if $\min X_1 > \min X_2$ then there is no $x \in X_1$ such that $x \leqslant \min X_2$ and $\min X_2$ cannot appear in any witness for $\ll$.

Note in particular that if $\min X_1 = \min X_2$ and $\max X_1 = \max X_2$, even though $X_1 \neq X_2$ monotonicity of ev implies $\mathrm{ev} X_1 = \mathrm{ev} X_2$. In particular, ev only depends on min and max. There exists $f \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ such that $\mathrm{ev} X = f(\min X, \max X)$. Furthermore by definition of $\ll$ f is monotone along both of its components. Note that in our use of f, its first component will always be smaller than its second.

Let us look at the Wasserstein lifting of $\mathcal{P}$ along ev. Let $r \colon X \times X \to \mathcal{V}$ be a $\mathcal{V}$-pseudometric and $T_1, T_2 \subseteq \mathcal{P}X$ be finite subsets,

$$\mathcal{P}^{\downarrow} r(T_1, T_2) = \bigvee \left\{ \widehat{\mathcal{P}} r(T) \mid \forall i, \ \mathcal{P}\pi_i T = T_i \right\}$$
$$= \inf \{\mathrm{ev} \circ \mathcal{P} r(T) \mid \forall i, \ \mathcal{P}\pi_i T = T_i\}$$
$$= \inf \{f(\min \mathcal{P} r(T), \max \mathcal{P} r(T)) \mid \forall i, \ \mathcal{P}\pi_i T = T_i\}$$
$$= \min \{f(\min \mathcal{P} r(T), \max \mathcal{P} r(T)) \mid \forall i, \ \mathcal{P}\pi_i T = T_i\}$$
$$\text{(as } T_1 \text{ and } T_2 \text{ are finite subsets there is a finite number of couplings)}$$

Because f is monotone in both arguments,

$$\mathcal{P}^{\downarrow} r(T_1, T_2) \geqslant f\left(\min \{r(x, y) \mid x \in T_1, \ y \in T_2\}, \max \left(\bigcup_{x \in T_1} \{r(x, y) \mid y \in T_2\} \cup \bigcup_{y \in T_2} \{r(x, y) \mid x \in T_1\} \right)\right)$$

We get an equality with the following coupling of T_1 and T_2 that we will note T from now on (this is the optimal coupling mentioned in [4]),

$$T = \bigcup_{x \in T_1} \{r(x, y) \mid y \in T_2\} \cup \bigcup_{y \in T_2} \{r(x, y) \mid x \in T_1\}$$

There are $r(x_1, y_1)$ and $r(x_2, y_2)$ such that

$$\mathcal{P}^{\downarrow} r(T_1, T_2) = f(r(x_1, y_1), r(x_2, y_2))$$

Now let us look at the Kantorovich lifting of $\mathcal{P}$ along ev:

$$\mathcal{P}^{\uparrow} r(T_1, T_2) = \bigwedge \left\{ d_e(\widehat{\mathcal{P}}g(T_1), \widehat{\mathcal{P}}g(T_2)) \mid g \colon r \to d_e \right\}$$
$$= \sup \left\{ |f(\min \mathcal{P}g(T_1), \max \mathcal{P}g(T_1)) - f(\min \mathcal{P}g(T_2), \max \mathcal{P}g(T_2))|; g \colon r \to d_e \right\}$$

If we find g giving equality:

$$|f(\min \mathcal{P}g(T_1), \max \mathcal{P}g(T_1)) - f(\min \mathcal{P}g(T_2), \max \mathcal{P}g(T_2))| = f(r(x_1, y_1), r(x_2, y_2))$$

Withoug loss of generality let us suppose that,

$$r(x_2, y_2) = \min \{r(x_2, y) \mid y \in T_2\}$$

Then, we set,

$$\forall x \in T_1, \ g(x) = \min \{r(x, y) \mid y \in T_2\}$$
$$\forall y \in T_2, \ g(y) = 0$$

