Shock reflection in gas dynamics

Denis Serre
École Normale Supérieure de Lyon*

À ma Mère

Abstract

This paper is about multi-dimensional shocks and their interactions. The latter take place either between two shocks or between a shock and a boundary. Our ultimate goal is the analysis of the reflection of a shock wave along a ramp, and then at a wedge. Various models may be considered, from the full Euler equations of a compressible fluid, to the Unsteady Transonic Small Disturbance (UTSD) equation. The reflection at a wedge displays a self-similar pattern that may be viewed as a two-dimensional Riemann problem. Most of mathematical problems remain open. Regular Reflection is the simplest situation and is well-understood along an infinite ramp.

More complicated reflections occur when the strength of the incident shock increases and/or the angle between the material boundary and the shock front becomes large. This is the realm of Mach Reflection. Mach Reflection involves a so-called triple shock pattern, where typically the reflection of the incident shock detaches from the boundary, and a secondary shock, the Mach stem, ties the interaction point to the wall. The triple shock pattern is *pure* if it is made only of the incident, reflected and secondary shocks, but of no other wave. As predicted by J. von Neumann, pure triple shock structures are impossible. A common belief was that this impossibility is of thermodynamical nature. We prove here that the obstruction is of kinematical nature, thus is independent either of an equation of state or of an admissibility condition. This holds true for all situations: Euler models, irrotational flows and UTSD, the latter case having been known for a decade.

Because the Regular Reflection problem along a wedge gathers several major technical difficulties (a free boundary, a domain singularity, a solution singularity, a mixed-type system of PDEs, a type degeneracy across the sonic line), its solvability is still far from our knowledge, except in the simplest context of potential flows with small incidence, for which G.-Q. Chen & M. Feldman announced recently a solution. Good though partial results have been obtained by S. Čanić & al. for the UTSD model and by Y. Zheng for the Euler system.

As far as the Euler equations are concerned, we improve and derive with higher mathematical rigour our *pointwise* estimates of 1994. Our improvements concern most of the estimates:

*UMPA (UMR 5669 CNRS), ENS de Lyon, 46, allée d'Italie, F–69364 Lyon, cedex 07, FRANCE. The research of the author was partially supported by the European IHP project "HYKE", contract # HPRN–CT–2002-00282.

- We give a now rigorous proof of the minimum principle for the pressure,
- Our new bound of the size of the subsonic domain applies now to data of arbitrary strength and incidence,
- This together with the observation that the entropy increases, yields much better pointwise estimates of field variables,
- We prove that there must exist a vortical singularity, at least in the barotropic case: the vorticity of the flow may not be square integrable,
- Last but not least, we give a rigorous justification that the flow is uniform between the ramp, the pseudo-sonic line and the reflected shock, the latter being straight.

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Definitions

Let $d = 2$ or 3 be the dimension of the physical space. Although $d = 3$ is relevant for the real world, many phenomena are two-dimensional at the leading order and we shall often assume that $d = 2$. We denote by t the time variable and by $x = (x_1, ..., x_d) \in \mathbb{R}^d$ the space variable.

Gases in thermodynamical equilibrium are described by a velocity field $u \in \mathbb{R}^d$ and scalar internal variables, namely the *mass density* ρ, the *specific internal energy* e, the *pressure* p, the *temperature* T, the *entropy* S and a few others. These are functions of t and x. Thermodynamics tells us that the internal variables are not fully independent but are determined by two of them. It is classical to make the choice of (ρ, e) as the primary variables, but other choices are useful in some questions, as (p, T) or (p, τ) where $\tau := 1/\rho$ denotes the *specific volume*. The way ρ and e determine the other quantities proceeds from an *equation of state*

$$p = P(\rho, e), \tag{1}$$

where the function P describes the gas under consideration. A *perfect gas*, also called an *ideal gas*[1], obeys $p = (\gamma - 1)\rho e$, where the *adiabatic constant* $\gamma > 1$ equals $1 + 2/N$, N being the number of freedom degrees (translational, rotational, vibrational,...) of a molecule. For instance, $\gamma = 5/3$ for a monatomic gas (Helium, Argon,...) and $\gamma = 7/5$ for a di-atomic gas (hydrogen, oxygen, nitrogen,...). At moderate pressure and temperature, air may be considered as a perfect diatomic gas. The temperature and entropy are then obtained through the identity between differentials

$$T\mathbf{d}S = \mathbf{d}e + p\mathbf{d}\frac{1}{\rho}. \tag{2}$$

The evolution of the field $U := (\rho, u, e)$ obeys the Navier–Stokes equations for compressible fluids. In the absence of external forces, they consist in the conservation laws of:

- mass,
- momentum,
- energy.

For this reason, we shall call the densities of mass ρ, of momentum ρu and of mechanical energy $\frac{1}{2}\rho|u|^2 + \rho e$ the *conserved variables* of the system.

In suitable regimes, one may neglect the shear stress (Newton viscosity) and heat diffusion (Fourier law). Then the system reduces to first order in space and time and is called the *full system of Euler equations*:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{3}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \tag{4}$$

$$\partial_t \left(\frac{1}{2}\rho|u|^2 + \rho e\right) + \operatorname{div}\left(\left(\frac{1}{2}\rho|u|^2 + \rho e + p\right) u\right) = 0. \tag{5}$$

[1]Some people prefer saying that a gas is perfect if $p\tau = RT$ where R is a constant. Then it is ideal if moreover e is proportional to T.

As for general first-order systems of conservation laws, the Euler system is scale-invariant, in that if $(x,t) \mapsto U$ solves (3,4,5) and if λ is a positive real number, then $U^\lambda(x,t) := U(\lambda x, \lambda t)$ is a solution too. Together with the translation invariance and the Galilean invariance $(\rho, u, e, x, t) \mapsto (\rho, u - \bar{u}, e, x + t\bar{u}, t)$ ($\bar{u}$ a constant vector), this allows for a rather large set of explicit simple solutions. At the simplest level are all the constant fields. Next, there are simple centered waves, which are purely one-dimensional: planar rarefaction wave, contact discontinuities and shock waves. Beginners will find thorough descriptions of these waves in Dafermos' book [29] or in the Handbook article by G.-Q. Chen & D. Wang [22]. In this paper, we investigate truly two-dimensional simple patterns:

- Planar shocks reflecting along a planar wall,
- Planar shocks reflecting at a wedge.

These problems amount to solving a system of nonlinear partial differential equations, together with boundary conditions. The flow is *steady* in the first case and *pseudo-steady* in the second one. The latter terminology means that the governing PDEs are the same, up to lower order derivatives, as in the former case. We warn the reader however, that in pseudo-steady flows, the relevant unknown is $(\rho, u - \frac{x}{t}, e)$ instead of (ρ, u, e). The field $u - x/t$ is called the *pseudo-velocity*.

When the strength of the incident shock is moderate, or when its angle of incidence is not too small, the reflection along a planar wall is solved by means of two shocks (incident and reflected), separating three constant states. This pattern is known as a *Regular Reflection* and is designated by the acronym RR. Since the incident shock and the states that it separates are given data, one needs only to determine the reflected shock and the state behind. This is done through explicit algebraic manipulations. Such computations had been known as soon as 1940, and are due to von Neumann [56, 57]. As we shall see below, the nonlinear equation to solve has generally *two* solutions for a moderate incident strength, yielding two types of Regular Reflection, a *weak* and a *strong* one. Physical and numerical experiments suggest that the strong RR is unstable, its instability being presumably of Hadamard type. See the analytic study by Teshukov [73].

When the strength of the incident shock increases, the boundary-value problem fails to admit a RR and one needs to consider a more elaborate reflection configuration, called a *Mach Reflection* and designated by MR. This terminology was coined by J. von Neumann after the very first description by Ernst Mach [51] in 1878. The common feature to all kinds of MR is the presence of a *triple point* where three shock waves meet, together with a slip line (vortex sheet). Numerical or physical experiments suggest several types of MR, called *single* (SMR), *transitional* (TMR), *double* (DMR) or *complex* (CMR) ; this list is ordered by increasing complexity.

In some regimes, a Mach Reflection occurs where it is not allowed by the theory based on shock polar analysis. For some reason, the approximation of the flow by plane waves separating constant states is not valid. Numerical experiments suggest that slightly different configurations may occur, for which several scenarii have been proposed. In one of them, called *von Neumann* Reflection (vNR), the reflected shock degenerates into a compression wave near the triple point (see [7, 8, 16]) while the incident shock and the so-called *Mach stem* (the third shock) seem to have a common tangent at the triple point. In another one, called *Guderley* Mach Reflection, an array of triple points takes place along the Mach stem immediately after the reflection point; at each of these points one observes an interaction between the Mach shock and a wave that bounces between this Mach shock and the sonic line. This wave is alternatively a shock and an expansion fan (see [43, 71]). Since there is essentially no rigorous mathematics on such patterns, we shall mainly limit our study to RR, although MR is relevant in most of realistic situations. The reader interested in MR theory should consult the nice review by G. Ben-Dor [5] on the experimental side, and the papers by Hunter & coll. and by Čanić and coll. given in references, on the theoretical side. His book [4] displays a lot of experimental results and describes, at least on a phenomenological basis, the various strategies for transition from one kind of reflection to another. See also the review paper by Hornung [41]. The present paper is devoted on the one hand to the basics of the theory, like shock polars and terminology, and on the other hand to rigorous qualitative results that could be used in a strategy toward well-posedness of reflection problems.

