A finite presentation of graphs of treewidth at most 3

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Outline

1. Graphs, terms, and treewidth

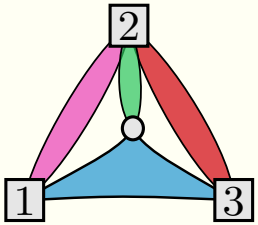
2. The main result

3. Recursively decomposing graphs of treewidth at most 3

4. Conclusion

Graphs, terms, and treewidth

Graphs



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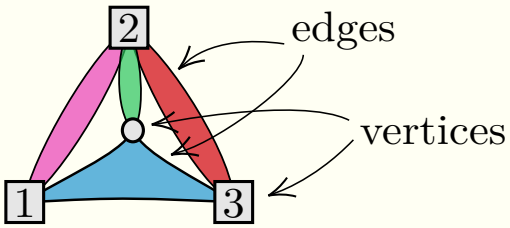

Graphs

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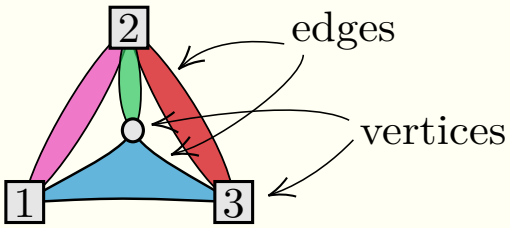

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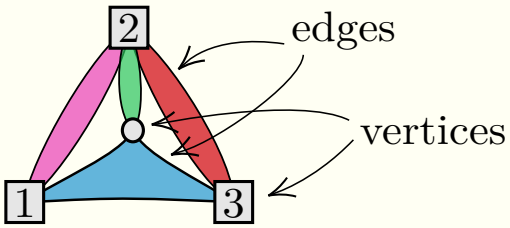

Specificities:

- **Sources**: squared vertices, form the interface of the graph,

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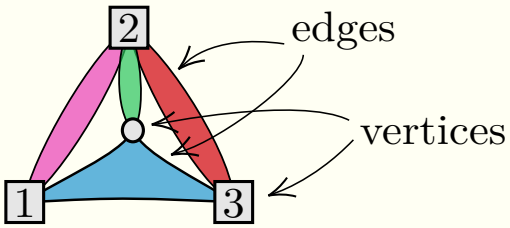

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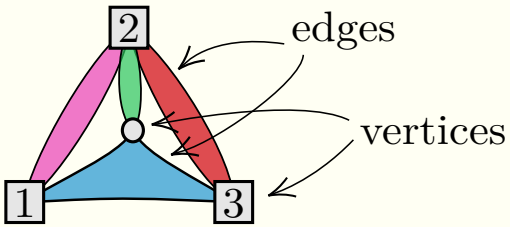

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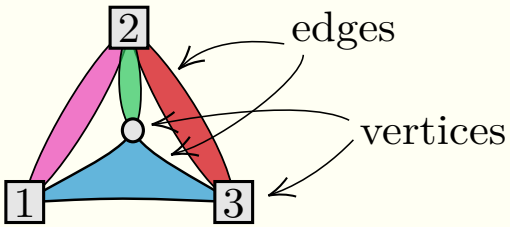

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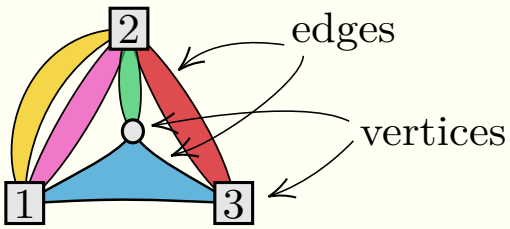

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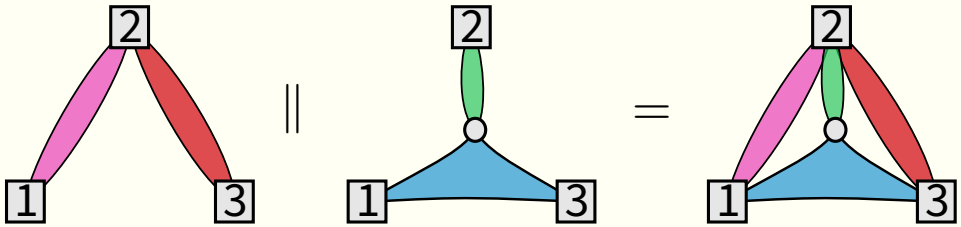
Graph operations and terms

Parallel operation, $- \parallel -$:



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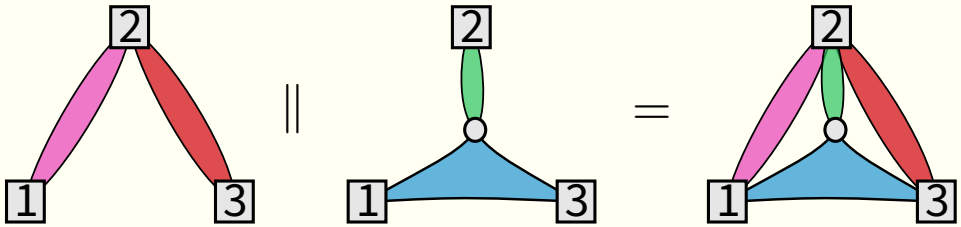

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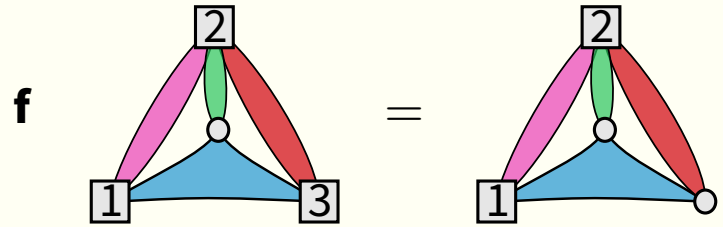

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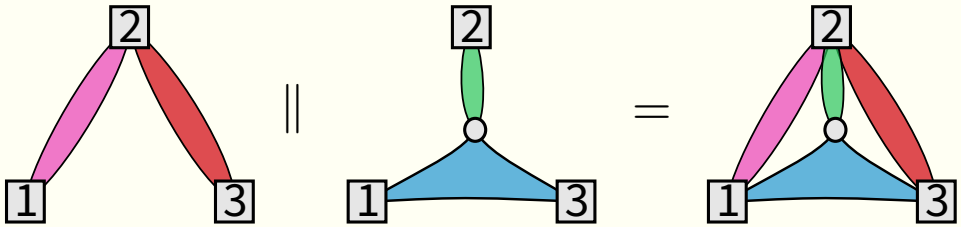

