INTEGRAL MODELS OF SHIMURA VARIETIES OF PEL TYPE

SANDRA ROZENSZTAJN

1. PEL data for integral models

1.1. **Data from the previous chapter.** In the previous chapter, we considered a set of Shimura data of PEL type. That is we had:

(1) a finite semisimple $\mathbb{Q}$-algebra B, endowed with a positive involution $*$
(2) a finite dimensional B-module V, endowed with a non-degenerate bilinear alternating pairing $\langle \cdot, \cdot \rangle$.
(3) an $\mathbb{R}$-morphism $h : \mathbb{C} \to \mathrm{End}_B(V)_\mathbb{R}$ such that complex conjugation on $\mathbb{C}$ corresponds by h to the adjunction in $\mathrm{End}_B(V)_\mathbb{R}$ with respect to the pairing $\langle \cdot, \cdot \rangle$, and such that $(u, v) \mapsto \langle u, h(i)v \rangle$ is a symmetric definite positive pairing over $V_\mathbb{R}$.

Let G be the reductive group over $\mathbb{Q}$ defined by

$$G(R) = \{g \in \mathrm{GL}(V \otimes R), \exists \mu \in R^*, \forall x, y \in V \otimes R, \langle gx, gy \rangle = \mu \langle x, y \rangle\}$$

We can attach to h a morphism $\mu_h : \mathbb{C}^* \to G_\mathbb{C}$ that induces a decomposition $V_\mathbb{C} = V_0 \oplus V_1$, where $\mu_h(z)$ acts by z on V_1 and by 1 on V_0. The reflex field E of the Shimura data is then the subfield of $\overline{\mathbb{Q}}$ generated by the traces of the elements of B acting on V_0.

Let $\mathcal{X}$ be the $G(\mathbb{R})$-conjugation class of the morphism $\mathbb{C}^* \to G_\mathbb{R}$.

Then for each compact open subgroup K of $G(\mathbb{A}_f)$, we have a Shimura variety $\mathrm{Sh}(G, \mathcal{X})_K$ such that $\mathrm{Sh}(G, \mathcal{X})_K(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{X} \times (G(\mathbb{A}_f)/K)$.

The Shimura variety has a model over the reflex field E of the Shimura datum. This model can be constructed as a moduli space parameterizing abelian varieties with polarization, endomorphisms and level structure.

We now want to construct an integral model at p of the Shimura variety, that is a smooth model over the ring of integers $\mathcal{O}_{E_\nu}$ of the completion E_ν of E at some place ν of E over p.

We need some extra data and assumptions in order to ensure that this is possible, and in order to define this integral model as a moduli space of abelian varieties.

1.2. **New data.** Let $\mathcal{O}_B$ be a $\mathbb{Z}_{(p)}$-order in B that is stable under the involution $*$ of B and becomes maximal after tensorization with $\mathbb{Z}_p$.

We require additional conditions:

(1) B is unramified at p, that is $B_{\mathbb{Q}_p} = B \otimes_\mathbb{Q} \mathbb{Q}_p$ is isomorphic to a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$.
(2) there exists a $\mathbb{Z}_p$-lattice Λ in $V_{\mathbb{Q}_p}$ that is stable under $\mathcal{O}_B$, and such that the pairing $\langle \cdot, \cdot \rangle$ induces a perfect duality of Λ with itself.

We fix such a lattice Λ as part of the data although the construction of the integral model doesn't depend on the choice of Λ.

Example 1.1. Let B be an imaginary quadratic extension of $\mathbb{Q}$. Then condition (1) simply means that p doesn't ramify in B. We can choose for $\mathcal{O}_B$ the $\mathbb{Z}_{(p)}$-order generated by the ring of integers of B.

1.3. **The reductive group.** Let G be the reductive group attached to the Shimura datum. Because of the additional conditions, $G_{\mathbb{Q}_p}$ is unramified, as we can define a smooth reductive model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. We denote by C_0 the hyperspecial subgroup consisting of the $\mathbb{Z}_p$-points of this model. It is the subgroup of $G_{\mathbb{Q}_p}$ that stabilizes the lattice Λ.

2. Preliminaries

2.1. **Polarized abelian schemes with an action of $\mathcal{O}_B$.** Let S be a $\operatorname{spec} \mathcal{O}_{E_\nu}$-scheme.

2.1.1. *Definition.*

Definition 2.1. Let R be a subring of $\mathbb{Q}$. An R-isogeny between two abelian schemes A and A' is an isomorphism in the category where the objects are abelian schemes and the set of morphisms from A to A' is $\operatorname{Hom}(A, A') \otimes_{\mathbb{Z}} R$. An R-polarization of A is a polarization of A that is also an R-isogeny from A to the dual abelian scheme A^t.

Definition 2.2. We say that (A, λ, ι) is a $\mathbb{Z}_{(p)}$-polarized abelian scheme with an action of $\mathcal{O}_B$ if:
(1) A is an abelian scheme over S.
(2) λ is a $\mathbb{Z}_{(p)}$-polarization.
(3) ι is an injective ring homorphism $\mathcal{O}_B \to \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ which respects involutions on both sides: the involution $*$ on the left side, and the Rosati involution $\dagger$ coming from λ on the right side.

2.1.2. *Rigidity for endomorphisms of abelian schemes.*

Proposition 2.3. *Let S be a normal noetherian scheme, U a dense open subset of S. Let A and B be abelian schemes over S, and $f_U : A_U \to B_U$ a morphism of abelian schemes. Then f_U extends uniquely to a morphism $f : A \to B$ over S.*

Proof. A complete proof can be found in [FC90], prop 2.9 of chapter I. In the case where S is of dimension one, it follows from the fact that A and B are Néron models of A_U and B_U respectively. □

The proposition has the following corollary:

Corollary 2.4. *Let S be a normal noetherian scheme, and A an abelian scheme over S. Suppose that there exists an open dense subset U of S such that A_U has a structure of polarized abelian scheme with an action of $\mathcal{O}_B$. Then A is uniquely endowed with a structure of polarized abelian scheme with an action of $\mathcal{O}_B$ with the polarization extending the previous polarization.*

2.2. **The determinant condition of Kottwitz.** We now have to find a way to explain how $\mathcal{O}_B$ acts on the abelian scheme. More precisely we want to be able to express the fact that $\mathcal{O}_B$ acts on $\operatorname{Lie}(A)$ the same way it acts on V_0.

2.2.1. *The determinant condition for projective modules.* We fix once and for all a generating family $\alpha_1, \dots \alpha_t$ of $\mathcal{O}_B$ as a $\mathbb{Z}_{(p)}$-module.

Let R be an algebra over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and M be a finitely generated projective R-module. Suppose that $\mathcal{O}_B$ acts on M by R-linear endomorphisms. We then say that M is a R-module with an action of $\mathcal{O}_B$.

We consider the action of $\mathcal{O}_B[X_1, \dots X_t]$ on $M \otimes_R R[X_1, \dots X_t]$. We denote by $\det_M \in R[X_1, \dots X_t]$ the determinant of the element $X_1\alpha_1 + \dots X_t\alpha_t$ for this action. Here the ring R is understood.

It is clear that $\det_M$ is functorial in R : that is, if $f : R \to R'$ is a homomorphism of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebras, and $M' = M \otimes_R R'$, then $\mathcal{O}_B$ acts on M' by R'-linear endomorphisms and $\det_{M'} = f(\det_M)$.