Galilean invariance. As mentioned above, the unsteady models are Galilean invariant: If (ρ, u, e) is a solution (even in the sense of distributions), and $\bar{u}$ is a constant vector, then

$$(\hat{\rho}, \hat{u}, \hat{e})(x, t) := (\rho(x + t\bar{u}, t), u(x + t\bar{u}, t) - \bar{u}, e(x + t\bar{u}, t))$$

defines another solution. This observation is particularly useful when dealing with simple flows as shock waves for instance: it will be enough to consider steady shock waves.

Remark that, because of Galilean invariance, the notions of supersonic, sonic and subsonic flows do not make sense in the absence of a specified reference frame, since the sound speed is independent of the Galilean frame, while the flow velocity is not.

1 Models for gas dynamics

Besides the full Euler system (3,4,5), there are a variety of models, each depending on some simplifying assumptions.

1.1 Barotropic models.

We begin by extracting from (3,4,5) equations that are not in conservative form. First of all, (3) and (4) imply

(6) $$(\partial_t + u \cdot \nabla)u + \frac{1}{\rho}\nabla p = 0, \qquad u \cdot \nabla := \sum_{\alpha=1}^{d} u_\alpha \frac{\partial}{\partial x_\alpha}.$$

Eliminating between (6) and (3,5), we obtain

(7) $$(\partial_t + u \cdot \nabla)e + \frac{p}{\rho}\operatorname{div} u = 0.$$

Finally, (3) and (7) yield the following transport equation for the entropy:

$$(\partial_t + u \cdot \nabla)S = 0. \tag{8}$$

We notice that all these computations involve the chain rule and do not make sense when the field experiences a discontinuity. Therefore the conclusion that S is constant along the *particle paths* (these are integral curves of the equation $dx/dt = u(x,t)$) is not correct when such a trajectory crosses a shock. However, it is known that the jump $[S]$ of the entropy at a shock is of the order of the *cube* of the shock strength. This reflects the transmission identity

$$[e] + \langle p \rangle \left[\frac{1}{\rho}\right] = 0, \qquad \langle p \rangle := \frac{p_+ + p_-}{2}, \tag{9}$$

which should be compared with (2). When the oscillations of the flow are small, it is therefore reasonable to assume that S does not vary across shocks. Under this simplification, and if S was constant at initial time, we conclude that S remains constant forever. The flow is then called *isentropic*. The constancy of S means that ρ and e are not any more independent of each other: the specific energy and therefore the pressure become functions of the density. We denote $p = P(\rho)$. For this reason, such models are also called *barotropic*. For a perfect gas, one has $P(\rho) = A(S)\rho^\gamma$. As in the full Euler equation, the equation of state P determines everything else. For instance, starting from (2) and expressing that $\mathrm{d}S = 0$, we obtain

$$\frac{\mathrm{d}e}{\mathrm{d}(1/\rho)} = -P,$$

or equivalently

$$e(\rho) = \int^\rho p(s) \frac{ds}{s^2}. \tag{10}$$

We check easily that for an isentropic flow, (5) is formally a consequence of (3) and (4). Therefore we feel free to drop it and retain the following shorter system, called improperly the *isentropic Euler equation*:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{11}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = 0. \tag{12}$$

Another way to end with a barotropic model is to assume that heat dissipation is strong enough drive the temperature to a constant value. The model is called *isothermal*. The constancy of the temperature is again a relation between ρ and e, and thus implies an equation of the form $p = P(\rho)$. The isothermal case of a perfect gas yields a linear P. In this situation, (5) is not any more a consequence of (3) and (4). However, it may be dropped with the following argument: the right-hand side of the conservation law of energy is not zero, but accounts for the heat diffusion. Since we assumed that heat diffusion is dominant, the convective part of this equation (the left-hand side of (5)) may be dropped. The rest just ensures that $T \equiv \mathrm{cst}$.

1.2 Irrotational models.

Barotropic models still satisfy formally Equation (6). However, the fact that $p = P(\rho)$ ensures that $\rho^{-1}\nabla p$ is curl-free. Taking the curl of (6), we thus obtain an equation for u only, which reads in dimension $d = 3$:

$$(\partial_t + u \cdot \nabla)\omega + (\operatorname{div} u)\omega = (\omega \cdot \nabla)u, \qquad \omega := \nabla \times u. \tag{13}$$

We notice that when $d = 2$ the right-hand side is absent in (13). Nevertheless, in both cases, (13) is a linear transport equation in ω. Therefore irrotationality propagates as long as the vector field u is Lipschitz continuous. Once again, Equation (13) is not valid across shocks[2] ; it is therefore an approximation to claim that the flow is irrotational forever.

When the flow is irrotational, one may introduce a velocity potential ϕ by $u = \nabla\phi$ and work in terms of the unknown (ρ, ϕ). A clever choice of ϕ and an integration of (6) yield the equation

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + i(\rho) = 0, \qquad i'(s) = p'(s)/s. \tag{14}$$

The function $\rho \mapsto i(\rho)$ is called the *enthalpy*. Equation (14) is to be coupled with the conservation of mass that we rewrite now

$$\partial_t\rho + \operatorname{div}(\rho\nabla\phi) = 0. \tag{15}$$

We notice that $i = e + \rho^{-1}p$, with $e(\rho)$ given by (10).

1.3 Steady flows, potential flows.

A flow is steady when (ρ, u, e) depend only on the space variable, but not on the time variable. For instance, the full Euler system becomes

$$\begin{aligned} \operatorname{div}(\rho u) &= 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla p &= 0, \\ \operatorname{div}\left(\left(\frac{1}{2}\rho|u|^2 + \rho e + p\right)u\right) &= 0. \end{aligned}$$

Combining the first and the third line above, we immediately obtain

$$u \cdot \nabla B = 0, \qquad B := \frac{1}{2}|u|^2 + e + \frac{p}{\rho}. \tag{16}$$

The quantity B is called the *Bernoulli invariant*. Equation (16) tells that B is constant along the particle paths, at least as long as the solution is smooth (see below for a better result). This constant may vary from one particle path to another. In some cases, it may be relevant to

[2]See Section 2.3. The jump of vorticity is proportional to the curvature of the shock front when the state is constant on one side.

assume that this constant does not depend on the particle and therefore $B \equiv \text{cst}$. For instance, the flow might be uniform in some remote region, while every particle path comes from this region. This assumption allows us to eliminate e and to work with the conservation of mass and momentum only. We warn the reader that such a reduced system is not of barotropic form, as the pressure becomes a function of both ρ and $|u|$.

In the barotropic case, the conservation of momentum reads formally

$$(u \cdot \nabla)u + \nabla i(\rho) = 0.$$

Multiplying by u^T, one obtains again

$$u \cdot \nabla B = 0, \qquad B := \frac{1}{2}|u|^2 + i(\rho) = \frac{1}{2}|u|^2 + e + \frac{p}{\rho}. \tag{17}$$

Steady irrotational flow. The same observations as above hold true when $\operatorname{curl} u \equiv 0$. The assumption that B equals a constant now yields an equation of the form $\rho = F(|u|)$. If moreover the flow is irrotational, then the barotropic model reduces to a single second order equation in the velocity potential:

$$\operatorname{div}(F(|\nabla\phi|)\nabla\phi) = 0. \tag{18}$$

Equation (18) may be rewritten in the quasilinear form

$$F(|\nabla\phi|)\Delta\phi + \frac{F'(|\nabla\phi|)}{|\nabla\phi|}\sum_{\alpha,\beta}\partial_\alpha\phi\partial_\beta\phi\partial_\alpha\partial_\beta\phi = 0. \tag{19}$$

It follows that its type depends on the local state of the fluid. Since F is positive, this equation is elliptic (respectively hyperbolic) whenever $F(|u|) + |u||F'(|u|)|$ is positive (resp. negative). From

$$i \circ F(r) + \frac{1}{2}r^2 \equiv i_0,$$

we derive $F'(r)i' \circ F(r) = -r$. In particular F' is negative. We may rewrite this as $F'(r)p' \circ F(r) = -rF(r)$, from which we find that ellipticity amounts to saying that $|u|^2 < p'(\rho)$. Likewise, hyperbolicity of (18) is equivalent to $|u|^2 > p'(\rho)$.