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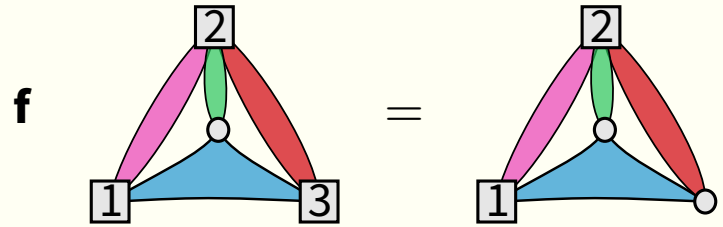

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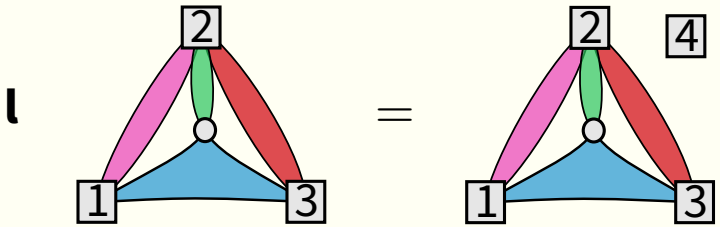

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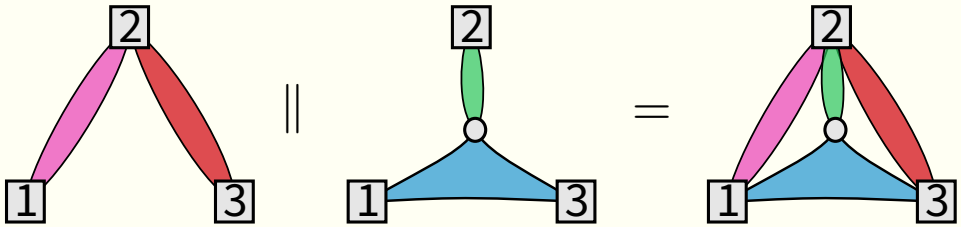

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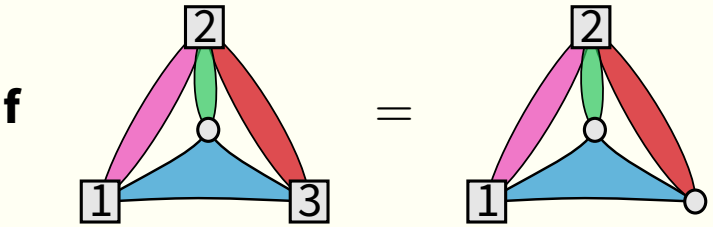

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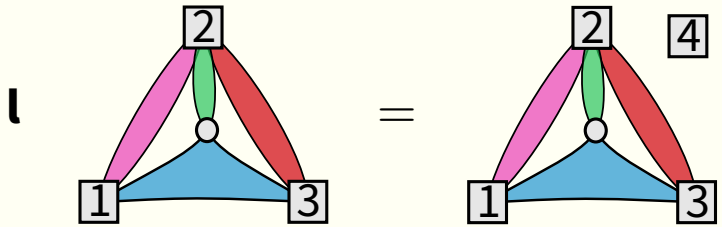
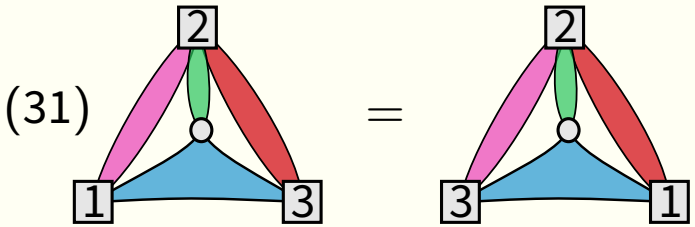


Permutation of sources:



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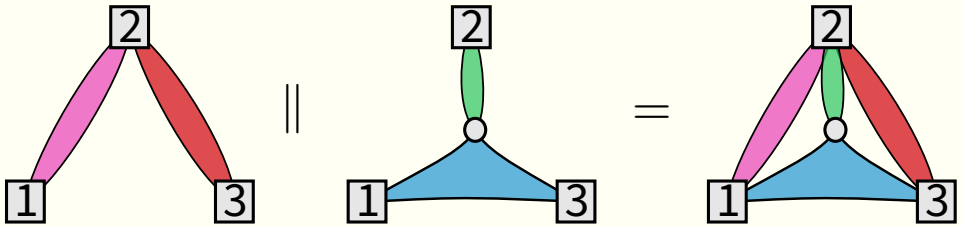

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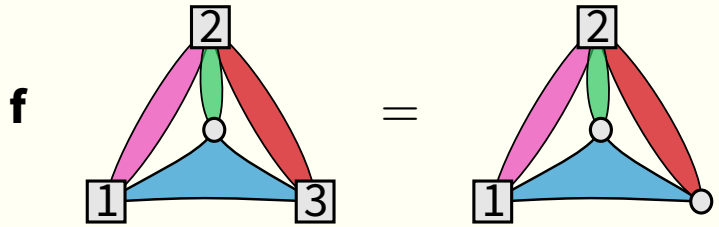

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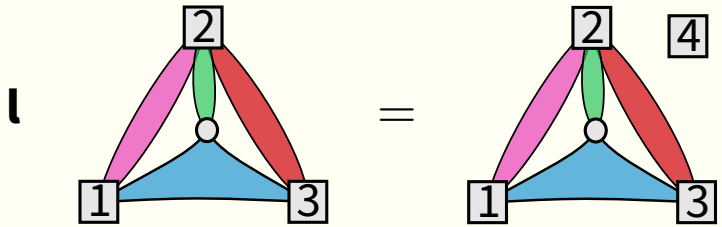

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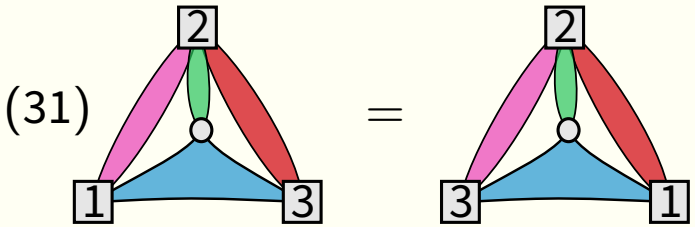

Corresponding grammar:

$$t, u ::= a \mid \emptyset \mid t \parallel u \mid \mathbf{l}\, t \mid \mathbf{f}\, t \mid \mathrm{p}\, t,$$

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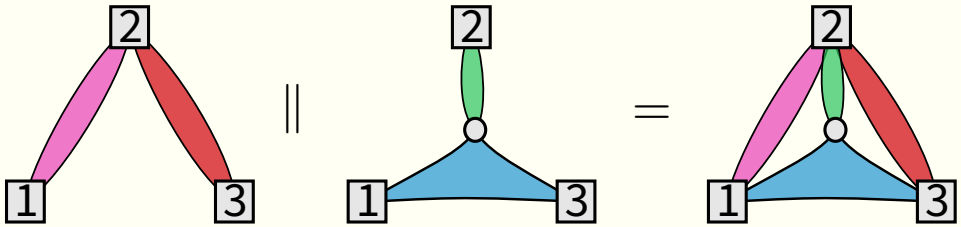