A special case of such a module with an action of $\mathcal{O}_B$ is given by V_0 (see definition in §1.1). We have the following result:

Lemma 2.5. $\det_{V_0} \in \left(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\right)[X_1, \dots X_t]$.

Proof. By definition of the reflex field E, all the elements $\det(b; V_0)$ lie in E, so the coefficients of $\det_{V_0}$ are in E. Let F be a number field such that the action of B on V_0 is defined on F. Then the image of $\mathcal{O}_B$ in the algebra of $g \times g$ matrices over F is an $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$-order. Hence $\det(b, V_0)$ is in $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$ for all $b \in \mathcal{O}_B$. So the coefficients of $\det_{V_0}$ are integral over $\mathbb{Z}_{(p)}$. This proves the lemma. □

Lemma 2.6. *There exists a finite unramified extension K of E such that there exists a free $\mathcal{O}_K$-module L with an action of $\mathcal{O}_B$ and* $\det_L = \det_{V_0}$.

Lemma 2.7. *Let k be a field, and V and W be two finite-dimensional k-vector spaces with an action of $\mathcal{O}_B$. Then V and W are isomorphic if and only if* $\det_V = \det_W$.

Proof. Let us denote $\mathcal{O}_B \otimes_{\mathbb{Z}} k$ by A. Then A is a finite dimensional semisimple algebra over k. Indeed, if $\mathrm{char}(k) = 0$ then $A = B \otimes_{\mathbb{Q}} k$, and if $\mathrm{char}(k) = p$ then A is a product of matrix algebras over extensions of $\mathbb{F}_p$, as $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a maximal order of $B \otimes \mathbb{Q}_p$, which is itself a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$. Moreover, V and W are isomorphic as k-vector spaces with an action of $\mathcal{O}_B$ if and only if they are isomorphic as A-modules.

We write $A = A_1 \times \cdots \times A_n$ where the A_i are simple k-algebras. We consider V and W as A-modules. Then we have decompositions $V = V_1 \times \cdots \times V_n$ and $W = W_1 \times \cdots \times W_n$ where V_i and W_i are A_i-modules. As A_i is simple it has only one isomorphism class of irreducible representation. Hence V and W are isomorphic if and only if $\dim V_i = \dim W_i$ for all i, and it is clear that this information can be recovered from $\det_V$ and $\det_W$. □

Definition 2.8. If R is an $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra, and M an R-module with an action of $\mathcal{O}_B$, then we say that M satisfies the determinant condition if $\det_M$ equals the image of $\det_{V_0}$ in $R[X_1, \dots X_t]$.

We then show how the isomorphism class varies under specialization:

Lemma 2.9. *Suppose R is an $\mathcal{O}_{E_\nu}$-algebra that is a local ring with residue field k, and let M be a finitely generated projective R-module with an action of O_B. We denote $M \otimes_R k$ by M_0. Then M satisfies the determinant condition if and only if M_0 does.*

Proof. We need only prove that M satisfies the determinant condition when M_0 does. Suppose first that R is in fact an $\mathcal{O}_K$-algebra, for K as in lemma 2.6. Then M_0 is isomorphic to $L \otimes_{\mathcal{O}_K} k$ as a k-module with action of $\mathcal{O}_B$, as $\det_{M_0} = \det_{L \otimes_{\mathcal{O}_K} k}$ by lemma 2.7.

We see then that $L \otimes_{\mathcal{O}_K} R$ is a projective $\mathcal{O}_B \otimes R$-module. It is true if $R = K$, as then $\mathcal{O}_B \otimes R = B \otimes K$ is semisimple. Hence by base change it is also true when k is a field of characteristic 0. When $R = \mathcal{O}_K$, then $\mathcal{O}_B \otimes R$ is a product of matrix algebras over extensions of $\mathbb{Z}_p$. The result then holds by Morita equivalence, as L is torsion free. Hence by base change it also holds when k is a field of characteristic p.

So the $\mathcal{O}_B \otimes k$-linear isomorphism $L \otimes_{\mathcal{O}_K} k \to M_0$ lifts to an $\mathcal{O}_B \otimes R$-linear morphism $L \otimes_{\mathcal{O}_K} R \to M$. Now forget the action of $\mathcal{O}_B$. Nakayama's lemma implies that this morphism is an isomorphism.

When R is not an $\mathcal{O}_K$-algebra, we can deduce the result from the preceding case by considering the localizations at maximal ideals of $R \otimes_{\mathbb{Z}_p} \mathcal{O}_K$. $\square$

We conclude that $\det_M$ depends only on specializations:

Corollary 2.10. *Let R be an $\mathcal{O}_{E_\nu}$-algebra such that $\operatorname{spec} R$ is connected, and let M be a finitely generated projective R-module. Then M satisfies the determinant condition if and only if there exists a maximal ideal $\mathfrak{m}$ of R, with residue field k, such that $M \otimes_R k$ satisfies the determinant condition.*

2.2.2. *The determinant condition for abelian schemes with an action of $\mathcal{O}_B$.* Let (A, λ, ι) be a polarized abelian scheme with an action of $\mathcal{O}_B$ over the base scheme S. Then $\mathcal{O}_B$ acts on $\operatorname{Lie} A$, which is a locally free $\mathcal{O}_S$-module. For each open affine subset U of S, we can define $\det_{\operatorname{Lie} A}(U) \in \Gamma(U, \mathcal{O}_S)[X_1, \dots X_t]$ as in §2.2.1. By functoriality of the definition of det, these sections are compatible, hence glue to define a global section $\det_{\operatorname{Lie} A} \in \Gamma(S, \mathcal{O}_S)[X_1, \dots X_t]$. As $\Gamma(S, \mathcal{O}_S)$ is naturally an $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra it makes sense to compare $\det_{\operatorname{Lie} A}$ to the image of $\det_{V_0}$ in $\Gamma(S, \mathcal{O}_S)[X_1, \dots X_t]$. Following [Kot92], we set the following definition:

Definition 2.11. The triple (A, λ, ι) satisfies the determinant condition of Kottwitz if $\det_{\operatorname{Lie} A}$ is the image of $\det_{V_0}$.

One consequence of the definition is the following: the dimension of $\operatorname{Lie} A$ and hence that of A is equal to that of V_0.

2.2.3. *Some geometric properties of the determinant condition.* We state what geometric consequences we can deduce from proposition 2.9 and corollary 2.10:

Proposition 2.12. *Let S be an $\mathcal{O}_{E_\nu}$-scheme, and S_0 a closed subscheme of S with nilpotent definition ideal. Let (A, λ, ι) be an abelian scheme over S with an action of $\mathcal{O}_B$. Suppose that the base change of (A, λ, ι) to S_0 satisfies the determinant condition. Then so does (A, λ, ι).*

Proposition 2.13. *Let S be an $\mathcal{O}_{E_\nu}$-scheme, and (A, λ, ι) an abelian scheme over S with an action of $\mathcal{O}_B$. Then there is a closed subscheme T of S that is a union of connected components, such that for all closed point x of S, $(A_x, \lambda_x, \iota_x)$ satisfies the determinant condition if and only if x is a point of T.*

2.2.4. *The example of unitary groups.* In the case of unitary groups over $\mathbb{Q}$, we give a condition on $\operatorname{Lie} A$ that is equivalent to the determinant condition of Kottwitz and that is simpler to state.