Remark. Since the conservation of energy reads $\operatorname{div}(\rho B u) = 0$ (for steady flows), B does not vary across steady shocks (*i.e.* discontinuities with non-zero mass flux). Thus the assumption that B is constant is reasonable when it is so in the far field, provided we know that the vortex sheets have small enough amplitude.

1.4 Characteristic curves

The previous analysis shows that $c(\rho) := \sqrt{p'(\rho)}$ plays a fundamental role and has the dimension of a velocity. This quantity turns is called the *sound speed*. In a non-barotropic flow (full Euler system), it is given by the expression

$$c(\rho, e) := \sqrt{\frac{\partial p}{\partial \rho} + \frac{p}{\rho^2}\frac{\partial p}{\partial e}}. \tag{20}$$

For a perfect gas we obtain the well-known formula

$$c = \sqrt{\frac{\gamma p}{\rho}}.$$

Coming back to Equation (19), we may eliminate F and F', and find its equivalent form

$$c^2 \Delta\phi - \sum_{\alpha,\beta} \partial_\alpha\phi\partial_\beta\phi\partial_\alpha\partial_\beta\phi = 0. \tag{21}$$

In unsteady models, the notion of sound speed is relevant too, and one can show that the corresponding system is hyperbolic whenever c^2 is positive, that is c is a non-zero real number. Then the Cauchy problem is locally well-posed in spaces of smooth functions. See [22, 29, 64] for instance.

Characteristic curves are well-defined in two independent variables, which may be either (t, x) if $d = 1$, or $x = (x_1, x_2)$ if $d = 2$. Let us begin with the time-dependent flows that depend only on one space variable. *Characteristic curves* are integral curves, that is solutions of differential equations

$$\frac{dx}{dt} = \lambda(x, t), \tag{22}$$

where λ, a function of U, is one of the characteristic velocities. For a general hyperbolic quasilinear system

$$\partial_t U + A(U)\partial_x U = 0, \tag{23}$$

the characteristic velocities are the eigenvalues of the matrix $A(U)$; hyperbolicity tells that these eigenvalues are real.

For the full Euler system, the characteristic speeds are $\lambda_- = u - c(\rho, e)$, $\lambda_0 = u$ and $\lambda_+ = u + c$, whence three characteristic families. Notice that the characteristic curves for the second family are particle paths. The other ones "propagate" sound waves. For the barotropic model, we still have $\lambda_- = u - c$ and $\lambda_+ = u + c$, and it seems that $\lambda_0 = u$ is dropped. This observation is correct if the conservation of momentum is restricted to momentum in the x variable, but it is not if we also take in account the momenta in the transverse directions (think of a gas in 3-d, of which the flow is planar).

The situation for steady flows in 2-d is significantly different, for the evolution is governed by some quasilinear system

$$A(U)\partial_x U + B(U)\partial_y U = 0. \tag{24}$$

If we take x (for instance) as a time-like variable, the characteristic velocities are generalized eigenvalues:

$$\det(B(U) - \lambda(U)A(U)) = 0. \tag{25}$$

Characteristic curves are therefore integral curves of

$$\frac{dy}{dx} = \lambda(x, y), \tag{26}$$

or in other words and more generally

$$\det(A(U)dy - B(U)dx) = 0. \tag{27}$$

Although hyperbolicity tells that $\xi_1 A(U) + \xi_2 B(U)$ is diagonalizable with real eigenvalues for every choice of $\xi \in \mathbb{R}^2$, there is no reason why (25) would admit real solutions. The only general remark that can be made is that there is at least one such solution in the barotropic case because the matrices are 3×3 and 3 is an odd integer.

To go further, let us denote $\lambda_j(U; \xi)$ $(j = 1, ...)$ the eigenvalues of $\xi_1 A(U) + \xi_2 B(U)$. Then $\lambda = -\xi_2/\xi_1$ solves the problem (25) as soon as $\lambda_j(U; \xi) = 0$ for some j. Since there holds

$$\lambda_-(U; \xi) = u \cdot \xi - c|\xi|, \qquad \lambda_-(U; \xi) = u \cdot \xi, \qquad \lambda_+(U; \xi) = u \cdot \xi + c|\xi|, \tag{28}$$

we find the following characteristic directions $\mathbb{R}(1, \lambda) = \mathbb{R}(\xi_2, -\xi_1)$ for the steady problem:

- In all situations, $\mathbb{R}u$, the direction of the particle paths.
- When $|u| > c$ (supersonic flow), and only in this case, the directions V such that $\det(u, V) = c|V|$. There are two such directions, which make an angle α with the particle path, where $|\sin \alpha| = c/|u|$.
- In the borderline case where $|u| = c$, these two directions coincide with the direction of the flow. This is a rather degenerate situation.
- If the flow is subsonic (that is $|u| < c$), we still have two such vectors V, but they are complex conjugate and cannot serve to define characteristic directions.

We conclude that a steady supersonic flow is hyperbolic, in every direction that is not characteristic. For a subsonic flow, the model is of mixed type hyperbolic-elliptic: the hyperbolic mode corresponds to the sole real characteristic direction u, while the elliptic one accounts for the two complex conjugate characteristic directions V.

Real characteristic directions are crucial in the study of propagation of singularities. Let us assume that U is a continuous, piecewise $\mathcal{C}^1$, solution of a first-order system in two independent variables. Then the locus of the singularities (here discontinuities of the first derivatives) is a union of characteristic curves. In particular, for a smooth solution that is constant on some open domain, the boundary of that domain consists of a union of characteristic curves that have a very simple geometry. This is an important remark when constructing explicit steady flows, for it is not hard to match local flows along characteristic lines as long as they are supersonic, but it is harder to deal with subsonic flows, because of the rigidity of solutions of elliptic problems (analyticity, maximum principle,...) In particular, a flow must have some regularity properties in its subsonic domain, though not in the supersonic region.

Let us end this paragraph by warning the reader: when a discontinuity separates two regions in which the flow is smooth, the discontinuity does not need to be a characteristic line. The characteristic property of interfaces is only valid for discontinuities of derivatives of order one at least.

Remark. In Section 4, we shall encounter systems of the form

$$A(U)\partial_x U + B(U)\partial_y U = g(U), \tag{29}$$

where g is some smooth function. The same theory as above applies. The characteristics of the system (24) do propagate the weak singularities (*i.e.* those of derivatives) of the piecewise smooth solutions of (29), in spite of the presence of an additional lower order term. These characteristics are defined by the same equation (27).

1.5 Entropy inequality

Since the unsteady Euler equations (full or barotropic) form a first-order hyperbolic system of conservation laws, its solutions usually develop singularities in finite time. These singularities are most often discontinuities along moving hypersurfaces. Of course, the conservation laws have to be understood in the sense of distributions. It turns out that too many piecewise smooth solutions exist, so that the uniqueness for the Cauchy problem fails dramatically.

Parallel to this mathematical difficulty, thermodynamics tells us that discontinuous solutions are not reversible, despite the formal reversibility of the conservation laws.

These two remarks suggest that some of the discontinuities that solve the PDEs must be ruled out by some criterion, which has to be of mathematical nature while having a physical relevance. The most popular criterion, the *entropy inequality*, was introduced by Jouguet [46] for gas dynamics and then generalized to hyperbolic first-order systems by Kruzkov [48] and Lax [49]. It states that for every conservation law

$$\partial_t \eta + \operatorname{div} Q = 0$$

that is formally consistent with the model, where η and Q are given functions of the conserved variables with η *convex*, a weak solution is admissible if and only if it satisfies moreover

$$\partial_t \eta + \operatorname{div} Q \leq 0. \tag{30}$$

A function η as above is called an *entropy* by mathematicians (sometimes a convex entropy). This terminology differs from that of physicists. As a matter of fact, there is essentially one non-trivial entropy inequality for gases, where $\eta := -\rho S$ for the full Euler system (with $Q = \eta u = -\rho S u$), and $\eta := \frac{1}{2}\rho|u|^2 + \rho e$ for the barotropic model (with $Q = (\eta + p)u$). It is worth noticing that in the latter case, the energy is not conserved across shocks, and its global decay plays the role of an entropy condition!

For a perfect gas, one has $S = \log e - (\gamma - 1)\log\rho$. It is an interesting exercise to prove that

$$\left(\rho, \rho u, \frac{1}{2}\rho|u|^2 + \rho e\right) \mapsto -\rho S$$

is a convex function.

For a general equation of state, (30) is a necessary condition for admissibility but might not be sufficient. A wide literature exists on this topic, for which we refer to Chapter VIII of [29]. Several refined conditions are known, with various degrees of efficiency. For perfect gases however, all admissibility conditions are equivalent to (30). For this reason, we shall content ourself with the entropy inequality, assuming that the gas is perfect or close to be so.