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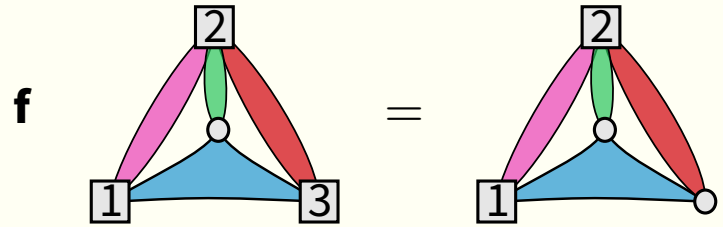

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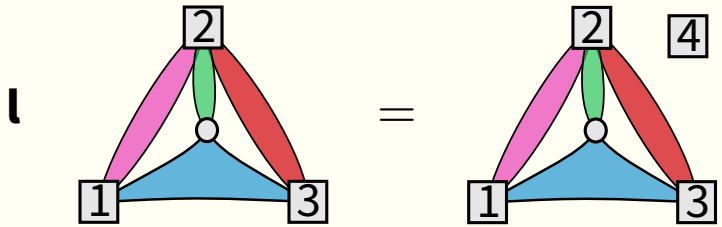

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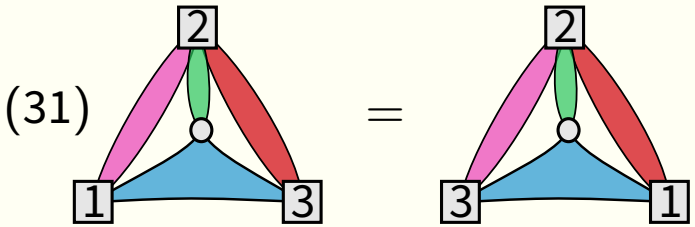

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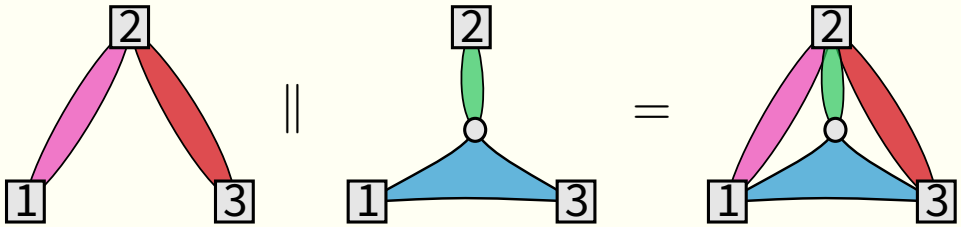

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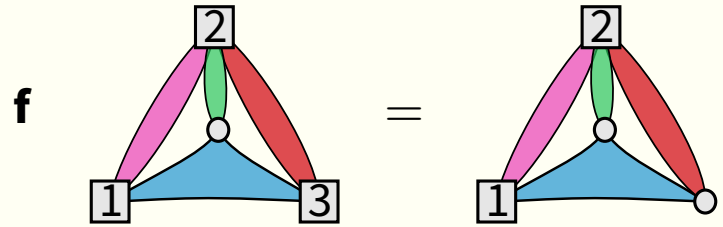

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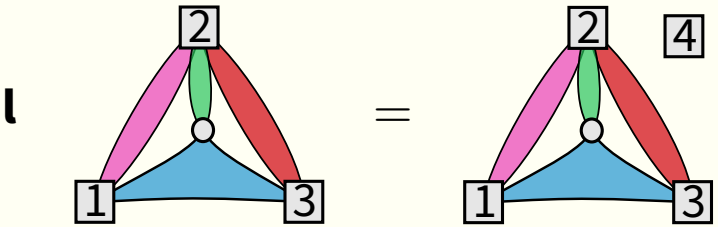

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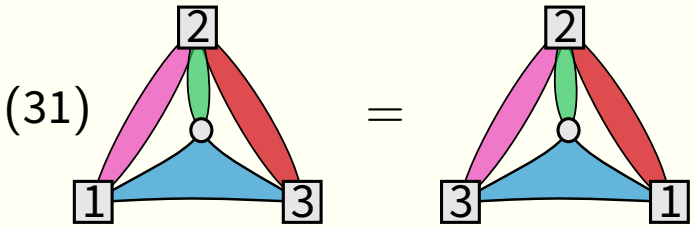

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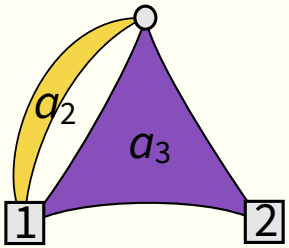
An example with two parsings



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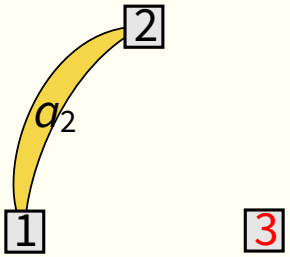

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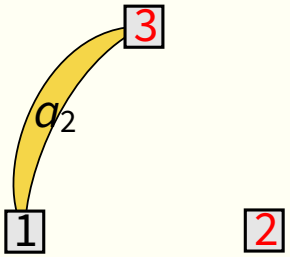

$(32)\ \mathbf{l}\, a_2$

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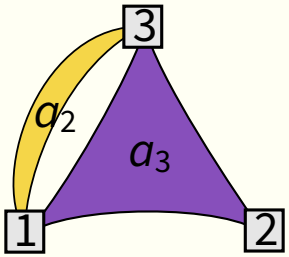

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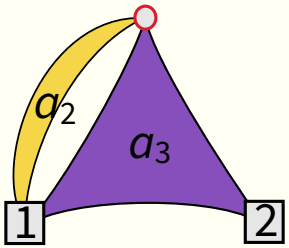

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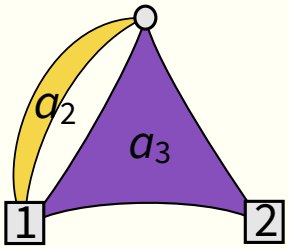

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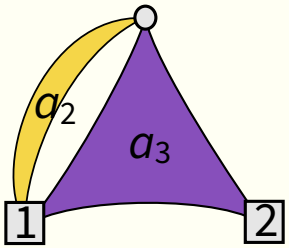

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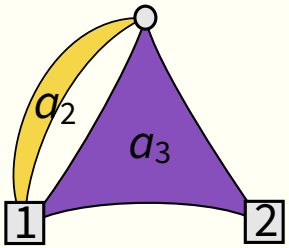

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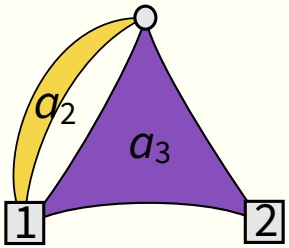

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Graphs can have multiple parsings: $\mathbf{f}(32)(\mathbf{l}\,a_2 \parallel (32)a_3)$.