With this definition,

$$|f(\min \mathcal{P}g(T_1), \max \mathcal{P}g(T_1)) - f(\min \mathcal{P}g(T_2), \max \mathcal{P}g(T_2))|$$
$$= f(\min \mathcal{P}g(T_1), \max \mathcal{P}g(T_1)) = f(r(x_1, y_1), r(x_2, y_2))$$

Also, for all $y, y' \in T_2$, $r(y, y') \geqslant \top = d_e(g(y), g(y'))$, for all $y \in T_2$ and $x \in T_1$,

$$d_e(g(x), g(y)) = g(x)$$
$$= \min \{r(x, y') \mid y' \in T_2\}$$
$$\leqslant r(x, y)$$

and finally for all $x, x' \in T_1$, if $g(x) = r(x, y)$ and $g(x') = r(x', y')$ following the definition of g,

$$r(x, y') \leqslant r(x, x') + r(x', y')$$
$$r(x, y') \geqslant r(x, y)$$

giving,

$$r(x, x') \geqslant r(x, y) - r(x', y')$$

and by symmetry,

$$r(x', x) \geqslant r(x', y') - r(x, y)$$

meaning that $d_e(g(x), g(x')) \leqslant r(x, x')$. Thus, we are in the situation of the proposition **??**, and there exists $g \colon r \to d_e$ such that duality holds. $\qquad\square$

Following our previous interpretation, there is no observable, implicit or explicit, that could be defined on $\mathcal{P}$, explaining why duality always hold. In [5], it is shown that min is not well-behaved for the powerset functor. Here we know that a well-behaved evaluation map for the powerset functor must depend only on the maximum and on the minimum of the given subset of $\mathcal{V}$. A question is, "is there a well-behaved evaluation map that do not only depend on max ?". To this end, a way of knowing when an evaluation map is well-behaved that depend not on subsets but on values would be great. We give here such a result:

Proposition 28. *Let* ev *an evaluation map for the finite powerset functor. It is well-behaved if and only if:*

- *it is monotone in both its arguments;*

- *for all $x_m \leqslant x_M$ and $y_m \leqslant y_M$ in $\mathcal{V}$ such that $y_m + x_M \leqslant x_m + y_M$,*

$$\mathrm{ev}(y_m + x_M, x_M + y_M) \leqslant \mathrm{ev}(y_m, y_M) + \mathrm{ev}(x_m, x_M)$$

- $\mathrm{ev}(0,0) = 0.$

Proof. In the definition 21 of well-behaved evaluation maps, there are three conditions. The first one is that it should be monotone. This is, here, equivalent to it being monotone in each of its component. The third one is that precomposing by $\mathcal{P}\kappa_X$ gives a map superior or equal to $\kappa \mathcal{P} X$, that is that the direct image of $\{\top\}$ is equal to $\{\top\}$. Here this is equivalent to the third condition: $\mathrm{ev}(0,0) = 0$.
Finally there is the condition:

$$\forall p\colon X \to \mathcal{V}, \ \mathrm{ev} \circ \mathcal{P}(p+q) \leqslant_{\mathbb{R}} \mathrm{ev} \circ \mathcal{P}(p) + \mathrm{ev} \circ \mathcal{P}(q)$$

As p and q are arbitrary predicates, this amounts to say that for all finite subsets $Y, Z \subseteq \mathcal{V}$ that are in bijection $f\colon Y \to Z$,

$$\mathrm{ev}(\min\left\{z + f^{-1}(z)|z \in Z\right\}, \max\left\{z + f^{-1}(z)|z \in Z\right\}) \leqslant \mathrm{ev}(\min\left\{z|z \in Z\right\}, \max\left\{z|z \in Z\right\}) + \mathrm{ev}(\min\left\{f^{-1}(\right.$$

Noting $x_m, x_M = \min X, \max X$, and $y_m, y_M = \min Y, \max Y$, this is equivalent to, the second part of this inequation can be rewritten

$$\mathrm{ev}(x_m, x_M) + \mathrm{ev}(y_m, y_M)$$

Without loss of generality, suppose $y_m + x_M \leqslant y_M + x_m$. Now, suppose that the condition of this proposition holds. Then as $\min\left\{x + f^{-1}(x)|x \in X\right\} \leqslant x_M + y_m$ and as $\max\left\{x + f^{-1}(x)|x \in X\right\} \leqslant x_M + y_M$, by monotonicity of ev we get that ev is well-behaved.