Let B be a quadratic imaginary extension of $\mathbb{Q}$, and let τ be in B such that $\mathcal{O}_B = \mathbb{Z}[\tau]$. Fix a prime p that is unramified in B.

Let R be an $\mathcal{O}_{B,(p)}$-algebra, and M a locally free R-module with an action of $\mathcal{O}_B$. Then we have a decomposition $M = M^+ \oplus M^-$ where M^+ and M^- are also locally free. Here M^+ is defined as the submodule of M where the action of τ from the action of $\mathcal{O}_B$ and the action of the image of τ in R coincide, and M^- is the submodule where these action differ by conjugation in $\mathcal{O}_B$. When R is connected we can then define the type of M as the pair of integers $(\operatorname{rk} M^+, \operatorname{rk} M^-)$.

Then an abelian scheme A with an action of $\mathcal{O}_B$ over a base S that is an $\mathcal{O}_B$-scheme satisfies the determinant condition if and only if $\operatorname{Lie} A$ has the same type as the B-module V_0.

2.3. **Level structure.**

2.3.1. *Level subgroups.* We fix K^p a compact open subgroup of $G(\mathbf{A}_f^p)$. Here $\mathbf{A}_f^p$ denotes the ring of finite adeles of $\mathbb{Q}$ away from p.

Example 2.14. Fix a prime-to-p integer $N > 0$. We define a compact open subgroup of $G(\mathbf{A}_f^p)$ by: $K(N) = \{g \in G(\mathbf{A}_f^p), (g-1)(\Lambda \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(p)}) \subset N(\Lambda \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{(p)})\}$. This subgroup is the principal level subgroup of level N.

2.3.2. *Tate modules.* We denote $\Lambda \otimes \hat{\mathbb{Z}}^{(p)}$ by $\Lambda^{(p)}$, and $\Lambda \otimes \mathbf{A}_f^p$ by $V^{(p)}$.

Let (A, λ, ι) be a polarized abelian scheme with an action of $\mathcal{O}_B$, defined on a $\operatorname{spec} \mathcal{O}_{E_\nu}$-scheme. Let s be any geometric point of S, and consider the Tate modules: $T(A_s) = \varprojlim_N A_s[N]$, $T^{(p)}(A_s) = \varprojlim_{N \text{ prime to } p} A_s[N] = T(A_s) \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}^{(p)}$ and $V^{(p)}(A_s) = H_1(A_s, \mathbf{A}_f^p) = T^{(p)}(A_s) \otimes_{\hat{\mathbb{Z}}^{(p)}} \mathbf{A}_f^p$.

They are endowed with a non-degenerate bilinear form, coming from the polarization λ, and an action of $\mathcal{O}_B$, coming from the action of $\mathcal{O}_B$ on A itself.

Let f be a separable isogeny from A to A' with kernel C. Then f induces a morphism $T(f) : T(A_s) \to T(A'_s)$ which is injective with cokernel isomorphic to C_s. If f is of prime-to-p degree then f induces $T^{(p)}(f) : T^{(p)}(A_s) \to T^{(p)}(A'_s)$ which is injective with cokernel C. We also have an isomorphism $V^{(p)}(f) : V^{(p)}(A_s) \to V^{(p)}(A'_s)$.

Suppose f is a separable R-isogeny for some subring R of $\mathbb{Q}$. Then $V(f)$ is still well-defined but f doesn't necessarily map $T(A_s)$ into $T(A'_s)$. In fact f maps $T(A_s)$ into $T(A'_s)$ if and only if f is an isogeny in the usual sense.

2.3.3. *Definition of the level structures.* Let S be a $\operatorname{spec} \mathcal{O}_{E_\nu}$-scheme, and s a geometric point of S. Let (A, λ, ι) be a polarized abelian scheme with an action of B. We say that a map η^p from $V^{(p)}$ to $V^{(p)}(A_s)$ respects the structures on both sides if it respects the bilinear forms up to a scalar in $(\mathbf{A}_f^p)^*$, and if it is compatible with the $\mathcal{O}_B$-action on both sides.

Let g be in $G(\mathbf{A}_f^p)$. If η^p respects the structures on both sides then so does $\eta^p \circ g$. Hence $G(\mathbf{A}_f^p)$ acts on the set of such maps.

Definition 2.15. Let K^p be a compact open subgroup of $G(\mathbf{A}_f^p)$. A level structure of level K^p on (A, λ, ι) is a choice of a geometric point s for each connected component of S, and for each s a choice of a K^p-orbit $\overline{\eta}^p$ of morphisms $\eta^p : V^{(p)} \to V^{(p)}(A_s)$ respecting the structures on both sides and such that the orbit is fixed under the action of $\pi_1(s, S)$.

Remark 2.16. The last condition ensures that a level structure is in fact independant of the choice of s. Moreover, a level structure exists at some point s if and only if for any geometric point s' in the same connected component as s there exists a level structure at s'.

3. The integral model as a moduli scheme

3.1. **Definition of the moduli problem.** Let us fix a set of PEL data $(B, \Lambda, *)$ as in section 1. We also fix a compact open subgroup K^p of $G(\mathbf{A}_f^p)$. We will define a moduli problem classifying abelian schemes with an action of $\mathcal{O}_B$ and K^p level structure.

Definition 3.1. Let $\mathcal{F}_{K^p}$ be the following category fibered in groupoids over the category $(Sch/\operatorname{spec} \mathcal{O}_{E_\nu})$ of $\operatorname{spec} \mathcal{O}_{E_\nu}$-schemes:

- The objects over a scheme S are quadruples $\underline{A} = (A, \lambda, \iota; \overline{\eta}^p)$, where (A, λ, ι) is a $\mathbb{Z}_{(p)}$-polarized projective abelian scheme over S with an action of $\mathcal{O}_B$ which respects the determinant condition of Kottwitz (definition 2.11 of

§2.2), and $\overline{\eta}^p$ is a level structure of level K^p over each connected component of S.

- The morphisms from $\underline{A} = (A, \lambda, \iota; \overline{\eta}^p)$ to $\underline{A'} = (A', \lambda', \iota'; \overline{\eta}'^p)$ over S are given by a $\mathbb{Z}_{(p)}$-isogeny $f : A \to A'$ compatible with the action of $\mathcal{O}_B$ and the level structures, that is:
  (1) there exists a locally constant function r with values in $\mathbb{Z}_{(p)}^*$ such that $\lambda = r(f^t \circ \lambda' \circ f)$.
  (2) f induces a morphism from $\mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ to $\mathrm{End}\,(A') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, that we still denote by f; then for all $b \in \mathcal{O}_B$, $f \circ \iota(b) = \iota'(b)$.
  (3) $\overline{\eta}'^p = V^{(p)}(f) \circ \overline{\eta}^p$, where we denote by $V^{(p)}(f)$ the morphism induced from $V^{(p)}(A_s)$ to $V^{(p)}(A'_s)$.

3.2. **Known results about the moduli problem.**

Theorem 3.2. *$\mathcal{F}_{K^p}$ is a smooth Deligne-Mumford stack and it is representable by a quasi-projective scheme when K^p is small enough.*

The part of the theorem concerning representability will be proved in section 4, where we will also explain what "small enough" means. The part about smoothness will be proved in section 5.

When $\mathcal{F}_{K^p}$ is representable by a scheme, we denote this scheme by S_{K^p}.

Theorem 3.3. *The family of schemes S_{K^p}, for K^p the small enough compact open subgroups of $G(\mathbf{A}_f^p)$, form a tower of schemes with finite smooth transition morphisms. The group $G(\mathbf{A}_f^p)$ acts on the tower via Hecke operators.*

This is the object of section 6.