Minimum principle for the full Euler system. For the full Euler system, it has been shown by L. Tartar [70] and independently by A. Harten & coll. (see also Exercise 3.18 in ([64])) that, besides the conservation laws, the mathematical entropies are all of the form $\rho f(S)$ with $f : \mathbb{R} \to \mathbb{R}$. For a perfect gas, the convex ones correspond to non-increasing functions f such that $s \mapsto e^{-s/\gamma} f'(s)$ is non-decreasing. For such functions, the solutions must obey the generalized entropy inequality

$$\partial_t(\rho f(S)) + \operatorname{div}(\rho f(S) u) \leq 0. \tag{31}$$

In particular, choosing for f a function of the form $s \mapsto \max\{0, S_0 - s\}$, we obtain (see in particular [67] and also [28, 62]) a minimum principle that fits with the physical intuition that the entropy S tends to increase:

Theorem 1.1 *For an entropy solution of the full Euler system in an isolated domain (meaning that $u \cdot \nu = 0$ along the boundary), the physical entropy S is globally non-decreasing:*

$$t \mapsto \inf\{S(x,t)\,;\, x \in \Omega\} \tag{32}$$

is a non-decreasing function of time.

1.6 Other models

Since the Euler equations form a rather complicated system of PDEs, with strong interaction between nonlinearity and the characteristics, simplified models have been used in order to reduce the complexity, while catching the main features of the flows under consideration.

The unsteady transonic disturbance equation. In the course of this article, we shall be concerned with the transition between Regular Reflection (RR) and Mach Reflection (MR) at a wedge of angle 2α. Following [42, 54, 43], and assuming that the strength of the incident shock is weak, this transition occurs when

$$M - 1 = O(\alpha^2), \tag{33}$$

M being the Mach number of the incident shock. This regime allows for a competition between nonlinearity and diffraction. In the sequel, the parameter

$$a := \frac{\alpha}{\sqrt{M-1}} \tag{34}$$

is kept fixed.

Since only small disturbances of the rest state $(\rho_0, 0, p_0)^T$ are considered, we may assume that the gas is ideal for some $\gamma > 1$. With $\epsilon := 2(M-1)$, one defines new unknowns v and w by the expansion

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = \begin{pmatrix} \rho_0 \\ 0 \\ p_0 \end{pmatrix} + \epsilon \frac{2}{\gamma+1} v \begin{pmatrix} \rho_0 \\ c_0 \vec{e}_x \\ \gamma p_0 \end{pmatrix} + \epsilon^{3/2} \frac{2\sqrt{2}}{\gamma+1} w \begin{pmatrix} 0 \\ c_0 \vec{e}_y \\ 0 \end{pmatrix} + O(\epsilon^2). \tag{35}$$

Rescaling now the space and time variables by

$$X := \frac{x - c_0 t}{\epsilon}, \qquad Y := \sqrt{2} \frac{y}{\epsilon^{1/2}}, \qquad T = c_0 t, \tag{36}$$

we obtain the *Unsteady Transonic Small Disturbance equation* (UTSD)

$$\partial_T v + \partial_X \left(\frac{1}{2} v^2 \right) + \partial_Y w = 0, \tag{37}$$

$$\partial_Y v - \partial_X w = 0. \tag{38}$$

This system is sometimes called the *2-D Burgers equation*. We notice that, because of the anisotropic change of scales, (37,38) is neither Galilean nor rotationally invariant. We point out that the parameter a in (34) determines the slope dX/dY of the incident shock.

Since the UTSD admits shock waves, there must be a selection criterion, in order to eliminate those discontinuities that are physically irrelevant. This can be done thanks to an entropy inequality as for general first-order systems of conservation laws. For every parameter $k \in \mathbb{R}$, classical solutions of (37,38) satisfy

$$\partial_T \frac{(v-k)^2}{2} + \partial_X \left(\frac{v^3}{3} - \frac{kv^2 + w^2}{2} \right) + \partial_Y ((v-k)w) = 0,$$

and the admissible discontinuous solutions should satisfy the inequality

$$\partial_T \frac{(v-k)^2}{2} + \partial_X \left(\frac{v^3}{3} - \frac{kv^2 + w^2}{2} \right) + \partial_Y ((v-k)w) \le 0 \tag{39}$$

in the distributional sense. On a single shock, it just tells that the jump of v is negative from the left to the right in the X-direction.

The UTSD is especially useful in that it catches many of the features of the Euler equations in the presence of shock reflection. For large a, it admits a strong and a weak Regular Reflection (see Paragraph 3.3.2), while it displays a transition to Mach Reflection when a diminishes.

Like the Euler models, it does not admit pure triple-point configurations (see Section 2.2). At last, fewer computational ressources are required to solve it numerically. It has been studied intensively at the theoretical and numerical level as well ; see for instance [7, 10, 11, 13, 15, 16, 17, 30, 54, 66, 71] and the references cited above. This series of papers culminates with the proof of existence of a local solution to the transonic (see [11]) and the supersonic Regular Reflection (see [13]). A major difficulty in these works is of course the presence of a free boundary (the reflected shock, see Section 4), for which tools were elaborated in [15]. In the supersonic case, one also faces an elliptic equation whose symbol degenerates at the boundary, say along the sonic line.

The pressure-gradient model. For numerical purposes (see [1]), one may be tempted to split the full Euler system into two evolution problems, one in which we drop the pressure and energy terms, and one in which we drop the convection term. The latter reads

$$\begin{aligned} \partial_t \rho &= 0, \\ \partial_t(\rho u) + \nabla p &= 0, \\ \partial_t(\rho e) + p \,\mathrm{div} u &= 0, \end{aligned}$$

where $\rho e =: \epsilon(\rho, p)$. Assuming that ρ was a constant ρ_0 at initial time, it stays a constant forever because of the first equation. Elimination of u between the last two equations yields

$$\partial_t \left(\frac{\rho_0}{p} \partial_t \epsilon(\rho_0, p) \right) = \Delta p.$$

This is a non-linear wave equation of the form

$$\partial_t^2 \psi(p) = \Delta p,$$

with

$$\psi' = \frac{\rho_0}{p} \frac{\partial \epsilon(\rho_0, p)}{\partial p}.$$

For an ideal gas, $\psi(p)$ is a positive constant times $\log p$. After a rescaling, the pressure-gradient system may be rewritten as

$$\partial_t u + \nabla p = 0, \quad \partial_t E + \mathrm{div}(pu) = 0, \qquad E := \frac{1}{2}|u|^2 + p.$$

According to Zheng [81], this model is a good approximation of the Euler system when the velocity is small and γ is large[3].

[3] A large γ is not realistic when speaking of gases, of course.

2 Multi-dimensional shocks

2.1 Jump relations for a single shock

Beyond constant flows, we consider piecewise constant flows of the form

$$U(x,t) = \begin{pmatrix} U_-, & \text{for} & x\cdot\nu < \sigma t, \\ U_+, & \text{for} & x\cdot\nu > \sigma t, \end{pmatrix} \tag{40}$$

where ν, the direction of propagation, is a unit vector and $\sigma \in \mathbb{R}$ is the normal velocity of the shock. Hereabove, U stands for the set of unknowns, say (ρ, u, e) in the full system, (ρ, u) in the isentropic case and the irrotational case. Since a constant flow is obiously a solution of any of these models, U is a solution if and only if it satisfies the correct transmission conditions, called *Rankine–Hugoniot relations.*

The first one is that associated to the conservation of mass (3),

$$[\rho(u\cdot\nu - \sigma)] = 0, \tag{41}$$

from which we may define the *mass flux* across the interface:

$$j := \rho_-(u_-\cdot\nu - \sigma) = \rho_+(u_+\cdot\nu - \sigma).$$

It is positive when the fluid flows from the negative side (that defined by $x\cdot\nu < \sigma t$) to the positive side, and negative in the opposite configuration.

Besides (41), we write jump relations that depend on the model under consideration:

Full system: We write the Rankine–Hugoniot conditions for (4,5), namely

$$\begin{aligned} [\rho(u\cdot\nu - \sigma)u] + [p]\nu &= 0, \\ \left[\left(\frac{1}{2}\rho|u|^2 + \rho e\right)(u\cdot\nu - \sigma)\right] + [pu\cdot\nu] &= 0. \end{aligned}$$

Because of (41), they read equivalently:

$$j[u] + [p]\nu = 0, \tag{42}$$

$$j\left[\frac{1}{2}|u|^2 + e\right] + [pu\cdot\nu] = 0. \tag{43}$$

Barotropic case: We only write the first two jump conditions (41,42).