Conversely, suppose that ev is well-behaved. Then consider $X = (x_M, x_m, x_M)$ and $Y = (y_M, y_M, y_m)$ Then, well-behavness on these sets implies

$$\mathrm{ev}(y_m + x_M, x_M + y_M) \leqslant \mathrm{ev}(x_m, x_M) + \mathrm{ev}(y_m, y_M)$$

ending the proof. $\square$

This result can be used to prove some evaluation maps are not well-behaved:

Example 7. As $(1 + 2) + (2 + 2) \geqslant (1 + 2) + (1 + 2)$, $\mathrm{ev}(x, y) = x + y$, that is $\mathrm{ev}(X) = \min X + \max X$ is not well-behaved.

It can be noted that in this example, composing by the square roots gives $\sqrt{(1 + 2) + (2 + 2)} = \sqrt{7} \leqslant 2\sqrt{3}$. A good candidate for a well-behaved evaluation map is thus $\mathrm{ev}(X) = \sqrt{\min X + \max X}$. Interestingly enough, it is well-behaved;

Proposition 29. *The* evaluation map *$\mathrm{ev} \colon \mathcal{PV} \to \mathcal{V}$ associating to a subset $X \subseteq \mathcal{V}$ the value $\sqrt{\min X + \max X}$ is* well-behaved.

Proof. This proof will be done using the proposition 28. The goal is to show that for all $x_m, x_M, y_m, y_M \in \mathcal{V}$ such that $x_m \leqslant x_M$, $y_m \leqslant y_M$ and $y_m + x_M \leqslant y_M + x_m$,

$$\sqrt{y_m + 2x_M + y_M} \leqslant \sqrt{x_m, x_M} + \sqrt{y_m, y_M}$$

We prove this is always true:

$$\sqrt{y_m + 2x_M + y_M} \leqslant \sqrt{x_m + x_M} + \sqrt{y_m + y_M}$$
$$\Leftrightarrow y_m + 2x_M + y_M \leqslant x_m + x_M + y_m + y_M + 2\sqrt{(x_m + x_M)(y_m + y_M)}$$
$$\Leftrightarrow x_M - x_m \leqslant 2\sqrt{(x_m + x_M)(y_m + y_M)}$$
$$\Leftrightarrow x_M^2 - 2x_m x_M + x_m^2 \leqslant 4(x_m + x_M)(y_m + y_M)$$
$$\Leftrightarrow x_M^2 - 2(x_m + 2(y_m + y_M))x_M + x_m^2 - 4x_m(y_m + y_M) \leqslant 0$$

From now on, we will consider y_m, y_M, and x_m fixed. Note that $x_M \in [x_m, y_M - y_m + x_m]$. Let us study the corresponding quadratic equation in x_M. The discriminant is

$$\begin{aligned}
\Delta &= 4(x_m + 2(y_m + y_M))^2 - 4(x_m^2 - 4x_m(y_m + y_M)) \\
&= 4x_m^2 + 16x_m(y_m + y_M) + 16(y_m + y_M)^2 - 4x_m^2 + 16x_m(y_m + y_M) \\
&= 32x_m(y_m + y_M) + 16(y_m + y_M)^2
\end{aligned}$$

which is obviously positive. Thus, if the interval for x_M, that is $[x_m, y_M - y_m + x_m]$ is included in the interval in which our quadratic equation is negative, then the condition for well-behavness is always true and the proof is over.

Firs, we want

$$\frac{-(-2(x_m + 2(y_m + y_M))) - \sqrt{\Delta}}{2} \leqslant x_m$$
$$\Leftrightarrow x_m + 2(y_m + y_M) - \frac{\sqrt{\Delta}}{2} \leqslant x_m$$
$$\Leftrightarrow 16(y_m + y_M)^2 \leqslant \Delta$$
$$\Leftrightarrow 16(y_m + y_M)^2 \leqslant 32x_M(y_m + y_M) + 16(y_m + y_M)^2$$
$$\Leftrightarrow 0 \leqslant 32x_M(y_m + y_M)$$

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which is always true.