The generic fibre of the schemes we have constructed are isomorphic (except for some special cases) to the Shimura varieties S_K in characteristic zero that were defined in the preceding chapter, when $K = K^p C_0$. Moreover this is compatible with the action of Hecke operators. More precisely we will show in section 7:

Theorem 3.4. *When the algebra B has no factor of type D we have the following isomorphism for each compact open subgroup K^p of $G(\mathbf{A}_f^p)$:*

$$S_{K^p} \otimes_{\mathcal{O}_{E_\nu}} E_\nu \xrightarrow{\sim} S_K \otimes_E E_\nu$$

These isomorphisms are compatible with the action of $G(\mathbf{A}_f^p)$ on both sides.

4. Representability of the moduli problem

4.1. **Statement of the theorem.**

Theorem 4.1. *For all level subgroups K, $\mathcal{F}_K$ is a Deligne-Mumford stack. Moreover if K is small enough so that it is contained in a principal level subgroup of level $N \geq 3$, the functor $\mathcal{F}_K$ is representable by a quasi-projective scheme over $\mathcal{O}_{E_\nu}$.*

We will prove this theorem by comparing our moduli problem to the case of the Siegel modular varieties, that is the case where the endomorphism ring is trivial, which is already known by the results of [MFK94]. This is the proof outlined in [Kot92]. Another strategy to study the representability of the moduli problem would be via Artin's criterion. This has the advantage of being more direct and not to rely on the difficult results of [MFK94], but it has the drawback that it only shows that the moduli problem is representable by an algebraic space when the level is small enough. To prove that it is in fact representable by a quasi-projective scheme one then has to use the theory of compactifications of integral models of Shimura varieties. A detailed proof using this strategy can be found in [Lan08].

4.2. **The Siegel case.**

4.2.1. *The result of* [MFK94]. We first study the case where $B = \mathbb{Q}$. Then the group G is the symplectic group and the scheme we obtain is the Siegel moduli space of abelian varieties. This case is studied in [MFK94]. We describe the result.

Fix an integer $g \geq 1$ and an integer $N \geq 3$.

Definition 4.2. Let $\mathcal{F}$ be the category fibered in groupoids on $\mathbb{Z}[1/N]$-schemes such that: For any $\mathbb{Z}[1/N]$-scheme S, the set of objects of $\mathcal{F}_S$ is the set of triples $\underline{A} = (A, \lambda; \alpha)$ where A in a projective abelian scheme of dimension g, λ is a principal polarization, and α is a symplectic similitude (with multiplicator in $(\mathbb{Z}/N\mathbb{Z})^*$) between $(\mathbb{Z}/N\mathbb{Z})^2_S$ and $A[N]_S$. Here we consider $(\mathbb{Z}/N\mathbb{Z})^2$ to be endowed with the standard symplectic form. If $\underline{A}$, $\underline{A'}$ are objects of $\mathcal{F}_S$, then the morphisms from $\underline{A}$ to $\underline{A'}$ are the isomorphisms $f : A \to A'$ such that $\lambda = f^t \circ \lambda' \circ f$ and $\alpha' = f \circ \alpha$.

The main result is the following ([MFK94], theorem 7.9, see also [MB85], theorem 3.2 of chapter VII):

Proposition 4.3. *The category fibered in groupoids $\mathcal{F}$ is representable by a (smooth) quasi-projective scheme.*

4.2.2. *Reformulation of the moduli problem.* In order to compare more easily our situation to that studied in [MFK94] we give another formulation of the moduli problem. We endow $\mathbb{Q}^{2g} = V$ with the standard symplectic form, so that $\Lambda = \mathbb{Z}^{2g}_{(p)}$ is a self-dual lattice. Then the reductive group G is the symplectic group Gsp_{2g}. Let $K = K(N)$ be the principal level subgroup of level N in the symplectic group for some N prime to p.

Let $\mathcal{F}'$ be the category fibered in groupoids as in definition 3.1, with this set of Shimura data.

Proposition 4.4. *When N is prime to p, $\mathcal{F}'$ and the restriction of $\mathcal{F}$ to the subcategory of $\mathbb{Z}_p$-schemes are isomorphic.*

Proof. Let S be a $(\operatorname{spec} \mathbb{Z}_p)$-scheme, and let $\mathcal{F}_S$ and $\mathcal{F}'_S$ be the categories of objects of $\mathcal{F}$ and $\mathcal{F}'$ over S.

Let $(A, \lambda; \alpha)$ be an object of $\mathcal{F}_S$, and choose for each connected component of S a geometric point s. The morphism $\alpha : (\mathbb{Z}/N\mathbb{Z})^2_S \to A[N]_S$ gives an isomorphism $\alpha_s : (\mathbb{Z}/N\mathbb{Z})^2 = \Lambda \otimes (\mathbb{Z}/N\mathbb{Z}) \to A[N]_s$ that is invariant under the action of $\pi_1(s, S)$. This morphism α_s then extends to an isomorphism of symplectic modules η^p : $\Lambda \otimes \hat{\mathbb{Z}}^{(p)} \to T^p(A_s)$. This morphism isn't unique, but its $K(N)$-conjugation class is uniquely determined by α_s, and hence invariant under the action of $\pi_1(s, S)$. Hence the orbit $\overline{\eta}^p$ of $\eta^p : V^{(p)} \to V^{(p)}(A_s)$ defines a level structure $\overline{\eta}^p$ extending α. Hence to each $(A, \lambda; \alpha)$ we can attach $(A, \lambda; \overline{\eta}^p)$ which is an object of $\mathcal{F}'_S$. Morphisms in $\mathcal{F}_S$ also define morphisms in $\mathcal{F}'_S$.

Let $\underline{A} = (A, \lambda; \alpha)$ and $\underline{A'} = (A', \lambda'; \alpha')$ be objects in $\mathcal{F}_S$. Let $f : A \to A'$ be a $\mathbb{Z}_{(p)}$-isogeny which is a morphism in the category $\mathcal{F}'_S$. Then f is an isomorphism, as by construction of the level structures f induces an isomorphism from $T^{(p)}(A)$ to $T^{(p)}(A')$. Moreover we necessarily have $\lambda' = f^t \circ \lambda \circ f$. Hence f is a morphism in the category $\mathcal{F}_S$.

The last point is to see that any object in $\mathcal{F}'_S$ is isomorphic to an object coming from $\mathcal{F}_S$. Let $\underline{A} = (A, \lambda; \overline{\eta}^p)$ be an object of $\mathcal{F}'_S$. We need to find $\underline{A'} = (A', \lambda', \overline{\eta}^{p\prime})$ and a morphism $\underline{A} \to \underline{A'}$ in $\mathcal{F}'_S$ such that λ' is a principal polarization and $\overline{\eta}'^p$: $V^{(p)} \to V^{(p)}(A'_s)$ induces a symplectic similitude between $(\mathbb{Z}/N\mathbb{Z})^2_S$ and $A[N]_S$.