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Goal: finding axioms that prove equal terms parsing the same graph,

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Proposition

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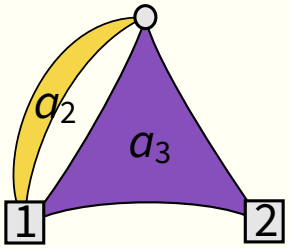
Here, this proposition serves as definition for treewidth.

The main result

From one parsing to another ?



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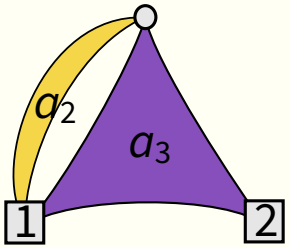

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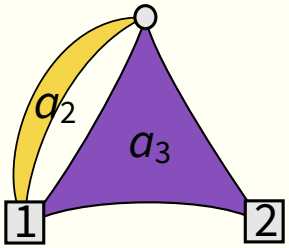

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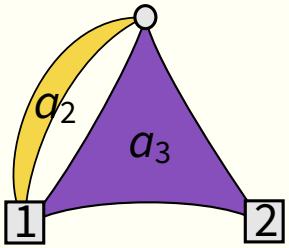

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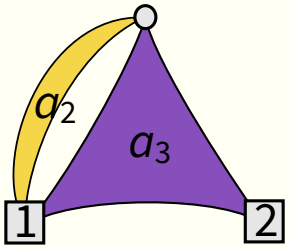

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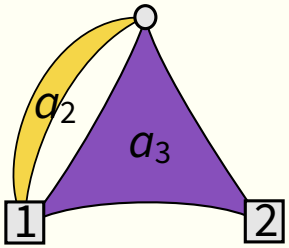

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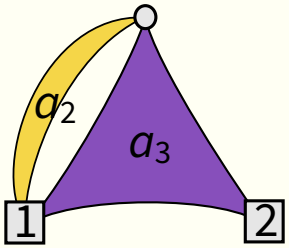

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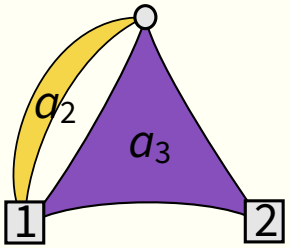

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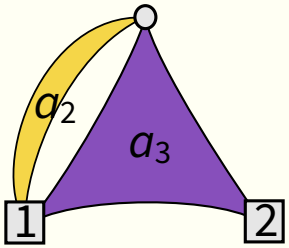

$$\mathbf{f}(32)(\mathbf{l}\,a_2 \parallel (32)a_3)$$
$$\mathbf{f}(32)(\mathbf{l}\,a_2 \parallel (32)a_3)$$

Axiom: a pair of terms that can be used to prove two terms construct the same graph.

Problem

Is there a finite list of axiom Ax such that for all terms t, u of width at most 3

$$\mathrm{Ax} \vdash t = u \Leftrightarrow \mathbf{g}\,t \simeq \mathbf{g}\,u$$

holds ?

The axioms, the main result, and the method

Axioms Ax given by:

1. $a \parallel (b \parallel c) = (a \parallel b) \parallel c$,

The axioms, the main result, and the method

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1. $a \parallel (b \parallel c) = (a \parallel b) \parallel c$, $\quad a \parallel b = b \parallel a$, and $\quad a \parallel \emptyset = a$,
2. $pqa = (p \circ q)a$, and $\quad \mathrm{id}\, a = a$,
3. $p(a \parallel b) = pa \parallel pb$ and $\quad p\emptyset = \emptyset$,
4. $\mathbf{l}(a \parallel b) = \mathbf{l}\, a \parallel \mathbf{l}\, b$ and $\quad \mathbf{l}\, \emptyset = \emptyset$,
5. $p\, \mathbf{f}\, a = \mathbf{f}\, \dot{p} a$ and $\quad \mathbf{l}\, pa = \dot{p}\, \mathbf{l}\, a$,
6. $\mathbf{l}\, \mathbf{f}\, a = \mathbf{f}\, r\, \mathbf{l}\, a$ and $\quad \mathbf{l}\, \mathbf{l}\, a = r\, \mathbf{l}\, \mathbf{l}\, a$,
7. $\mathbf{f}\, a \parallel b = \mathbf{f}(a \parallel \mathbf{l}\, b)$,
8. + 4 other axioms not detailed here.

The axioms, the main result, and the method

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For all terms t, u of width at most 3 $\mathrm{Ax} \vdash t = u \Leftrightarrow \mathbf{g}\, t \simeq \mathbf{g}\, u$.

The axioms, the main result, and the method

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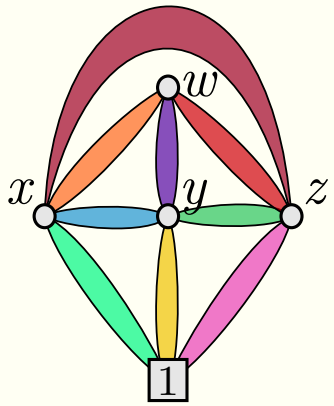
Theorem

For all terms t, u of width at most 3 $\mathrm{Ax} \vdash t = u \Leftrightarrow \mathbf{g}\, t \simeq \mathbf{g}\, u$.

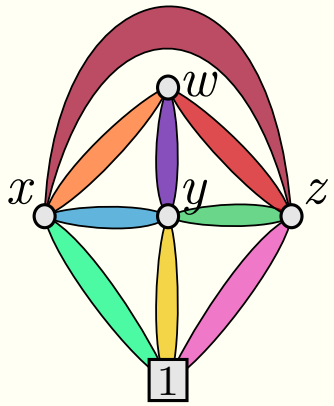
Proof idea: we use axioms to recursively decompose terms following connectivity structures adapted to our graphs.

The main difficulty: forget points non-determinism



- making w a source gives a graph of treewidth 4 (because we would have K_5),

The main difficulty: forget points non-determinism



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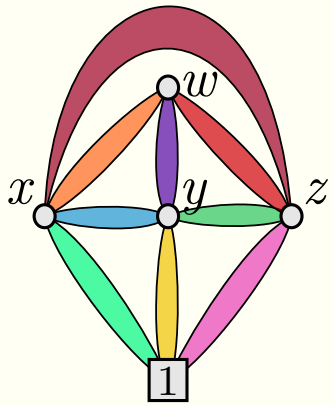
The main difficulty: forget points non-determinism



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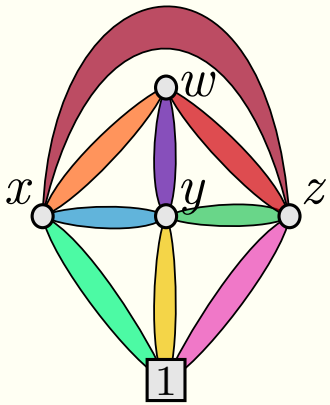
The main difficulty: forget points non-determinism



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x, y, and z (but not w) are called **forget points**.