The second condition is

$$y_M - y_m + x_m \leqslant \frac{-(-2(x_m + 2(y_m + y_M))) + \sqrt{\Delta}}{2}$$

$$\Leftrightarrow y_M - y_m + x_m \leqslant x_m + 2(y_m + y_M) + \frac{\sqrt{\Delta}}{2}$$

$$\Leftrightarrow 0 \leqslant 3y_m + \frac{\sqrt{\Delta}}{2}$$

which is always true. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2.4 The diagonal functor Δ

The counter-example given in [3] is for the *diagonal functor* Δ mapping a set X to $X \times X$ and a map f to $f \times f$. Following our interpretation, an implicit observable associated to Δ is the symmetry of elements in $X \times X$. Because the evaluation map given by the sum $\mathrm{ev} \colon (x, y) \in \mathcal{V} \times \mathcal{V} \to x + y\mathcal{V}$ does not distinguish symmetric elements, duality is prevented from holding. We first generalizes this result:

Proposition 30. *Let* ev *a well-behaved evaluation map for* Δ *such that for all* $x, y \in \mathcal{V}$, $\mathrm{ev}(x, y) = \mathrm{ev}(y, x)$. *Then* duality *holds if and only if* ev *is constant and equal to* $\top$.

Here we give an example of well-behaved evaluation map for Δ for which duality holds.

Proposition 31. *Let* $(\mathcal{V}, \otimes, I)$ *be a symmetric unital quantale in which* $I = \top$. *Consider the functor* Δ. *Then, any of the two projection maps* π_i *is well-behaved and* duality *holds for the associated liftings.*

Proof. π_i is obviously monotone. Furthermore, $\pi_i(\top, \top) = \top$ and $\pi_i(x \otimes y, x \otimes y) = x \otimes y = \pi_i(x, x) \otimes \pi_i(y, y)$ proving π_i is well-behaved. Now consider $r \colon X \times X \to \mathcal{V}$ a $\mathcal{V}$-metric. Then there is only one coupling $((x_1, y_1), (x_2, y_2))$ having (x_1, x_2) and (y_1, y_2) as projections through Δ. Thus,

$$\Delta^{\downarrow} r(x, y) = \pi_i \Delta r((x_1, x_2), (y_1, y_2)) = r(x_1, x_2)$$

Furthermore,

$$\Delta^{\uparrow} r(x, y) = \sup \{|\pi_i \circ (f \times f)(x) - \pi_i \circ (f \times f)(y)|; f \colon r \to \mathrm{d_e}\}$$
$$= \sup \{|f(x_1) - f(y_1)|; f \colon r \to \mathrm{d_e}\}$$

We already saw in the proof 24 that using the proposition **??** there exists an $f \colon r \to \mathrm{d_e}$ such that

$$|f(x_1) - f(y_1)| = r(x_1, x_2)$$

proving the Kantorovich-Rubinstein duality holds in this case. $\qquad\qquad\qquad\square$

Projections beak the symmetry given by the symmetry observable in a way that symmetric elements cannot relate to one another through ev. This is the explanation, through our interpretation of duality, of why duality holds in this case. One could wonder what would happens if ev was not symmetric, but still did depend on both variables. If we do not have a general result in this setting, we know that in some cases duality does not hold. We interpret duality not holding because symmetric elements still relate too much to one another through ev:

Proposition 32. *Let ev a well-behaved evaluation map for Δ. Then duality holds if and only if ev is of the form*

$$\mathrm{ev} = f \circ \pi_i + \mathbb{1}_{\{\pi_j = \infty\}} g \circ \pi_i$$

for $i \neq j$.