Irrotational flow: We write the jump condition (41), plus that associated to $\nabla\times u = 0$, namely

$$[u]\times\nu = 0. \tag{44}$$

From this, ϕ is given by $\phi(x,t) = u\cdot(x-\sigma t\nu) + \text{cst}$. We insert this formula into (14) and get our last relations

$$\left[\frac{1}{2}|u|^2 - \sigma u\cdot\nu\right] + [i(\rho)] = 0. \tag{45}$$

We notice that thanks to (44), (45) is equivalent to

$$\frac{1}{2}[(u\cdot\nu-\sigma)^2] + [i(\rho)] = 0. \tag{46}$$

In the subsequent analysis, we must distinguish the *contact discontinuities* (CD, for which $j=0$), from the *shock waves* (or just *shocks*), which correspond to $j\neq 0$. CDs do not happen in the irrotational case, because i' is positive for realistic gases and therefore

$$(j=0) \Longrightarrow ([i(\rho)]=0) \Longrightarrow ([\rho]=0).$$

In a CD of the full or a barotropic model, there holds $[p]=0$ and $u_\pm\cdot\nu=\sigma$. Conversely, any set $(U_-,U_+;\nu,\sigma)$, satisfying these two identities, defines a CD. In the barotropic case, the constancy of the pressure implies $[\rho]=0$ and therefore CDs are slip lines, also called *vortex sheets*. In the full Euler case, one may have $[u]=0$ since $[p]=0$ does not imply $(\rho_+,e_+)=(\rho_-,e_-)$.

The description of shocks is a bit more involved. On the one hand, (42) implies

$$j^2\left[\frac{1}{\rho}\right] + [p] = 0, \tag{47}$$

$$[u]\times\nu = 0. \tag{48}$$

On the other hand, combining (42,43) and the following identities

$$\left[\frac{1}{2}|u|^2\right] = \langle u\rangle[u], \qquad [pu] = \langle p\rangle[u] + [p]\langle u\rangle,$$

gives

$$j[e] + \langle p\rangle[u\cdot\nu] = 0. \tag{49}$$

This proves (9), because of $[u\cdot\nu] = j[1/\rho]$, which is nothing but the definition of j.

We show now that *the jump of the entropy is of cubic order in the shock strength.* It is not hard to see that (9) defines a curve in the (ρ,e)-plane when one of the states, say (ρ_0,e_0) is fixed. Let s be a smooth, non-degenerate parameter along this Hugoniot curve. With dots standing for derivations along the curve, we have

$$T\dot S = \dot e + p\dot\tau, \qquad \tau := \frac{1}{\rho}.$$

On the other hand, we have

$$\dot e + \frac{1}{2}\dot p[\tau] + \langle p\rangle\dot\tau = 0,$$

whence

$$2T\dot{S} = [p]\dot{\tau} - [\tau]\dot{p}.$$

Differentiating once more, we have

$$2\frac{d}{ds}\left(T\frac{dS}{ds}\right) = [p]\frac{d^2\tau}{ds^2} - [\tau]\frac{d^2p}{ds^2}.$$

These identities imply

$$\frac{dS}{ds} = \frac{d^2S}{ds^2} = 0$$

at the origin (ρ_0, e_0) of the curve. Whence $[S] = \mathcal{O}(s^3)$. This estimate is accurate for most of equations of state, for instance that of an ideal gas, but a degeneracy could cause the jump of the entropy to be of higher order.

Entropy conditions for shocks. Besides the Rankine–Hugoniot conditions, a shock has to satisfy the jump relations associated to the entropy inequality (30). Since Lipschitz solutions of the Euler equations do satisfy the entropy *equality*, selecting admissible shocks among all discontinuities is precisely the role of the entropy *inequality*. The jump condition that is obtained is, of course, an inequality, which reads either

$$j[S] \geq 0, \tag{50}$$

in the full Euler case (meaning that the entropy of a small part of fluid increases when one crosses a shock), or

$$j\left[\frac{1}{2}|u|^2 + e\right] + [pu \cdot \nu] \leq 0 \tag{51}$$

in the barotropic case. The latter apparently violates the conservation of energy, but this is due to the fact that we do not take in account one form of the bulk energy ; the inequality just tells that there is some heat release. Of course, one may eliminate u from (51) as we did in the full Euler case. We obtain the equivalent form of entropy inequality across a discontinuity:

$$j\left([e] + \langle p\rangle\left[\frac{1}{\rho}\right]\right) \leq 0. \tag{52}$$

Shocks are compressive. Let us restrict to the equation of state of an ideal gas. In the full Euler case, $p = (\gamma - 1)\rho e$ and $S = \log e - (\gamma - 1)\log\rho$. Thanks to (9), we may eliminate $e_\pm$ and obtain

$$\exp(S_+ - S_-) = X^\gamma \frac{1 - \mu^2 X}{X - \mu^2}, \qquad X := \frac{\rho_-}{\rho_+},\ \mu^2 := \frac{\gamma - 1}{\gamma + 1}. \tag{53}$$

As a matter of fact, (9) implies also that the ratio of the densities is bounded by $1/\mu^2$. From (53), we immediately find the equivalent formulation of the entropy inequality:

$$j[\rho] \geq 0. \tag{54}$$

We examine next the barotropic case. We have $p = A\rho^\gamma$ where A is a positive constant. Then $e = p/((\gamma-1)\rho)$. We may assume $A = 1$. We eliminate again $e_\pm$, now in inequality (51) and we end with the same entropy condition (54).

In conclusion, *shocks are compressive* in ideal gases: the density is higher in the domain into which the gas flows. This domain is usually named *downstream* while the region from which the gas flows is named *upstream*.

Because of the above analysis, we also impose (51) in the irrotational model.

Shocks are supersonic on the front side. Given a steady shock (U_-, U_+), one may always modify the velocities by adding the same constant vector, provided the latter is parallel to the shock front. Since the tangential component of the velocity is continuous, we see that it may be cancelled simultaneously on both sides. Hence, not only there holds $|u| \geq |u \cdot \nu|$, but also we may choose a reference frame in such a way that the shock be still steady, and that it hold $|u| = |u \cdot \nu|$. This explains why the super-/sub-sonic property of one of the states with respect to the shock, is encoded only in the normal component of the velocity: We say that a steady shock between U_- and U_+ is *supersonic* (resp. *subsonic*) with respect to a neighbouring state U if $|u \cdot \nu| > c(\rho, e)$ (resp. $|u \cdot \nu| < c(\rho, e)$). Accordingly, a state $U = U_\pm$ is said to be *subsonic relatively* (resp. *supersonic relatively*) to the the shock if $|u \cdot \nu|$ is smaller (resp. larger) than the sound speed.

Theorem 2.1 *Consider a steady shock for a perfect gas with the full Euler model. Say that $p_- < p_+$. Then there holds $\rho_+ > \rho_-$. Furthermore, the shock is relatively supersonic on the front side (upstream) and relatively subsonic on the back side (downstream):*

$$|u_- \cdot \nu| > c_-, \qquad |u_+ \cdot \nu| < c_+.$$

Proof

Inserting the assumption $\rho_+ e_+ > \rho_- e_-$ in (9), we obtain $\rho_+ > \rho_-$.

Let us define $c_* = \sqrt{[p][\rho]}$. Eliminating j, we have

$$(u_- \cdot \nu)(u_+ \cdot \nu) = c_*^2. \tag{55}$$

On the other hand, simple manipulations transform (9) into

$$(1 - \mu^2)((u \cdot \nu)^2 - c^2) = (u \cdot \nu)^2 - c_*^2 \tag{56}$$

on both sides.

From $\rho_- < \rho_+$ we have $|u_+ \cdot \nu| < |u_- \cdot \nu|$. With (55), this implies

$$|u_+ \cdot \nu| < c_* < |u_- \cdot \nu|. \tag{57}$$

Finally, (57) and (56) yield

$$|u_+ \cdot \nu| < c_+, \qquad c_- < |u_- \cdot \nu|. \tag{58}$$

■

We point out that the above proof works for a more general equation of state, provided that the ratio

$$\frac{p - p_0}{\rho - \rho_0} \qquad (p = p(\rho, e),\, p_0 = p(\rho_0, e_0))$$

increases with ρ when (ρ_0, e_0) is kept fixed, along the *Hugoniot curve* defined by

$$e - e_0 + \frac{p + p_0}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) = 0.$$

As a matter of fact, such a property yields $c_- < c_* < c_+$. Since (57) still holds true, we deduce (58).