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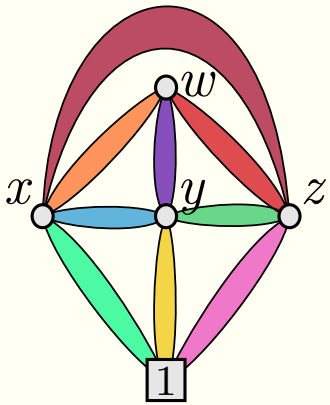

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The forget operation is associated to forget points. The main difficulty in the proof of the main result is handling the non-determinism of forget points

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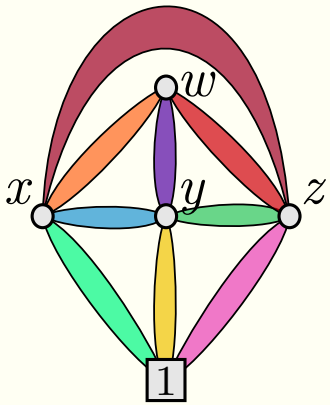

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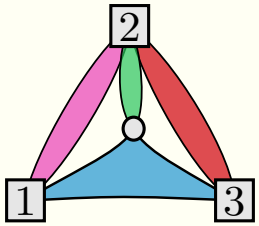
The forget operation is associated to forget points. The main difficulty in the proof of the main result is handling the non-determinism of forget points: a graph may have many of them as well as vertices that are not some.

Recursively decomposing graphs of treewidth at most 3

Decomposing graphs: prime decompositions

Consider the following graph:



Decomposing graphs: prime decompositions

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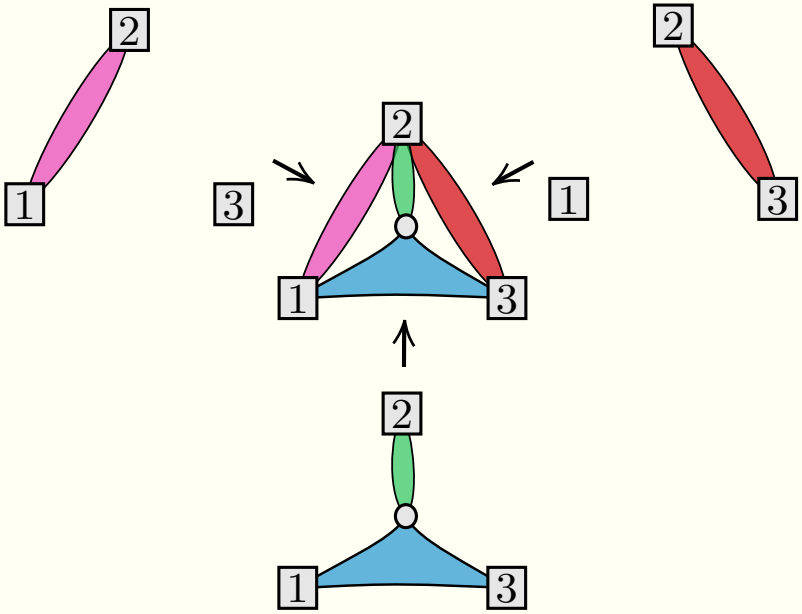

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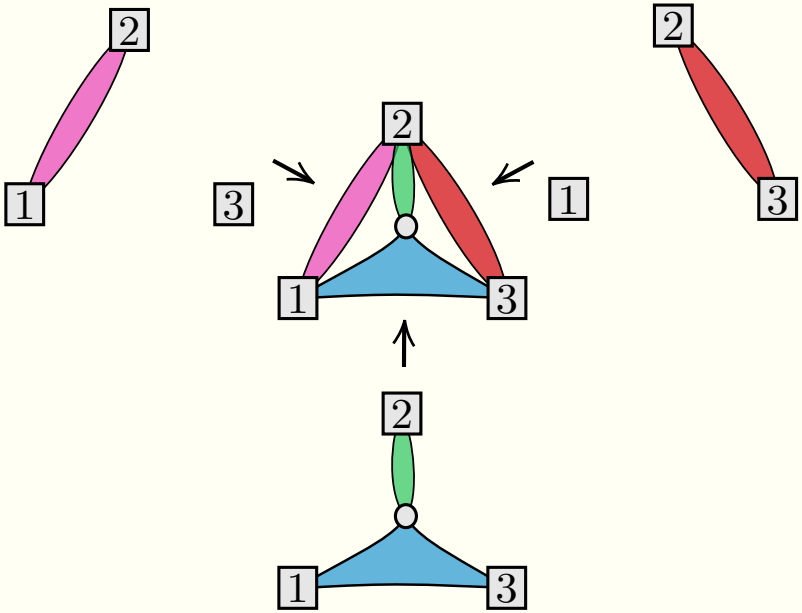
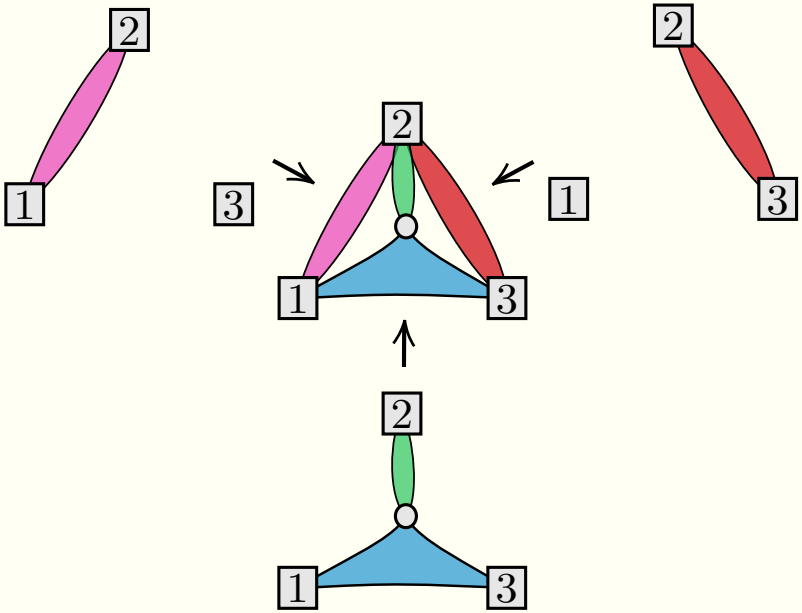