Proof. Suppose ev gives duality for Δ. Let us look at what the "symmetry observable" implies. Consider a pseudometric $r \colon X \times X \to [0, \infty]$. Let x, y be elements of X. Then, by definition,

$$\Delta^{\downarrow} r((x, y), (y, x)) = \mathrm{ev}(r(x, y), r(x, y))$$

By hypotheses,

$$\sup \{|\mathrm{ev}(fx, fy) - \mathrm{ev}(fy, fx)|; f \colon r \to \mathrm{d_e}\} = \mathrm{ev}(r(x, y), r(x, y))$$

Because we are on $[0, \infty]$, we can consider a sequence $(f_n)_n$ of functions from r to $\mathrm{d_e}$ that will give the above equality as its limit. From this we can extract a subsequence $(x_n, y_n)_n$ such that $\lim_{n \to \infty} |\mathrm{ev}(x_n, y_n) - \mathrm{ev}(y_n, x_n)| = \mathrm{ev}(r(x, y), r(x, y))$ and such that both $(x_n)_n$ and $(y_n)_n$ have a limit x_l and y_l and for all n, $|x_n - y_n| \leqslant r(x, y)$. Setting a function to be equal to x_l on x and y_l on y, and using the proposition **??**, we get that there actually exists $f \colon r \to \mathrm{d_e}$ such that $|\mathrm{ev}(fx, fy) - \mathrm{ev}(fy, fx)| = \mathrm{ev}(r(x, y), r(x, y))$. There after, we will note $r(x, y) = a$ and $fx = v$ and $fy = w$. The constraint on these values is $|v - w| \leqslant a$.

First, let us distinguish different cases. We suppose here that $0 < a < \infty$. If $\mathrm{ev}(a, a) = \top = 0$, as ev is subadditive in both its arguments, then for all $x, y < \infty$, $\mathrm{ev}(x, y) = 0$. Else, if $\mathrm{ev}(a, a) = \bot = \infty$, then for all $n \in \mathbb{N}$, $\mathrm{ev}\left(\frac{a}{2^n}, \frac{a}{2^n}\right) = \infty$, and for all $x, y \neq \top$, $\mathrm{ev}(x, y) = \infty$. Those two cases will be treated later. Here we suppose that $0 < \mathrm{ev}(a, a) < \infty$. Without loss of generality, we can suppose that

$$\mathrm{ev}(v, w) - \mathrm{ev}(w, v) = \mathrm{ev}(a, a)$$

If $v \geqslant w$, as ev is well-behaved,

$$\mathrm{ev}(v, w) \leqslant \mathrm{ev}(v, v) \leqslant \mathrm{ev}(w, v) + \mathrm{ev}(v - w, 0)$$

giving that,

$$\mathrm{ev}(v, w) - \mathrm{ev}(w, v) \leqslant \mathrm{ev}(v - w, 0) \leqslant \mathrm{ev}(a, 0) \leqslant \mathrm{ev}(a, a)$$

Because duality holds, all these inequalities are in fact equalities. In particular, for all $0 \leqslant c \leqslant a$, $\mathrm{ev}(a,c) = \mathrm{ev}(a,0)$ and

$$\mathrm{ev}(v,w) \leqslant \mathrm{ev}(a,w) = \mathrm{ev}(a,0)$$

so that we can write,

$$\mathrm{ev}(a,0) - \mathrm{ev}(w,v) = \mathrm{ev}(a,0)$$

proving $\mathrm{ev}(w,v) = 0$. In particular this implies that $\mathrm{ev}(0,v) = 0$, and by subadditivity, that $\mathrm{ev}(0,c) = 0$ for all $0 \leqslant c < \infty$. Now consider values c, d, d' with $d \leqslant d' < \infty$. By subadditivity,

$$\mathrm{ev}(c,d') \leqslant \mathrm{ev}(c,d) + \mathrm{ev}(0,d'-d) = \mathrm{ev}(c,d)$$

But by monotonicity, $\mathrm{ev}(c,d) \leqslant \mathrm{ev}(c,d')$ and by antisymmetry, $\mathrm{ev}(c,d) = \mathrm{ev}(c,d')$, so that ev only depends on one variable, exept on ∞, giving the form of the proposition. Let us prove that with such a form, duality holds. Suppose $\mathrm{ev}(x,y) = f(x) + \mathbb{1}_{\{y=\infty\}} g(x)$. Let $r \colon X \times X \to [0,\infty]$ a pseudometric. Consider $x, y, z, w \in X$. Then,

$$\Delta^{\downarrow} r((x,y),(z,w)) = \mathrm{ev}(r(x,z), r(y,w))$$
$$= f(r(x,z)) + \mathbb{1}_{\{r(y,w)=\infty\}} g(r(x,z))$$