Let us turn towards the barotropic case, with $p = A\rho^\gamma$. We still have (55) and $|u_+ \cdot \nu| < |u_- \cdot \nu|$, hence (57). On the other hand, p being convex, there holds $c_- < c_* < c_+$. Whence (58). Remark that this proof needs only the convexity of p. This proves

Theorem 2.2 *Consider a steady shock for a barotropic gas with $p' > 0$ and $p'' > 0$. Say that $p_- < p_+$. Then there holds $\rho_+ > \rho_-$. Furthermore, the shock is relatively supersonic on the front side (upstream) and relatively subsonic on the back side (downstream):*

$$|u_- \cdot \nu| > c_-, \qquad |u_+ \cdot \nu| < c_+.$$

For more general equations of state, it may be necessary to reinforce the admissibility condition for shock waves. The most popular admissibility condition is the *Lax shock condition*, which tells that if a hyperbolic system of conservation laws consists of n scalar conservation laws, there must be exactly $n+1$ incoming characteristic curves in a shock. This is an obvious necessary condition, at the linearized level, for the well-determination of the solution and of the shock front, from the PDEs on each side and their associated Rankine–Hugoniot relations only (see for instance [52]). In gas dynamics, this means that with ν the unit normal to the shock front pointing towards the plus region, then

$$\text{either } (u \cdot \nu)_+ - c_+ < 0 < (u \cdot \nu)_- - c_-, \text{ or } (u \cdot \nu)_+ + c_+ < 0 < (u \cdot \nu)_- + c_-\,.$$

This condition ensures that the shock is supersonic with respect to one of the states, the minus one in the first case, or the plus one in the second case. We unify these cases by saying that a steady shock is always supersonic with respect to the *upstream flow*. Since on the other hand the pressure is lower upstream than downstream, we have that the shock is supersonic with respect to the side of lower pressure. An equivalent statement is that if a steady shock is subsonic with respect to the state on one side, then the pressure on this side is higher than on the other one.

A subtlety about subsonic states. We warn the reader that the state on the subsonic side of a steady shock may be either subsonic or supersonic in the coordinate frame! The subsonicity holds with respect to the shock, thus exactly means that $|u \cdot \nu| < c$, while the supersonicity with respect to the coordinate frame means that $|u| > c$. Obviously these two inequalities are compatible. On the other hand, the state on the supersonic side of the shock is supersonic in every sense, since we have

$$|u| \geq |u \cdot \nu| > c.$$

This subtlety is of some importance, as supersonicity in the reference frame means precisely that the model, which is a system of PDEs in the (x, y)-plane, is hyperbolic in some direction, though not in every one. When we consider a 3-D steady flow, saying that the state is subsonic with respect to the steady shock means that the model is not hyperbolic in the direction normal to the shock. More generally, *the steady Euler system is hyperbolic in a direction $N \in \mathbb{R}^3$ if and only if $|u \cdot N| > c|N|$*. For the sake of simplicity, we illustrate this claim with the barotropic case. Working with unknowns $V = (\rho, u_1, u_2, u_3)^T$, the system reads in the quasilinear form

$$\left(A^1(V)\partial_1 + A^2(V)\partial_2 + A^3(V)\partial_3\right) V = 0.$$

Defining $A(V;\xi) = \sum_\alpha \xi_\alpha A^\alpha(V)$, we have

$$A(V;\xi) = (u \cdot \xi) I_4 + \begin{pmatrix} 0 & \rho \xi^T \\ \frac{c^2}{\rho}\xi & 0 \end{pmatrix}.$$

Let $P(V;\xi) := \det(A(V;\xi))$ be the principal symbol of the operator $\sum_\alpha A^\alpha \partial_\alpha$. Let $N \in \mathbb{R}^3$ be such that $P(N) \neq 0$ (the normal plane to N is not characteristic). Then the system is hyperbolic in the direction N if the roots of the polynomial $s \mapsto P(sN + \xi)$ are real for every ξ. In our problem, P is written

$$P(V;\xi) = (u \cdot \xi)^2 \left((u \cdot \xi)^2 - c^2 |\xi|^2\right),$$

thus there is always a double real root $-(u \cdot \xi)/(u \cdot N)$ in the fourth degree polynomial $P(sN + \xi)$. Taking ξ normal to N and u, we observe that the two remaining roots are real if and only if $|u \cdot N| \geq c$. Actually, these roots are the eigenvalues of some matrix and hyperbolicity requires in addition that this matrix be diagonalizable . This rules out the borderline case $|u \cdot N| = c|N|$.

About notations. When a steady shock is given, we always label U_+ the subsonic state and U_- the supersonic one. Therefore we always have $\rho_- < \rho_+$ and $p_- < p_+$. Also, we choose the unit normal ν that points toward the "plus" zone. Since U_- is upstream and U_+ downstream, we have $u_\pm \cdot \nu > 0$. In particular, there holds

$$j > 0, \qquad c_- < u_- \cdot \nu, \qquad u_+ \cdot \nu < c_+.$$

In many shock problems, a state U_0 is given and a pair $(U_1; \nu)$ is sought so that $(U_0, U_1; \nu)$ is a steady shock. Such notations are employed as long as one does not know whether the zero state is sub- or super-sonic (mind that this depends on ν). In some circumstances, it may be necessary to employ both notations in the same paragraph.

2.2 Triple shock structures

A shock interaction is the simplest pattern that is genuinely two-dimensional. At first glance, it would be a point where three shocks meet. We shall see below that it must be of even higher complexity, because of a very general obstruction result. Typically, such a configuration needs at least an additional wave: a vortex sheet or an expansion wave.

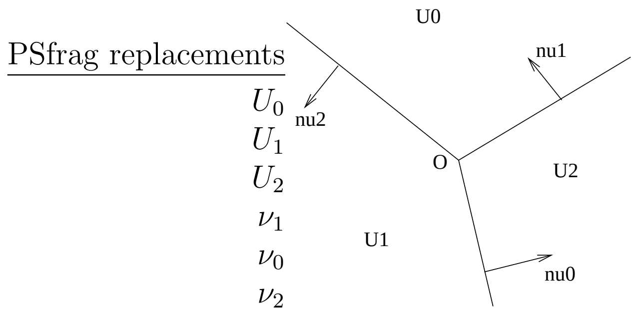

Figure 1: The (impossible) pure triple shock structure.

2.2.1 Vorticity flows

In this paragraph, the flow is governed by either the full Euler system, or that of a barotropic fluid. As a matter of fact, we only use the conservation of mass and momentum, without any information about the pressure. To our knowledge, the following result is new in its generality, especially because the obstruction appears to be of kinematical nature, rather than thermodynamical.

Theorem 2.3 *Consider a planar steady flow. There does not exist a pure triple shock structure, that is a piecewise constant flow with only three states separated by straight shocks.*

Amazingly enough, the proof involves only the kinematics, that is the conservations of mass and momentum.

Proof

Let us denote U_0, U_1, U_2 the constant states. We choose an orientation around the triple point. The unit normal to the shocks are, in cyclic order, ν_0, ν_1, ν_2, meaning that the line of equation $x \cdot \nu_\alpha = 0$ separates $U_{\alpha+1}$ from $U_{\alpha+2}$, the normal being oriented from the $(\alpha+1)$-zone to the $(\alpha+2)$-zone ; see Figure 1. The mass fluxes across the shocks are denoted as well ; for instance,

$$j_0 = \rho_1 u_1 \cdot \nu_0 = \rho_2 u_2 \cdot \nu_0.$$

We recall that $j_\alpha \neq 0$ for every α. Notice that the ν_α's are pairwise distinct unit vectors. We do not exclude that two normals be parallel, for instance $\nu_1 = -\nu_0$, but at least two of them are linearly independent.

We make use of (42) and of its equivalent form (thanks to (47) and to $j \neq 0$)

$$[u] = j \left[\frac{1}{\rho}\right] \nu. \tag{59}$$

In both (42) and (59), we sum circularly on the triple of shocks, in order to eliminate the $[u]$'s. We obtain an identity ($[h]_0$ denotes $h_2 - h_1$, and so on) that we may write in two forms:

$$\sum_{\rm circ} [p]_\alpha \frac{\nu_\alpha}{j_\alpha} = 0 \qquad \text{or} \qquad \sum_{\rm circ} \left[\frac{1}{\rho}\right]_\alpha j_\alpha \nu_\alpha = 0. \tag{60}$$

Since on the other hand we have

$$\sum_{\text{circ}}[p]_\alpha = 0, \qquad \sum_{\text{circ}}\left[\frac{1}{\rho}\right]_\alpha = 0,$$

where the jumps of p and of $1/\rho$ do not vanish[4], (60) tells us that the vectors $P_\alpha := j_\alpha^{-1}\nu_\alpha$ are collinear, and that the $Q_\alpha = j_\alpha\nu_\alpha$ are collinear too. Let L be the line passing through the P_α's. Since there is a pair of linearly independent ν_α's, L does not pass through the origin. Thus its image under inversion with respect to the unit circle is a circle C passing through the origin. Since the Q_α's belong to this circle and are collinear, two of them must be equal: say we have $j_1\nu_1 = j_2\nu_2$. The ν's being pairwise distinct, this implies $\nu_1 = -\nu_2$ (and thus $j_\beta = -j_\alpha$). Now, any of the equalities in (60) tells that ν_3 is collinear to the other ones, a contradiction. ■

Remarks.