We need only find $\underline{A'}$ such that $\overline{\eta}'^p$ induces an isomorphism between $\Lambda^{(p)}$ and $T^{(p)}(A'_s)$. Indeed such a level structure then induces a $\pi_1(s, S)$-invariant symplectic isomorphism $\alpha_s : (\mathbb{Z}/N\mathbb{Z})^2 \to A[N]_s$ that gives us the isomorphism $\alpha : (\mathbb{Z}/N\mathbb{Z})^2_S \to$

$A[N]_S$. Moreover we can change the polarization λ' to make it principal. We know that the bilinear forms on $\Lambda^{(p)}$ and $T^{(p)}(A'_s)$ differ by a scalar a in $(\mathbf{A}_f^p)^*$. By multipliying λ' by some prime-to-p integer, we can suppose that a is in $(\mathbf{A}_f^p)^* \cap \hat{\mathbb{Z}}^{(p)}$. Let ℓ be a prime not divising p. We know that λ' is divisible by ℓ^n if and only if the pairing on $T_\ell A_s$ is divisible by ℓ^n. Hence we can divide λ' by some integer n prime to p so that $a \in \hat{\mathbb{Z}}^{(p)*}$. But then the new polarization induces an isomorphism $T^{(p)}(A'_s) \to T^{(p)}({A'_s}^t)$ and so is principal.

Fix a submodule $M \subset V^{(p)}(A_s)$ that is invariant under the action of $\pi_1(s, S)$. Suppose that $T^{(p)}(A_s) \subset M$ with finite index. Then there exists an étale subgroup scheme $C \subset A$ with C_s isomorphic to $M/T^{(p)}(A_s)$, such that the isogeny $f : A \to A' = A/C$ induces an isomorphism from M to $T^{(p)}(A'_s)$. There is some prime-to-p integer n such that (nf^{-1}) is an isogeny from A' to A, then we can endow the abelian scheme A' with a polarization $\lambda' = (nf^{-1})^t \circ \lambda \circ (nf^{-1})$. We can define a level structure on (A', λ') by $\overline{\eta}'^p = f \circ \overline{\eta}^p$. Then f defines a morphism in $\mathcal{F}'_S$ from $(A, \lambda; \overline{\eta}^p)$ to $(A', \lambda'; \overline{\eta}'^p)$. Moreover $\overline{\eta}'^p$ has the property that $\overline{\eta}'^p(\Lambda^{(p)}) = T^{(p)}(A')$.

In the case where $\overline{\eta}(\Lambda^{(p)}) \subset T^{(p)}(A)$ we can also find a triple $(A', \lambda'; \overline{\eta}')$ and an isogeny $f : A' \to A$ that induces a morphism between $(A', \lambda'; \overline{\eta}'^p)$ and $(A, \lambda; \overline{\eta}^p)$ such that moreover $\overline{\eta}'^p(\Lambda^{(p)}) = T^{(p)}(A')$. The existence of f comes from the interpretation of the Tate module as a π_1. Then we define λ' as $f^t \circ \lambda \circ f$ and $\overline{\eta}'^p$ by the condition that $\overline{\eta}^p = f \circ \overline{\eta}'^p$.

By combining the two preceding cases we see that we can always find such an $(A', \lambda'; \overline{\eta'}^p)$. □

4.3. Reduction to the case of principal level structures.

Proposition 4.5. *If $\mathcal{F}_K$ is representable by a scheme whenever K is a principal level subgroup of level $N \geq 3$ then theorem 4.1 is true.*

Proof. We first show that the condition implies that $\mathcal{F}_K$ is a Deligne-Mumford stack for any level subgroup K. Indeed, let K' be a principal level subgroup of level $N \geq 3$ contained in K. Then we have a functor $\mathcal{F}_{K'} \to \mathcal{F}_K$ that sends the object $(A, \lambda, \iota; \overline{\eta}^p)/S$ to the object $(A, \lambda, \iota; \tilde{\overline{\eta}}^p)/S$ where $\tilde{\overline{\eta}}^p$ is the K-orbit generated by $\overline{\eta}^p$ (see also section 6). This functor makes the scheme $\mathcal{F}_{K'}$ an étale presentation of the stack $\mathcal{F}_K$. Hence $\mathcal{F}_K$ is a Deligne-Mumford stack (see [LMB00], proposition 4.3.1).

We then observe that whenever $K \subset K(N)$ for an $N \geq 3$, then $\mathcal{F}_K$ is representable by an algebraic space. This follows from lemma 4.6 below and [LMB00], corollary 8.1.1: a Deligne-Mumford stack where the objects have only the trivial automorphism is representable by an algebraic space. But then we have a finite morphism $\mathcal{F}_K \to \mathcal{F}_{K(N)}$, which is hence schematic, so $\mathcal{F}_K$ is representable by a scheme as $\mathcal{F}_{K(N)}$ is. □

We used the following rigidity lemma:

Lemma 4.6. *Let K be a level subgroup contained in a principal level subgroup $K(N)$ with $N \geq 3$. Then for any scheme S over $\mathcal{O}_{E_\nu}$ and any object $\underline{A}$ of $\mathcal{F}_{KS}$, $\underline{A}$ has only the trivial automorphism.*

Proof. It follows from the fact that an automorphism of a polarized abelian variety over an algebraically closed field that acts as the identity on the N-torsion subgroup for some $N \geq 3$ is the identity automorphism (see [Ser], or [Mil86] in the book [CS86] for a proof). □

4.4. **The general case.** We fix an integer $N \geq 3$, N prime to p, and we consider in what follows the level subgroup $K = K(N)$.

We have a natural transformation from $\mathcal{F}_B$ to $\mathcal{F}_\mathbb{Q}$, which is defined by sending the quadruple $(A, \lambda, \iota; \overline{\eta})$ over the S-scheme T to $(A, \lambda; \tilde{\overline{\eta}})$. Here $\tilde{\overline{\eta}}$ is the $\mathrm{GSp}_{2g}(N)$-orbit generated by $\overline{\eta}$.

We have to prove that the functor $\mathcal{F}_B$ is relatively representable over $\mathcal{F}_\mathbb{Q}$, and that $\mathcal{F}_B$ is projective over $\mathcal{F}_\mathbb{Q}$. More precisely:

Proposition 4.7. *Let K be small enough so that $\mathcal{F}_\mathbb{Q}$ is representable by a scheme. Then $\mathcal{F}_B$ is relatively representable over $\mathcal{F}_\mathbb{Q}$ by a scheme that is projective over $\mathcal{F}_\mathbb{Q}$.*

4.4.1. *A scheme over $\mathcal{F}_\mathbb{Q}$.* In this section we fix a $\operatorname{spec} \mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$-scheme S, an abelian scheme A over S, a $\mathbb{Z}_{(p)}$-polarization λ of A, and an isomorphism of étale sheaves $\phi_N : \Lambda/N\Lambda \xrightarrow{\sim} A[N]$ that respects the alternating forms on both sides up to a constant.

We will need the following result:

Lemma 4.8. *Let S be a locally noetherian scheme, and A a projective abelian scheme over S. Then the functor from S-schemes to sets that attaches $\mathrm{End}(A_T)$ to T is representable by a union of projective schemes over S. We will denote by $\mathcal{E}$ the scheme representing the functor $T \mapsto \mathrm{End}(A_T) \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$. It is also a union of projective schemes over S.*

This follows from the theory of Hilbert schemes, as an endomorphism of A_T is a special case of a subscheme of $A \times_T A$. A detailed proof of this lemma can be found in [Hid04], section 6.1.