Then, we have to distinguish cases. If $r(y,w)$ is finite, then either:

- if $r(x,y)$ is infinite, then $r(x,w)$ is infinite, and we set $f(x) = r(x,z)$ and $f(y) = f(z) = f(w) = 0$ meaning $|\mathrm{ev}(fx,fy) - \mathrm{ev}(fz,fw)| = \mathrm{ev}(fx,fy) = \mathrm{ev}(r(x,z),0) = \mathrm{ev}(r(x,z),r(y,w))$ and duality holds.

- if $r(x,y)$ is finite, then $r(x,w)$ is finite. Then we set $fx = r(x,z)$, and $fz = 0$. Because $r(x,y)$ and $r(x,w)$ are both finite, the value that f will take on y and w in the proof of the proposition ?? will be finite, giving, $|\mathrm{ev}(fx,fy) - \mathrm{ev}(fz,fw)| = \mathrm{ev}(fx,fy) = \mathrm{ev}(r(x,z),r(y,w))$ and duality holds.

Now, if $r(y,w)$ is infinite; by subbaditivity of ev

$$r(y,w) \leqslant r(y,x) + r(x,w)$$
$$\leqslant r(y,z) + r(z,w)$$

If this implies that either $r(y,x)$ or $r(x,w)$ and either $r(y,z)$ or $r(z,w)$ are infinite. Let us suppose that $r(y,x)$ and $r(y,z)$ are infinite (the case $r(x,w)$ and $r(z,w)$ is treated in the same way). Then we set $fx = r(x,z)$, $fz = 0$ and $fy = \infty$. Then, if $r(x,w) = r(y,w) = r(z,w) = \infty$, we set $fw = 0$. Using the proposition ?? we get duality. Else there is $v \in \{x,z\}$ such that $r(v,w)$ is finite. Then, the value that the proof of the proposition ?? will asign to fw is finite, giving duality. In any case, duality holds, proving the proposition. $\qquad \square$

Remark 13. This proof is remarkable. All the constraints ev must comply with are given just by looking at ev on symmetric elements, meaning the "implicit observable" we associated to Δ is the only thing that can prevent duality from holding. It seems that duality holds in general, except when the interaction between the liftings and the functor implies some very specific structural constraints.

3.2.5 The finite probability distribution functor $\mathcal{D}$

A seemingly good choice for ev is the canonical map. But duality does not hold in general for the canonical evaluation map. The counterexample we give is on the *finite probability distribution functor* which associates to a set X the set of finite probability distribution on X, $\mathcal{D}X = \{p \colon X \to [0;1] \mid \mathrm{supp}\,p \text{ is finite and } \sum_{x \in X} p(x) = 1\}$, and to any map $f \colon X \to Y$ the map $\mathcal{D}f \colon \mathcal{D}X \to \mathcal{D}Y$, defined by $\mathcal{D}f(p) = (\lambda y. \sum_{x \in f^{-1}(y)} p(x))$.

Proposition 33. *Consider the setting with the finite probability distribution functor, the canonical evaluation map, and the quantale $([0, \infty], +, 0)$ with reversed order. On this setting duality does not hold.*

Proof. Consider an arbitrary metric (a usual one in this setting) $d \colon X \times X \to [0, \infty]$ on X. The liftings on d are defined by, with $P_1, P_2 \in \mathcal{D}X$,