Basic simple graphs

- **Paths** between vertices

Decomposing graphs: prime decompositions

Consider the following graph:



Basic simple graphs

- **Paths** between vertices,
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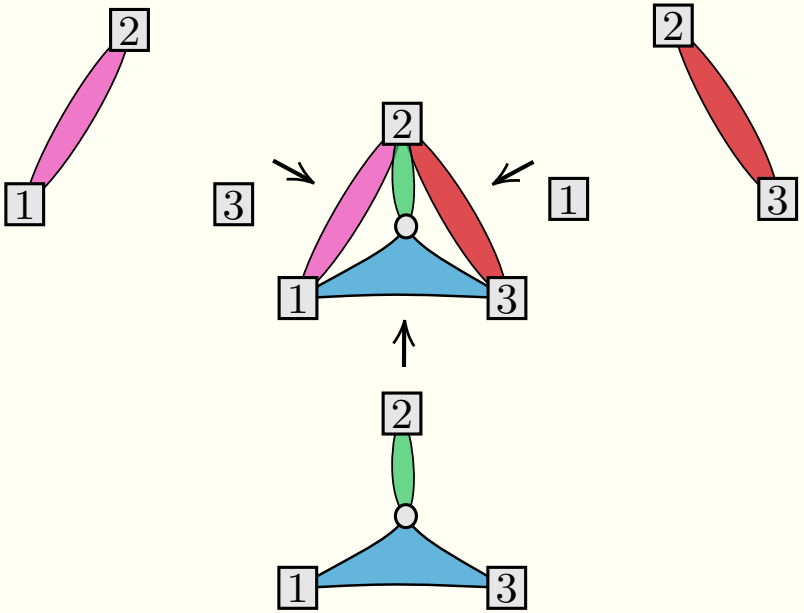

Basic simple graphs

- **Paths** between vertices,
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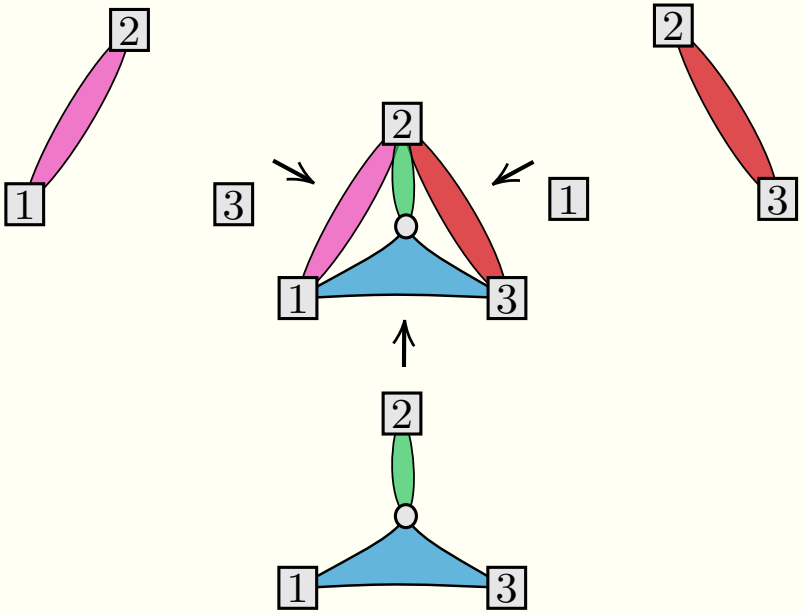
- **Proposition:** for a graph G, $G \simeq \uplus_i H_i$ with H_i its connected components.

Decomposing graphs: prime decompositions

Consider the following graph:



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Sourced graphs

- **Paths without sources**

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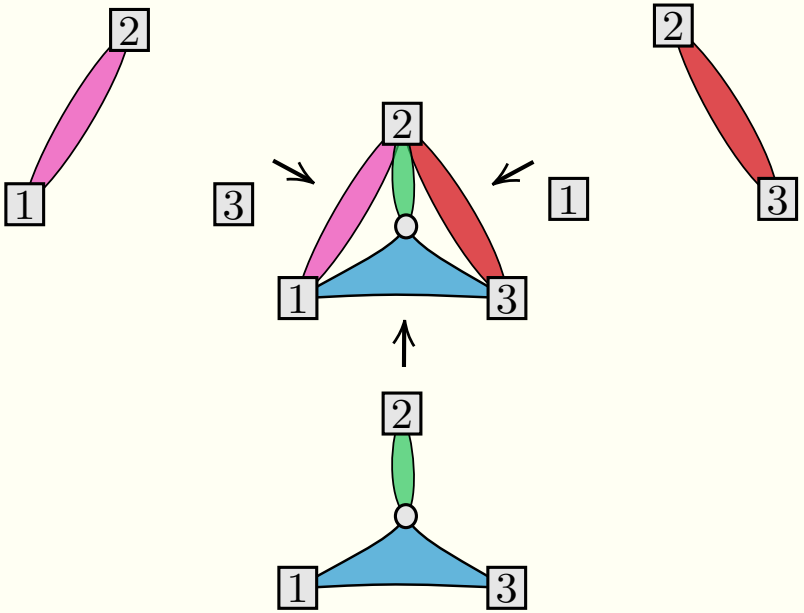

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Decomposing graphs: prime decompositions

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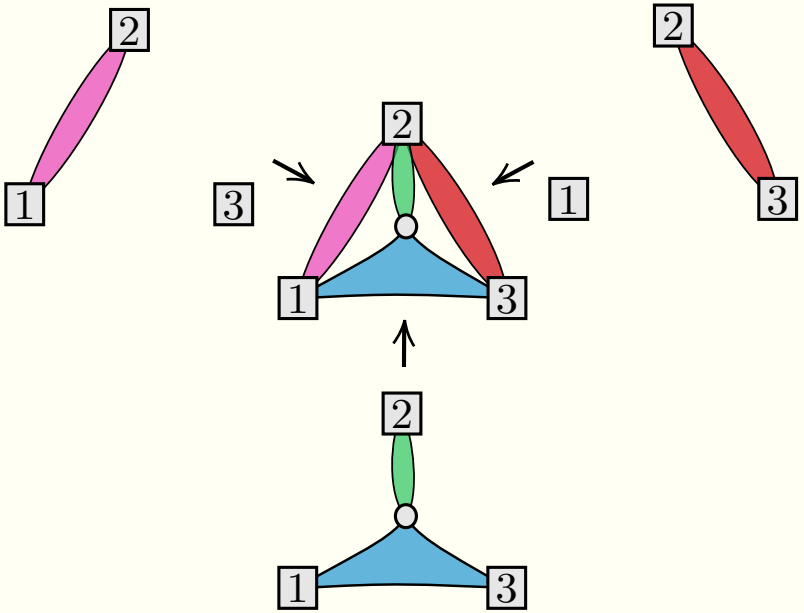