In our special case, the abelian scheme A is endowed with a prime-to-p polarization λ. So $\mathcal{E}$ naturally comes with an involution r, which is the Rosati involution. Let $m = 2n$ and $a_1, \dots a_m$ be a set of generators of $\mathcal{O}_B$ as a $\mathbb{Z}_{(p)}$-algebra with $a_{n+i} = a_i^*$. We define a closed subscheme Z of $\mathcal{E}^m$: let T be an S-scheme, and $(x_1, \dots x_m) \in \mathcal{E}^m(T)$. Then $(x_1, \dots x_m)$ is in Z if and only if any relationship verified by $(a_1, \dots a_m)$ is also verified by $(x_1, \dots x_m)$ and $r(x_i) = x_{n+i}$.

The abelian scheme A_Z is endowed with an algebra homomorphism $\mathcal{O}_B \to \mathrm{End}(A_Z) \otimes_\mathbb{Z} \mathbb{Z}_{(p)}$, which is compatible with the Rosati involution. That is, A_Z is a polarized abelian scheme with an action of $\mathcal{O}_B$ as in 2.1.

We know thanks to proposition 2.13 that the locus where the $\mathcal{O}_B$-action on A_Z satisfies the determinant condition is a union of connected components of Z. We also have an isomorphism of étale sheaves $\phi_N : (\Lambda/N\Lambda)_Z \xrightarrow{\sim} A_Z[N]$. Then the locus where this isomorphism extends to an $\mathcal{O}_B$ level N structure is a union of connected components of Z, as follows from remark 2.16. Moreover there is at most one $K(N)$-orbit of such liftings for each connected component. We denote by X_B the union of the connected components of Z where the determinant condition holds and the isomorphism ϕ lifts to a level structure.

Let A_{X_B} the abelian scheme over X_B coming from A. As follows from the construction of X_B, we have:

Lemma 4.9. *The abelian scheme A_{X_B} is naturally endowed with a structure of a polarized abelian scheme with $K(N)$-level.*

4.4.2. *Comparing $\mathcal{F}_B$ to $\mathcal{F}_\mathbb{Q}$.* We now show the relative representability of $\mathcal{F}_B$ over $\mathcal{F}_\mathbb{Q}$ when $\mathcal{F}_\mathbb{Q}$ is representable by a scheme. We fix a scheme S, and a morphism $S \to \mathcal{F}_\mathbb{Q}$, and consider the functor $\mathcal{F}' = \mathcal{F}_B \times_{\mathcal{F}_\mathbb{Q}} S$. We have to show that $\mathcal{F}'$ is representable by a scheme.

The given morphism $S \to \mathcal{F}_\mathbb{Q}$ amounts to an equivalence class of triples $(\mathcal{A}, \lambda; \overline{\eta})$ where $\mathcal{A}$ is an abelian variety over S, endowed with a prime-to-p polarization λ, and a level structure $\overline{\eta}$. We choose a representant of this equivalence class. We can then construct a scheme X_B over S as in §4.4.1.

We then define a natural transformation $\mathcal{F}' \to X_B$. Let T be an S-scheme. An element of $\mathcal{F}'(T)$ is an equivalence class of quadruples $(A, \lambda, \iota; \overline{\eta})$, such that its image by the forgetful functor $\mathcal{F}_B \to \mathcal{F}_{\mathbb{Q}}$ is in the same equivalence class as $(\mathcal{A}, \lambda; \overline{\eta})_T$. That is, there is a prime-to-p isogeny $f : A \to \mathcal{A}_T$, compatible with the polarizations and the level structures. Then f induces an isomorphism between $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ and $\mathrm{End}(\mathcal{A}_T)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. We use this isomorphism to define a morphism $\iota : \mathcal{O}_B \to \mathrm{End}(\mathcal{A}_T) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Hence we get a point in $X_B(T)$.

We have to show that this construction is well-defined, that is, it doesn't depend on the choice of $(A, \lambda, \iota; \overline{\eta})$ in the equivalence class. But this comes from the fact that any element of $(\mathcal{A}, \lambda; \overline{\eta})_T$ has no non-trivial automorphism, as we have chosen the level such that $\mathcal{F}_{\mathbb{Q}}$ is representable by a scheme.

Lemma 4.10. *This natural transformation is an isomorphism.*

Proof. We only have to find a natural transformation $X_B \to \mathcal{F}'$ that is a quasi-inverse to the transformation we have just defined. But this is lemma 4.9. □

Hence $\mathcal{F}_B$ is representable by the scheme X_B. The connected components of X_B are projective over the scheme representing $\mathcal{F}_{\mathbb{Q}}$, which is itself quasi-projective over S. To finish the proof of proposition 4.7, we only have to show that X_B has only a finite number of connected components. But this comes from the fact that $\mathcal{F}_B$ is locally of finite presentation over $\operatorname{spec} \mathcal{O}_{E_\nu}$, as can be seen using the criterion of Proposition 4.15 of [LMB00].

In some cases we know a little more about X_B:

Proposition 4.11. *Suppose that G is globally anisotropic. Then X_B is projective over* $\operatorname{spec} \mathcal{O}_{E_\nu}$.

5. Smoothness

Theorem 5.1. *$\mathcal{F}_{K^p}$ is a smooth Deligne-Mumford stack. When K^p is small enough so that S_{K^p} is a scheme, then it is a smooth scheme.*

We need only prove this when K^p is small enough so that S_{K^p} is a scheme, as the transition morphisms between the S_{K^p} with varying level subgroups are étale. As S_{K^p} is locally of finite presentation, we only have to prove that S_{K^p} is formally smooth, that is:

Proposition 5.2. *Let R be an $\mathcal{O}_{E_\nu}$-algebra. Let $S_0 = \operatorname{spec} R_0$ and $S = \operatorname{spec} R$ such that $R_0 = R/I$ with $I^2 = 0$. If $(A_0, \lambda_0, \iota_0; \overline{\eta}_0)$ on S_0 satisfies the determinant condition of Kottwitz, then it lifts to a $(A, \lambda, \iota; \overline{\eta})$ on S that also satisfies the determinant condition.*

5.1. First reductions.

Let us first take care of the level structure:

Lemma 5.3. *If $(A_0, \lambda_0, \iota_0)$ lifts to (A, λ, ι) then any level structure $\overline{\eta}_0$ on $(A_0, \lambda_0, \iota_0)$ lifts to a level structure $\overline{\eta}$ on (A, λ, ι).*

Lifting $\overline{\eta}$ amounts to lifting some sections of $A_0[N]$ to sections of $A[N]$, for a family of integers N prime to p. $A[N]$ being étale over S, this is automatic.

We now take the determinant condition out of the picture: if $(A_0, \lambda_0, \iota_0)$ on S_0 satisfying the determinant condition of Kottwitz lifts to (A, λ, ι) on S, then the lift automatically satisfies the determinant condition, thanks to proposition 2.12.

Moreover we also know the following result, which is a consequence of the "rigidity lemma" (theorem 6.1 of [MFK94]) :

Lemma 5.4. *Let A and B be two abelian schemes over S, then the restriction* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A_0, B_0)$ *is injective.*

From this we can deduce that if λ_0 and ι_0 both extend to a lifting of A, then the compatibility condition between involutions is automatically satisfied.

5.2. **The theory of Grothendieck-Messing.** As we already know that the generic fiber of the moduli space is smooth, we can assume that p is nilpotent on S. Hence we can use the theory of Grothendieck-Messing to study the problem of lifting A_0. Let us recall the part of the theory relevant to the situation. The complete constructions and proofs can be found in [Mes72].

