From Wiener-Khinchin theorem

$$< x(\tau)x(0) >= C(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\exp(-i\omega\tau)\ d\omega \tag{1}$$

thus

$$\partial_\tau C(\tau) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega\ S(\omega)\exp(-i\omega\tau)\ d\omega \tag{2}$$

and

$$\int_{-\infty}^{\infty} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau = -i\omega\ S(\omega) \tag{3}$$

As a consequence we may write

$$S(\omega) = -\frac{1}{i\omega}\int_{-\infty}^{\infty} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau \tag{4}$$

$$= \frac{i}{\omega}\left[\int_{-\infty}^{0} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau + \int_{0}^{\infty} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau\right] \tag{5}$$

Let us consider $\int_{-\infty}^{0} \partial_\tau C(\tau)\exp(-i\omega\tau)\ d\tau$ and change $\tau \to -\tau'$

$$\int_{-\infty}^{0} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau = \int_{\infty}^{0} \partial_{\tau'} C(-\tau')\exp(-i\omega\tau')\ d\tau' \tag{6}$$

$$= -\int_{0}^{\infty} \partial_{\tau'} C(-\tau')\exp(-i\omega\tau')\ d\tau' \tag{7}$$

$$= -\int_{0}^{\infty} \partial_{\tau'} C(\tau')\exp(-i\omega\tau')\ d\tau' \tag{8}$$

$$= -\left[\int_{0}^{\infty} \partial_{\tau'} C(\tau')\exp(i\omega\tau')\ d\tau'\right]^* \tag{9}$$

where we used $C(-\tau') = C(\tau')$ and $[\cdot]^*$ stands for complex conjugate
Thus replacing this result in eq.5

$$S(\omega) = \frac{i}{\omega}\left[-\left(\int_{0}^{\infty} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau\right)^* + \int_{0}^{\infty} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau\right] \tag{10}$$

$$= \frac{-2}{\omega}\ \ Imag\left[\int_{0}^{\infty} \partial_\tau C(\tau)\exp(i\omega\tau)\ d\tau\right] \tag{11}$$

Now we know that FDT imposes that for $\tau > 0$

$$\partial_\tau C(\tau) = -K_B\ TR(\tau)$$

thus

$$S(\omega) = \frac{2K_BT}{\omega}\ \ Imag\left[\int_{0}^{\infty} R(\tau)\exp(i\omega\tau)\ d\tau\right] \tag{12}$$

$$= \frac{2K_BT}{\omega}\ \ Imag\left[R(\omega)\right] \tag{13}$$

Furthermore as $S(\omega) = S(-\omega)$ then

$$
\begin{aligned}
< x^2 > &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega && (14) \\
&= \frac{1}{2\pi} \int_{0}^{\infty} 2S(\omega) d\omega && (15) \\
&= \int_{0}^{\infty} S_x(f) df && (16)
\end{aligned}
$$

where we have defined $S_x(f) = 2S(\omega)$ and $2\pi f = \omega$. Finally we get

$$
S_x(f) = \frac{4K_B T}{\omega} \quad Imag\left[R(\omega)\right] \qquad (17)
$$