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Sourced graphs

- **Paths without sources**,
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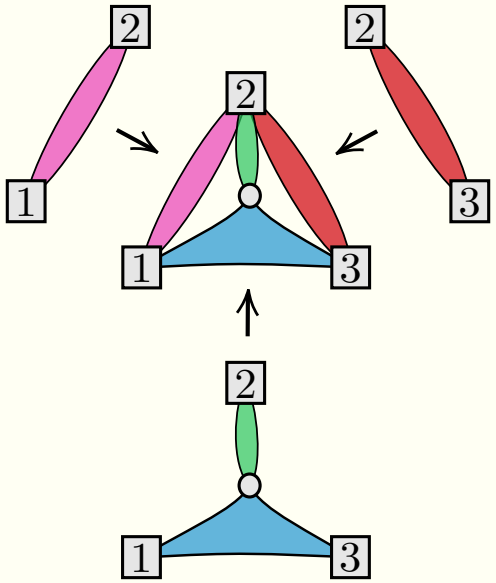
Decomposing graphs: full prime decompositions

Consider the following graph:



Basic simple graphs

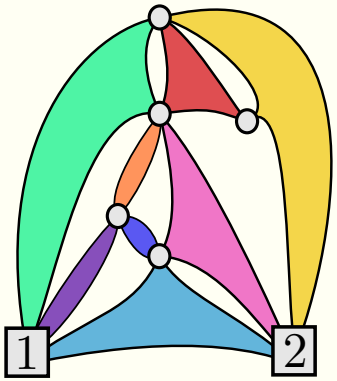
- **Paths** between vertices,
- **Connected components**
- **Proposition:** for a graph G, $G \simeq \uplus_i H_i$ with H_i its connected components.

Sourced graphs

- **Paths without sources**, except at endpoints
- **Full prime components**,
- **Proposition:** for a graph G, $G \simeq \|_i\, p_i \bullet^{j_i} H_i$ with H_i its full prime components

Decomposing graphs: anchors

Consider the following graph:



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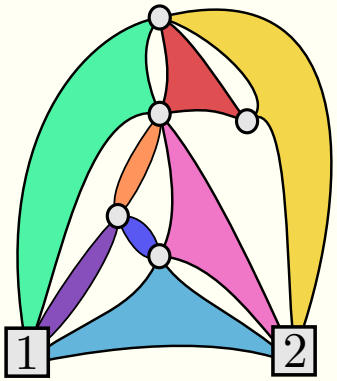

To decompose the graph further, we need to find some **forget points**, and do so as canonically as possible.

Decomposing graphs: anchors

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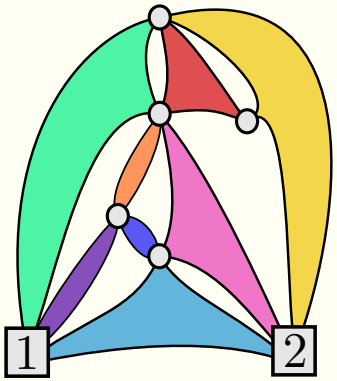

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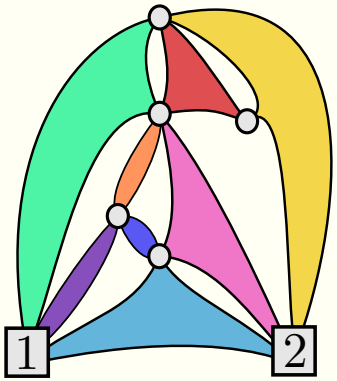

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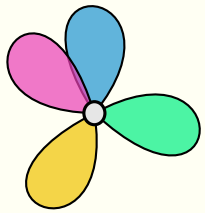
(petals are parts of the graphs)



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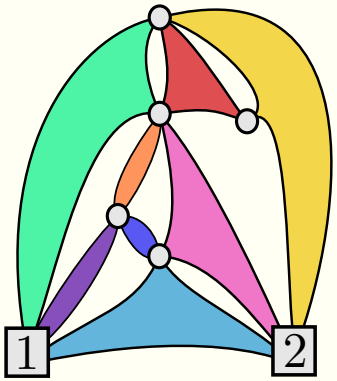

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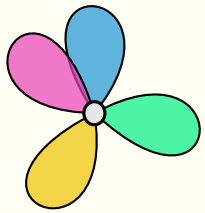

Sourced graphs:

Anchors:

Decomposing graphs: anchors

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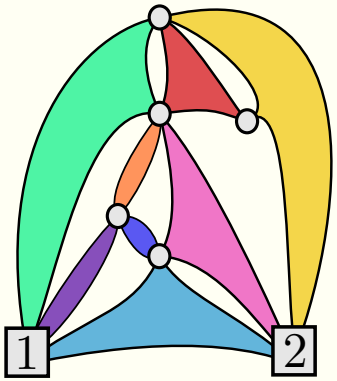

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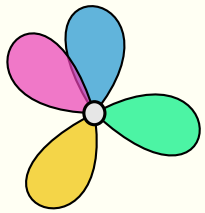
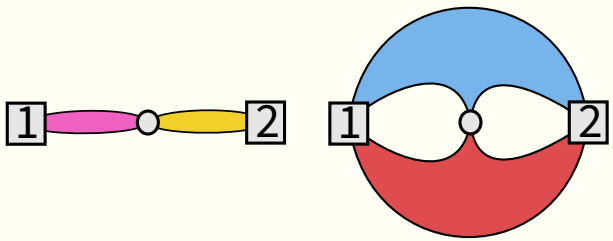


Sourced graphs:

Anchors:

(here for arity 2)



$+1$ case

Decomposing graphs: anchors

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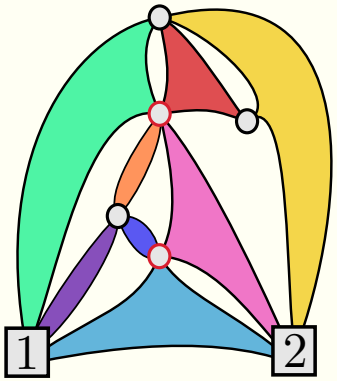

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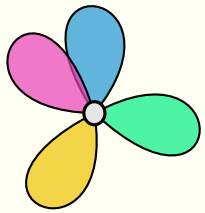

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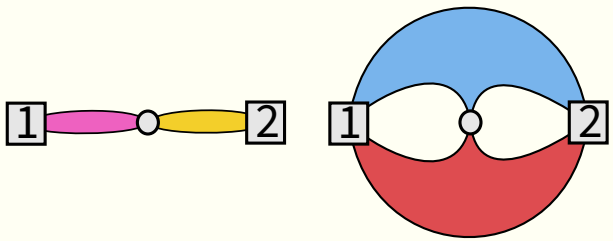

$+1$ case

Decomposing graphs: separation pairs

Hard graph: a full prime graph without anchors.