There is a functor from the category of abelian schemes over S_0 to the category of locally free sheaves on S associating to an abelian scheme A_0/S_0 the evaluation of the Dieudonné crystal $\mathbb{D}(A_0)$ on the inclusion $S_0 \to S$, that we will denote by $\mathbb{D}(A_0)_S$. For any abelian variety A/S lifting A_0, $\mathbb{D}(A_0)_S$ is canonically isomorphic to $\mathcal{H}^1_{DR}(A/S)$.

In the case where A_0 is a polarized abelian scheme with an action of $\mathcal{O}_B$, $\mathbb{D}(A_0)_S$ also has an action of $\mathcal{O}_B$. Moreover the polarization induces a morphism $\mathbb{D}(A_0)_S \to \mathbb{D}(A_0^t)_S = \mathbb{D}(A_0)_S^*$, which is an isomorphism because the polarization is separable, and which is compatible with the action of $\mathcal{O}_B$ on both sides. Hence the polarization induces a non-degenerate alternating form on $\mathbb{D}(A_0)_S$ that is skew-hermitian with respect to $\mathcal{O}_B$.

A submodule of $\mathbb{D}(A_0)_S$ is said to be admissible if it is locally a direct factor, and reduces to $(\mathrm{Lie}\, A_0)^*$ on S_0.

Theorem 5.5 (Grothendieck-Messing)**.** *There is an equivalence of categories between the category of abelian schemes over S and the category of pairs (A_0, F), where A_0 is an abelian scheme over S_0 and F an admissible submodule of $\mathbb{D}(A_0)_S$, given by $A \mapsto (A_{|S_0}, (\mathrm{Lie}\, A)^*)$.*

In order for the lifting A of A_0 to be polarized with an action of $\mathcal{O}_B$, it is enough that $(\mathrm{Lie}\, A)^*$ is an $\mathcal{O}_B$-stable totally isotropic submodule of $\mathbb{D}(A_0)_S$.

We are then reduced to the following linear algebra problem: Let M be a projective module of rank $2g$ over R with an action of $\mathcal{O}_B$ and a non-degenerate alternating form that is skew-hermitian with respect to $\mathcal{O}_B$. Let $M_0 = M \otimes_R R_0$, and let $N_0 \subset M_0$ be a locally direct factor submodule of M_0 of rank g stable under the action of $\mathcal{O}_B$ and totally isotropic for the alternating form. Find a lifting of N_0 to a submodule N of M that has the same properties.

The way to find such a submodule differ depending of the type of the group G. Details can be found in [LR87] and [Zin82]. We will only treat a simple example: the case of unitary groups over $\mathbb{Q}$.

5.3. **An example: unitary groups over** $\mathbb{Q}$**.** Let B be an imaginary quadratic extension of $\mathbb{Q}$, with involution the complex conjugation, and suppose that the prime p is split in B. Then $A = \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is $\mathbb{Z}_p \times \mathbb{Z}_p$ and the involution exchanges the factors. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then $e_i M$ is totally isotropic for $i = 1, 2$ as $e_1^* = e_2$. Moreover a submodule Q of an A-module is A-stable if and only if $Q = e_1 Q \oplus e_2 Q$.

We can further simplify the problem: Let A_0 be the universal abelian scheme over S_{K^p}. We know that $\mathcal{H}^1_{DR}(A_0/S)$ and $(\mathrm{Lie}\, A_0)^*$ are locally free modules on S_{K^p}. As smoothness is a local question on S_{K^p}, we can assume that the exact sequence $0 \to (\mathrm{Lie}\, A_0)^* \to \mathcal{H}^1_{DR}(A_0/S) \to \mathrm{Lie}(A_0^t) \to 0$ is split and that these modules are in fact free, This amounts to assuming that we have a decomposition $M_0 = N_0 \oplus P_0$, with N_0 and P_0 free. This implies that M is a free R-module with basis any lifting of a basis of M_0. Let us denote $e_i N_0$ by $N_{0,i}$ for $i = 1, 2$. Then the

$N_{0,i}$ are projective. We can also assume that they are free, by the same reasoning as before.

Let us choose a basis of M_0 consisting of the union of a basis of $N_{0,i}$, $i = 1, 2$, and a basis of P_0. We can lift the basis of $N_{0,i}$ to a family in $e_i M$, which gives us free liftings $N_i \subset e_i M$ of $N_{0,i}$. They are totally isotropic, but not necessarily orthogonal. As the bilinear form is non-degenerate, we can modify the lifting of the basis of $N_{0,2}$ such that N_2 is orthogonal to N_1, and still $N_1 \subset e_1 M$. Then $N = N_1 \oplus N_2$ is the lifting of N_0 we were looking for. Indeed, N is A-stable, totally isotropic, projective (even free), and M/N is projective, as it is isomorphic to the submodule P of M generated by any lifting of the chosen basis of P_0.

6. Hecke operators

We explain here the relation between the Shimura varieties when the level varies and the action of the Hecke operators.

6.1. **The tower of Shimura varieties.** Let K and K' be compact open subgroups of $G(\mathbf{A}_f^p)$, such that $K \subset K'$. Then we have a natural morphism from $\mathcal{F}_K$ to $\mathcal{F}_{K'}$ which sends a quadruple $(A, \lambda, \iota; \overline{\eta})$ over the base S to the quadruple $(A, \lambda, \iota; \overline{\eta}')$, where $\overline{\eta}'$ is the K'-orbit generated by η. Hence we have a morphism of moduli schemes $S_K \to S_{K'}$. As in the characteristic zero case, we then have a whole tower of integral models $(S_{K^p})_{K^p}$.

If K is a normal subgroup of K', then $S_K \to S_{K'}$ is an étale Galois covering of Galois group K'/K. More generally, for all $K \subset K'$ compact open subgroups of $G(\mathbf{A}_f^p)$, the morphism $S_K \to S_{K'}$ is finite étale and surjective.

The tower is smooth in the following sense: each of the schemes is smooth for K^p small enough, and the maps in the tower are also smooth.

6.2. **Action of the Hecke operators.** We also have Hecke operators: the group $G(\mathbf{A}_f^p)$ acts on the tower via its action on the level structure. That is: for each $g \in G(\mathbf{A}_f^p)$, g maps $\mathcal{F}_K \to \mathcal{F}_{g^{-1}Kg}$ by sending $(A, \lambda, \iota; \overline{\eta})$ to $(A, \lambda, \iota; \overline{\eta} \circ g)$.

7. Relation to the generic fiber

We will now see how the scheme S_{K^p} relates to the Shimura variety $\mathrm{Sh}(G, X)_K(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{X} \times G(\mathbf{A}_f)/K$ and to its canonical model. We first recall the construction of the canonical model.

7.1. **Modular definition of the canonical model.** Let K be a compact open subgroup of $G(\mathbf{A}_f)$, and let $\mathrm{Sh}(G, X)_K(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{X} \times G(\mathbf{A}_f)/K$.