- Theorem 2.3 is a folk result in multi-dimensional gas dynamics. However, it seems that all the previous proofs needed specific assumptions on the equation of state. Courant and Friedrichs ([27], paragraph 129) stated it for a polytropic gas. Henderson and Menikoff [40] considered a more general gas, but still with a restriction on the equation of state.

- Henderson and Menikoff's proof is based on the variation of the entropy across a sequence of shocks, at given final pressure. This estimate, which has its own interest, gives a rigorous basis to the following claim made several times by von Neumann [56, 57, 55]: Among the wedges separated by the shocks, there must be an upstream wedge and a downstream one. As the gas flows from upstream to downstream, it passes either on one side of the triple point or on the other side. Thus some molecules cross only one shock, while others cross two shocks. The increase of entropy should be higher in the former case than in the latter. Hence the state downstream may not be uniform. We point out that this entropy-based argument does not apply to the barotropic case. We do not exclude that a similar argument, based on the variation of total energy, might work in this latter case.

- Von Neumann's claim looks to be a kind of convexity inequality. Therefore it certainly needs some assumptions, like those of Henderson and Menikoff, in order to be proved. In particular, it applies only within the class of *entropy-admissible* shock waves.

- On the other hand, Courant and Friedrichs' proof, though being restricted to a small class of equations of state, is valid even for non-admissible shocks, since it does not involve an inequality at all. Thus it is completely independent from von Neumann's claim.

- The proof above only assumes that the stress tensor is of the form $-pI_d$ for some scalar p. This means that, given an arbitrary interface Σ, the force applied by the fluid on one side of Σ onto the fluid on the other side, is normal to Σ.

[4] If one of them vanished, we should find immediately $[\rho] = [p] = 0$ and $[u] = 0$, meaning that the shock was not present.

- It is remarkable that Theorem 2.3 holds true even if hyperbolicity fails in some region of the phase space, when an equation of state is given. For instance, it applies to a *van der Waals gas*.

- For the same reason, it is valid both for the full Euler model and the barotropic one.

- This two-dimensional result extends to three space dimensions, because the tangential velocity is constant across a shock.

- Actually, the result is valid for time-dependent flows, in the following form: In $2+1$ space-time dimensions, there does not exist a (locally defined) piecewise smooth solution $U(x,t)$ of the (full or barotropic) Euler system, where the discontinuities are three shocks (the normal velocities are non-zero) that meet transversally along a smooth curve $t \mapsto X(t)$. This because at a given time t_0, one may assume (thanks to Galilean invariance) that the velocity $\dot{X}(t_0)$ is zero. Then the field V defined by

  $$V(x,t) := \lim_{r \to 0+} U(rx + X(t_0), rt + t_0)$$

  is a steady triple shock pattern.

The most important consequence of Theorem 2.3 is the von Neumann paradox. Numerical simulations as well as laboratory experiments show that in some parameter regimes, a Mach Reflection takes place for which the flow seems to consist of three shocks separated by smooth regions, without any slip line or rarefaction fan. This is clearly in contradiction with our result. Collela & Henderson [26] suggested that such an irregular Mach Reflection actually contains a very small rarefaction fan that ties the diffracted shock to the now smooth curve formed by the incident shock and the Mach stem. Several authors have performed more and more accurate simulations in order to give a refined description of this pattern, called a *von Neumann Reflection*. An other plausible scenario is the one given by Tesdall & Hunter [71], after careful numerical experiments: There is a supersonic region behind the triple point, which consists of a sequence of supersonic patches formed by a sequence of expansion fans and shock waves that are reflected between the sonic line and the Mach shock. Each of the reflected shocks intersects the Mach shock, resulting in a sequence of triple points, rather than a single one. The numerical results do not indicate whether there are finitely many such triple points or not.

2.2.2 The one-dimensional case

The proof of Theorem 2.3 ressembles that of a well-known result about unsteady flows: In one-space dimension, the interaction of two shocks cannot produce a single shock (see Exercise 4.12 in [64]). Here, u is scalar and each shock has its own velocity. Let us denote $\tau := 1/\rho$. The Rankine–Hugoniot conditions for conservations of mass and momentum are

$$[\rho(u - \sigma)] = 0,$$

from which we define $j := (\rho(u-\sigma))_\pm$ (notice that the shocks are not steady), and

$$j[u] + [p] = 0.$$

We have again $j^2[\tau] + [p] = 0$, and in particular

$$[u] = j[\tau] = \pm\sqrt{[p][\tau]}.$$

Summing circularly, we infer

$$\sum_{\rm circ} \pm\sqrt{[p][\tau]} = 0.$$

The signs in this equality cannot be all equal. Therefore it reads

$$\sqrt{a_2 b_2} = \sqrt{a_0 b_0} + \sqrt{a_1 b_1},$$

where in addition

$$a_j b_j > 0, \qquad \sum a_j = 0, \qquad \sum b_j = 0.$$

Developing, we have

$$(a_0 + b_0)(a_1 + b_1) = a_2 b_2 = a_0 b_0 + a_1 b_1 + 2\sqrt{a_0 b_0 a_1 b_1},$$

from which we obtain $a_0 b_1 + a_1 b_0 = 2\sqrt{a_0 b_0 a_1 b_1}$, whence $a_0 b_1 = a_1 b_0$. Coming back to the definition of the a's and b's, this amounts to saying

$$p_2(\tau_1 - \tau_0) + p_1(\tau_0 - \tau_2) + p_0(\tau_2 - \tau_1) = 0.$$

In other words, the three points (p_j, τ_j) in the (p, τ)-plane are collinear. This implies that the slopes between them be equal. Since these are the j^2's, two of the j's must be equal, for instance $j_0 = j_1$. But this reads $\rho_2(u_2 - \sigma_0) = \rho_2(u_2 - \sigma_1)$. Whence $\sigma_0 = \sigma_1$: two among the three shocks have the same velocity. It is straightforward to conclude that the configuration is trivial.

Proposition 2.1 *Consider unsteady flows in one space dimension. Whatever the equation of state (which could be barotropic or not), there does not exist a pure triple shock configuration satisfying the conservation of mass and momentum.*

2.2.3 Irrotational flows

Slightly more difficult is the proof that a pure triple shock does not exist for potential flows. Again, it is unclear whether the following result has been stated before. The analysis in [54] (Appendix 2) proceeds only for weak shocks. It uses only approximate jump relations, thus it is not a genuine proof.

Theorem 2.4 *Consider a planar steady irrotational flow. There does not exist a pure triple shock structure, that is a piecewise constant flow with only three states separated by straight shocks.*

Proof

We proceed as in the proof of Theorem 2.3, except that the jump relations are

$$[u] \times \nu = 0, \qquad j^2 \left[\frac{1}{\rho^2} \right] + 2[i] = 0$$

and as usual

$$[u] = j \left[\frac{1}{\rho} \right] \nu.$$

Once again, we do not need to know the way i is determined[5] by the other parameters. We thus have

$$\sum_{\rm circ} j_\alpha^2 \left[\frac{1}{\rho^2} \right]_\alpha = 0, \qquad \sum_{\rm circ} \left[\frac{1}{\rho} \right]_\alpha j_\alpha \nu_\alpha = 0$$

while obviously

$$\sum_{\rm circ} \left[\frac{1}{\rho} \right]_\alpha = 0, \qquad \sum_{\rm circ} \left[\frac{1}{\rho^2} \right]_\alpha = 0.$$

Let us define the following nine points

$$P_\alpha := \left(j_\alpha \left\langle \frac{1}{\rho} \right\rangle_\alpha \right)^{-1} \nu_\alpha, \qquad Q_\alpha := j_\alpha \nu_\alpha, \qquad R_\alpha := j_\alpha \left\langle \frac{1}{\rho} \right\rangle_\alpha^{-1} \nu_\alpha$$

The following identities come from above

$$\sum_{\rm circ} j_\alpha^2 \left[\frac{1}{\rho^2} \right]_\alpha P_\alpha = 0, \qquad \sum_{\rm circ} \left[\frac{1}{\rho} \right]_\alpha Q_\alpha = 0, \qquad \sum_{\rm circ} \left[\frac{1}{\rho^2} \right]_\alpha R_\alpha = 0.$$

Since the sums of coefficients in each of these equalities vanish, we deduce that each triplet is collinear: the points P_α lie on a line L_P, the points Q_α belong to a line L_Q and the R_α's are on a line L_R. Since the ν_α's are distinct and unitary, none of these line pass through the origin.