Decomposing graphs: separation pairs

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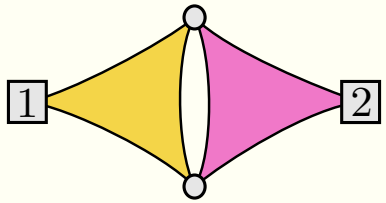
Most basic hard graph:



Decomposing graphs: separation pairs

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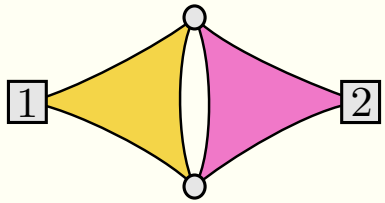
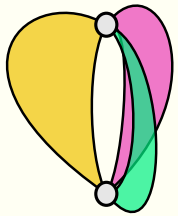


Basic simple graphs:
2-separators: pairs of vertices disconnecting the graph:



Decomposing graphs: separation pairs

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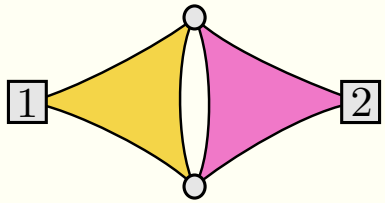

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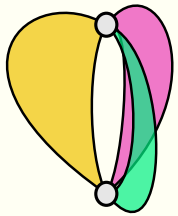
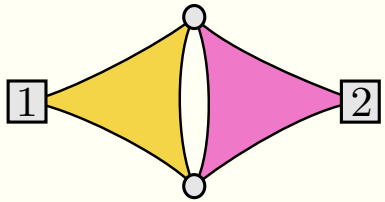


Sourced graphs:
Separation pair:



Decomposing graphs: separation pairs

Hard graph: a full prime graph without anchors.
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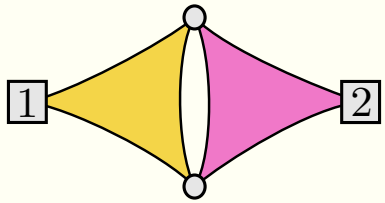

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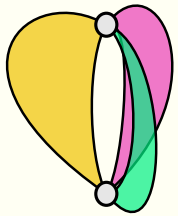

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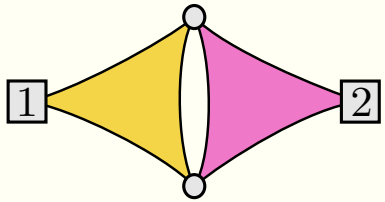


Proposition

Every hard graph of treewidth at most 3 has at least one separation pair consisting of forget points.

Classifying hard graphs of treewidth at most 3

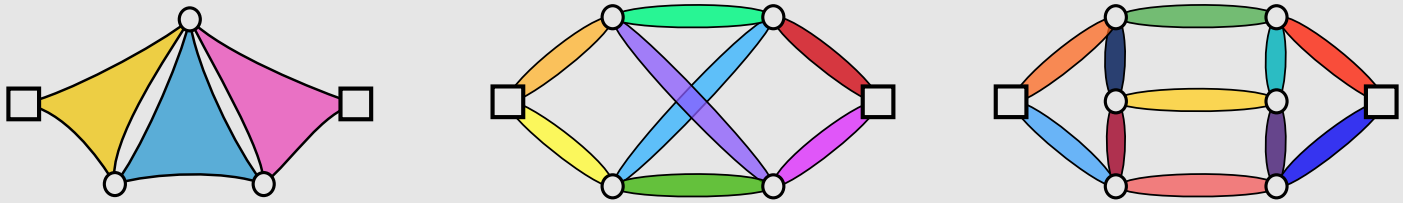
What happens if a hard graph has several separation pairs ?

Classifying hard graphs of treewidth at most 3

What happens if a hard graph has several separation pairs ?

Theorem

Whenever a hard graph has two distinct separation pairs whose vertices are forget points, then it must have one of the following shapes:



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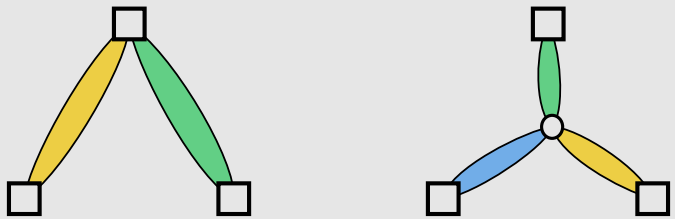
A graph with 3 sources either has the triangle K_3 as a minor on its sources, or it has one of the following shapes:



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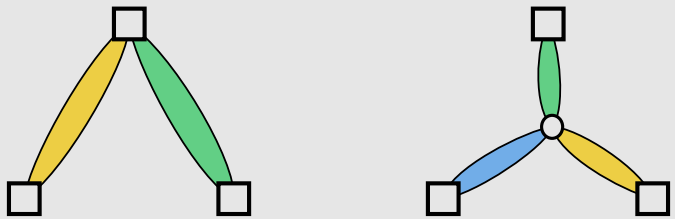

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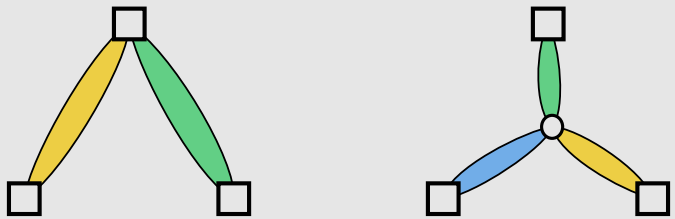

Graph minors: a way to see patterns in graphs, generalising the notion of subgraphs.
This is reminiscent of the proposition stating that a graph without a cycle (i.e. without K_3 as a minor) is a tree. Our lemma is stronger.

Conclusion

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- we can go up to 3-separators but no more,
- the number of analysed cases of the problem explodes,
- the main tool to reduce it here is the last lemma, which would need to be generalised for K_4 instead of K_3,
- but such a generalisation is not possible without getting a weaker statement.

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Two possible **algorithmic applications**: testing treewidth at most 3, solving the isomorphism problem for graphs of treewidth at most 3:

- no polynomial algorithms are known for these problems when treewidth is not fixed,
- linear algorithms are already known for both instances,
- but the structure our algorithms would follow is different from what exists already: we hope it can be used as heuristics for the problems on general graphs.

Summary

Reminders:

- graphs with sources, multi- and hyperedges,
- operations, terms, and treewidth,
- getting axioms for treewidth at most 3 ?
- Yes, proven by decomposing graphs following their connectivity structure (full prime decompositions, anchors, and separation pairs),
- two research lines: generalisation and algorithmic aspects (testing treewidth at most 3 and the isomorphism on graphs of treewidth at most 3).

Thank you for listening !