We can obtain a canonical model of this Shimura variety via a moduli space, as follows :

Definition 7.1. Let $\mathcal{F}'$ be the following category fibered in groupoids over the category $(Sch/\operatorname{spec} E)$ of $(\operatorname{spec} E)$-schemes:

- The objects over a scheme S are quadruples $\underline{A} = (A, \lambda, \iota; \overline{\eta})$, where (A, λ, ι) is a polarized projective abelian scheme over S with an action of $\mathcal{O}_B$ which respects the determinant condition of Kottwitz (definition 2.11 of §2.2), and $\overline{\eta}$ is a level structure of level K over each connected component of S, that is, a K orbit of isomorphism between $V \otimes \mathbf{A}_f$ and $H_1(A_s, \mathbf{A}_f)$, for s a geometric point of S.
- The morphisms from $\underline{A}$ to $\underline{A'}$ over S are given by a $\mathbb{Q}$-isogeny $f : A \to A'$ compatible with the action of $\mathcal{O}_B$ and the level structures, that is:
  1. there exists a locally constant function r with values in $\mathbb{Q}^*$ such that $\lambda = r(f^t \circ \lambda' \circ f)$.

(2) f induces a morphism from $\mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ to $\mathrm{End}\,(A') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, that we still denote by f; then for all $b \in \mathcal{O}_B$, $f \circ \iota(b) = \iota'(b)$.

(3) $\overline{\eta}' = V(f) \circ \overline{\eta}$, where we denote by $V(f)$ the morphism induced from $V(A_s)$ to $V(A'_s)$.

The functor $\mathcal{F}'$ is representable by a scheme S_K when K is small enough. Then S_K is a disjoint union of canonical models over E of the Shimura variety $\mathrm{Sh}(G, X)_K$. More precisely, as is explained in [Kot92], §8:

Proposition 7.2.

$$S_K = \sqcup_{\ker^1(\mathbb{Q},G)} Sh(G', \mathcal{X})_K$$

where $\ker^1(\mathbb{Q}, G)$ *is the set of locally trivial elements of* $H^1(\mathbb{Q}, G)$ *and parametrizes the interior forms* G' *of* G *that are locally isomorphic to* G *at every place.*

The failure of the Hasse principle is essentially harmless, as follows from the study of $\ker^1(\mathbb{Q}, G)$ in [Kot92], §7:

Proposition 7.3. *When* G *is of type C, or of type A with even* n, $\ker^1(\mathbb{Q}, G)$ *is trivial. When* G *is of type A with odd* n, *all the groups* G' *are isomorphic to* G.

We refer to [Kot92], §5 for the classification of the group G in types A, C, D. Let us just recall that unitary groups are of type A and the symplectic group is of type C.

In particular, under the hypotheses of the proposition, all the connected components of S_K are in fact isomorphic to the canonical model of the Shimura variety $\mathrm{Sh}(G, \mathcal{X})_K$.

7.2. **Relationship to the integral model.** Let C_0 be the maximal compact open subgroup of $G(\mathbf{A}_f)$ at p. If K^p is a compact open subgroup of $G(\mathbf{A}_f^p)$, then $K = K^p C_0$ is a compact open subgroup of $G(\mathbf{A}_f)$.

Theorem 7.4. *We have then the following isomorphism when* G *has no factor of type D:*

$$S_{K^p} \otimes_{\mathcal{O}_{E_\nu}} E_\nu \xrightarrow{\sim} S_K \otimes_E E_\nu$$

This isomorphisms are compatible with the action of $G(\mathbf{A}_f^p)$ *on both sides.*

It follows from this result that the generic fiber of the integral model S_{K^p} is a union of copies of the canonical model of the Shimura variety $\mathrm{Sh}(G, \mathcal{X})_K$.

We denote by $\mathcal{F}$ the category we introduced in definition 3.1 in order to define the moduli problem for the integral model of the Shimura variety. We write $\mathcal{F}'_{|E_\nu}$ and $\mathcal{F}_{|E_\nu}$ respectively for the restrictions of $\mathcal{F}'$ and $\mathcal{F}$ to the set of $(\mathrm{spec}\, E_\nu)$-schemes. Hence if $\mathcal{F}'$ is representable by the E-scheme S_K then $\mathcal{F}'_{|E_\nu}$ is representable by $S_K \otimes_E E_\nu$ and likewise for $\mathcal{F}_{|E_\nu}$.

We will prove the following proposition, which implies theorem 7.4:

Proposition 7.5. *When the algebra* B *has no factor of type D, the categories fibered in groupoids* $\mathcal{F}'_{|E_\nu}$ *and* $\mathcal{F}_{|E_\nu}$ *are isomorphic and the isomorphism is compatible with the action of the Hecke operators on both sides.*

Proof. Let S be a $(\mathrm{spec}\, E_\nu)$-scheme. Let us explain how to define an equivalence of categories from $\mathcal{F}_{|E_\nu S}$ to $\mathcal{F}'_{|E_\nu S}$. Let $\underline{A} = (A, \lambda, \iota; \overline{\eta}^p)$ an object of $\mathcal{F}_{|E_\nu S}$. The problem is in the definition of $\overline{\eta}'$: we already have a K^p-orbit of isomorphisms $\overline{\eta}$ between $V \otimes \mathbf{A}_f^p$ and $H_1(A_s, \mathbf{A}_f^p)$ and we have to extend it to the whole of $\mathbf{A}_f$. That is, we have to find a C_0-orbit of isomorphisms between $V \otimes \mathbb{Z}_p$ and $H_1(A_s, \mathbb{Z}_p)$.

Observe that V and $H_1(A_s, \mathbb{Q})$ are isomorphic B-modules, as they become so after tensorization by $\mathbb{Q}_\ell$ for any $\ell \neq p$ (this follows from the existence of the level structure outside p). Then $V \otimes \mathbb{Q}_p$ and $H_1(A_s, \mathbb{Q}_p)$ are isomorphic as B-modules. Moreover both have self-dual $\mathcal{O}_B$-lattices. Now we use the condition on the algebra B: as it has no factor of type D, we know by [Kot92], lemma 7.2 that the lattices $\Lambda \otimes \mathbb{Z}_p$ and $H_1(A_s, \mathbb{Z}_p)$ are isomorphic as hermitian modules with an action of $\mathcal{O}_B$. Moreover the C_0-orbit of isomorphism is then well-defined independently of choices. Hence we can uniquely extend the level structure $\overline{\eta}^p$ to $\overline{\eta}$. □

References

[CS86] Gary Cornell and Joseph H. Silverman, editors. *Arithmetic geometry*. Springer-Verlag, New York, 1986. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[Hid04] Haruzo Hida. *p-adic automorphic forms on Shimura varieties*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004.

[Kot92] Robert E. Kottwitz. Points on some Shimura varieties over finite fields. *J. Amer. Math. Soc.*, 5(2):373–444, 1992.

[Lan08] Kai-Wen Lan. *Arithmetic compactifications of PEL-type Shimura varieties*. PhD thesis, Harvard University, 2008.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[LR87] R. P. Langlands and M. Rapoport. Shimuravarietäten und Gerben. *J. Reine Angew. Math.*, 378:113–220, 1987.

[MB85] Laurent Moret-Bailly. Pinceaux de variétés abéliennes. *Astérisque*, (129):266, 1985.

[Mes72] William Messing. *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin, 1972.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Mil86] J. S. Milne. Abelian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York, 1986.

[Ser] Jean-Pierre Serre. Rigidité du foncteur de jacobi d'échelon $n \geq 3$. Appendice à l'exposé 17 du séminaire Cartan 1960–1961.

[Zin82] Thomas Zink. Über die schlechte Reduktion einiger Shimuramannigfaltigkeiten. *Compositio Math.*, 45(1):15–107, 1982.

UMPA, ÉNS de Lyon, UMR 5669 du CNRS, 46, allée d'Italie, 69364 Lyon Cedex 07, France

E-mail address: sandra.rozensztajn@ens-lyon.fr