$$\mathcal{D}^{\downarrow}d(P_1, P_2) = \bigvee \left\{ \widehat{\mathcal{D}}d(P) \mid P \in \mathcal{D}(X \times X), \ \forall i, \ \mathcal{D}\pi_i P = P_i \right\}$$
$$= \inf \left\{ \mathrm{ev}_{\mathrm{can}}(\mathcal{D}d(P)) \mid P \in \mathcal{D}(X \times X), \ \forall i, \ \mathcal{D}\pi_i P = P_i \right\}$$
$$\mathcal{D}^{\uparrow}d(P_1, P_2) = \bigwedge \left\{ \mathrm{d_e}(\widehat{\mathcal{D}}f(P_1), \widehat{\mathcal{D}}f(P_2)) \mid f \colon d \to \mathrm{d_e} \right\}$$
$$= \sup \left\{ |\mathrm{ev}_{\mathrm{can}}(\mathcal{D}f(P_1)) - \mathrm{ev}_{\mathrm{can}}(\mathcal{D}f(P_2))| \mid f \colon X \to \mathcal{V}, \ \mathrm{d_e} \circ (f \times f) \geqslant_{\mathbb{R}} d \right\}$$

First let us find out what $\mathrm{ev}_{\mathrm{can}} \colon \mathcal{D}\mathcal{V} \to \mathcal{V}$ is doing here. Let us consider $P \in \mathcal{D}\mathcal{V}$. By definition,

$$\mathrm{ev}_{\mathrm{can}}(P) = \bigvee \{r \mid P \in \mathcal{D}(\uparrow r)\}$$
$$= \inf \{r \mid P \in \mathcal{D}(\mathrm{true}_r)[\mathcal{D}\uparrow r]\}$$
$$= \inf \left\{ r \mid P \in (\lambda D. (\lambda y. \sum_{x \in \mathrm{true}_r^{-1}(y)} D(x))[\mathcal{D}\{v \in [0, \infty] \mid v \leqslant_{\mathbb{R}} r\}]) \right\}$$
$$= \inf \left\{ r \mid P \in (\lambda D. \left(\lambda y. \begin{cases} D(y) \text{ if } y \leqslant_{\mathbb{R}} r \\ 0 \text{ else} \end{cases} \right)[\mathcal{D}[0, r]])i \right\}$$
$$= \inf \{r \mid P \in \mathcal{D}[0, r]\}$$
$$= \max \mathrm{supp}\,P$$

where we do have a max because $\mathcal{D}$ is the finite probability distribution functor. Now note that P_1 and P_2 have finite support. Because $\mathcal{D}\pi_i P = P_i$ we have $\mathrm{supp}\,P \subseteq \mathrm{supp}\,P_1 \times \mathrm{supp}\,P_2$, and thus, the couplings P have a finite number of possible supports. Because of the definition of $\mathrm{ev}_{\mathrm{can}}$ $\mathrm{ev}_{\mathrm{can}}(\mathcal{D}d(P))$ depends only on P's support, and finally there are a finite number of elements defining $\mathcal{D}^{\downarrow}d(P_1, P_2)$: this is defined by a minimum. Using the proposition 17. we know that, as F preserves weak-pullback, $\mathrm{ev}_{\mathrm{can}}$ is well-behaved such that

$$\mathrm{ev}_{\mathrm{can}}^{-1}[0] = \mathcal{D}\kappa_0[\mathcal{D}(\{0\})]$$

Using the proposition 18, we know $\mathcal{D}d$ is a metric.

Because the Wasserstein lifting preserves metrics, if the Kantorovich lifting does not,

then it proves that duality does not hold. Suppose P_1 and P_2 have the same support but different distributions. We know that $\mathcal{D}^{\downarrow}d(P_1, P_2) \neq 0$ as d is a metric. Now, for all $f \colon d \to d_e$, $|\text{ev}_{\text{can}}(\mathcal{D}f(P_1)) - \text{ev}_{\text{can}}(\mathcal{D}f(P_2))| = 0$ because P_1 and P_2 share the same support, and by definition of ev_{can}. Thus, $\mathcal{D}^{\uparrow}d(P_1, P_2) = 0$ and duality does not hold.$\square$

This is the same proof than the one used in [3] to prove that with the functor Δ and $+$ as evaluation map, duality does not hold. Following our interpretation, the support of a finite probability distribution is an implicit observable. We mean by that that different elements can have the same observable, i.e. support, so that the maps Ff in the definition of the Kantorovich lifting maps same support to same support, meaning that an evaluation map unable to distinguish elements having the same observable will prevent duality from holding.

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