On another hand, each triplet $(\nu_\alpha, P_\alpha, Q_\alpha, R_\alpha)$ is on a ray D_α passing through the origin. Each ray may be identified with the real line, on which we have $\nu_\alpha^2 R_\alpha = P_\alpha Q_\alpha^2$. Up to a rotation we may assume that L_Q has equation $x_1 = s$ with s a nonzero constant. Then the relation between the P's and the R's are

$$R = s^2 \frac{\|P\|^2}{x_{1P}^2} P.$$

Let now $ax_1 + bx_2 = c$ ($c \neq 0$) be an equation of L_R. The above formula shows that in the equation $Ax_1 + Bx_2 - C = 0$ ($C \neq 0$) of L_P, the form $Ax_1 + Bx_2 - C$ must divide the polynomial $s^2(x_1^2 + x_2^2)(ax_1 + bx_2) - cx_1^2$. The ratio would be of degree two, and of valuation two also, hence should be a quadratic form, and more precisely x_1^2. This is clearly impossible.

∎

[5]The context is ambiguous here. On the one hand, we do not use at all the way the enthalpy varies with the density. On the other hand, we assume an equation of state $p = p(\rho)$, otherwise (14) would not hold.

Remark. Theorem 2.4 is a bit astonishing, because there is no possibility to resolve this obstruction by inserting vortex sheets. The latter simply do not exist in potential flows. This suggests that a correct pattern in a Mach stem involves at least a centered rarefaction wave.

2.2.4 The UTSD model

Rosales and Tabak [66] have shown result similar to Theorem 2.3 for the UTSD system (37,38). We borrow our proof from Theorem 11.1 in [82]. Since (37,38) is not Galilean invariant[6], we must consider more general travelling waves $U(x-at, y-bt)$.

Theorem 2.5 *Consider travelling waves of the UTSD model. There does not exist a pure triple shock structure, that is a piecewise constant flow with only three states separated by straight shocks.*

Proof
We may suppose that the wave travels only in the direction y, since there is a Galilean invariance in the x direction. We assume that the shocks are non-trivial: $[u]_\alpha \neq 0$ for $\alpha = 1, 2, 3$. The Rankine–Hugoniot relations read

$$\left[\frac{1}{2}v^2\right]\nu_x + [w - bv]\nu_y = 0, \qquad [w]\nu_x = [v]\nu_y.$$

In particular, the component ν_x is nonzero for each shock. Noticing that $[v^2] = 2\langle v\rangle[v]$, we obtain

$$\langle v\rangle\nu_x^2 - b\nu_x\nu_y + \nu_y^2 = 0,$$

or equivalently

$$\langle v\rangle = bm - m^2,$$

where $m := \nu_y/\nu_x$ is the slope of the shock. Since $[v] = -2[\langle v\rangle]$, this yields

$$[v]_\alpha = 2[m^2 - bm]_\alpha.$$

Using now $[w] = m[v]$, we deduce

$$[w]_\alpha = 2p_\alpha[m^2 - bm]_\alpha.$$

Suming up, there comes

$$\sum_{\text{circ}} m_\alpha[m^2 - bm]_\alpha = 0,$$

which also reads

$$\prod_{\text{circ}}[m]_\alpha = 0.$$

[6]The UTSD model is Galilean invariant in the x direction but not in the y direction.

Therefore at least two shocks are aligned. Since $[u] \times \nu = 0$, this means that two of the jumps $[u]_\alpha$ are collinear. By difference, the third one is also collinear with them, meaning that the three normal are aligned. This is a contradiction.

■

Let us point out that the UTSD model does not allow any kind of contact discontinuity. Since numerical experiments strongly suggest that we do encounter triple-point configurations, we must decide which additional phenomenon prevents Theorem 2.5 to apply, in order to solve this *von Neumann paradox*. Guderley [33, 34] suggested that there is a supersonic region behind the triple point, in which an additional expansion fan develops. The presence of a tiny supersonic region was validated by careful numerical experiments of Hunter & Brio [43]. Next, Tesdall & Hunter [71] refined the calculations and found an array of at least four supplementary triple points, next to the main one along the Mach shock. They are produced by a wave bouncing between the Mach shock and the sonic line. At each bounce, the nature of the wave flips, from shock to expansion fan, and back.

The possibility of a rarefaction fan is ruled out in the transonic case because hyperbolicity of the stationary equations is lost. Then there remains the possibility that the solution be non-smooth at the triple-point. This situation is plausible, after a nice analysis by Gamba & al. [30]. Notice that the v component must remain bounded because of physical considerations, so that the singularity manifests itself at the leading order in the component w only. This is similar to the situation of an elliptic first-order linear system, say the Cauchy–Riemann equations, when one of the components is only piecewise continuous on the boundary. Then the other component experiences a logarithmic singularity at every point of discontinuity of its conjugate.

2.2.5 Four shocks and more

Let us go back to the obstruction found in Paragraph 2.2.1. It is interesting to know whether a pure shock interaction pattern exists with more than three shocks. This question turns out to have a simple answer, found by Bleakney & Taub [6].

Let us anticipate a little bit. We shall construct in Section 3.1 simple patterns called Regular Reflection along a planar wall, denoted by the acronym RR. This patterns is described in Figure 4. The flow is piecewise constant, varying only across two straight shocks that meet at a boundary point. Along the wall, the normal component of the velocity vanishes.

Having in hands such a RR, we may built a four-shocks pure pattern in the plane, by making a reflection across the wall:

$$(x_2 < 0) \Longrightarrow \left(\begin{array}{c} \rho(x) := \rho(x_1, -x_2), \\ e(x) := e(x_1, -x_2), \\ u_1(x) = u_1(x_1, -x_2), \\ u_2(x) = -u_2(x_1, -x_2) \end{array} \right).$$

Hence pure four-shocks patterns do exist, contrary to triple-shock patterns. We point out that the manifold of all pure four-shocks patterns is *a priori* of dimension four (the four states and four shocks being described by 16 scalar parameters, while the Rankine–Hugoniot conditions giving 12 scalar constraints). On the other hand, the symmetric patterns built by Bleakney & Taub are described by four scalar parameters, say the upstream flow and the angle that it makes with the incident shock. Hence it is likely (though this claim would need a more detailed analysis) that every pure four-shock pattern has the symmetric form described above.

This mandatory symmetry makes the class of pure four-shocks patterns useless. An alternative to it consists in introducing a rarefaction fan or a slip line, instead of a fourth shock. The choice of a slip line yields the so-called *Mach Reflection* (MR) ; see Section 3.2. The existence of patterns with three shocks and a rarefaction fan was proved by Bargmann & Montgomery [3]. However, this latter class does not seem as useful as the MR class. Notice that in old papers, as [3, 6], a rarefaction fan is called a *Prandtl–Meyer variation.* It has the property that the component of the flow normal to the radius vector is always sonic ($|u_\theta| = c(\rho)$).

Of course, we may consider a more general pattern organized around a center. Typically, the flow will depend only on the polar angle θ and not on the radius r. Such a flow is made of constant states, shocks, rarefaction fans and slip lines. For a barotropic flow with a reasonable equation of state $p = p(\rho)$, say with $p' > 0$ and $p'' \geq 0$, one proves easily that a shock is adjacent to constant states only. Except across slip lines, the flow is irrotational and thus we can it derive from a potential that is homogeneous of degree one. Along the unit circle, the local extrema of the potential must arise in zones where the flow is uniform.

2.3 The generation of vorticity across shocks

This section is devoted to the barotropic model, for which the vorticity $\omega := \nabla \times u$ obeys the transport equation (13). As seen in Section 1.2, this gives $\omega \equiv 0$, provided the initial vorticity vanishes and the flow remains Lipschitz continuous. In steady flows, (13) reduces to

$$(u \cdot \nabla)\omega + (\mathrm{div} u)\omega = (\omega \cdot \nabla)u,$$

again a transport equation along the flow. Thus ω vanishes downstream if it does upstream and the flow is Lipschitz continuous. Notice that in two space dimensions, ω equals $\partial_1 u_2 - \partial_2 u_1$ and satisfies

$$(u \cdot \nabla)\omega + (\mathrm{div} u)\omega = 0. \tag{61}$$

Notice that if we assume conversely that the flow be steady and irrotational, then we find that the fluid behaves locally like either a barotropic gas or an incompressible one, because of the identity

$$0 = \mathrm{curl}(\rho(u \cdot \nabla)u) = \mathrm{curl}\left(\frac{1}{2}\rho\nabla|u|^2 + \rho\omega u^\perp\right),$$

from which it follows, when $\omega \equiv 0$,

$$\nabla\rho \times \nabla|u|^2 = 0.$$

Thus either ρ is locally constant, or $|u|^2 = h(\rho)$ for some function h, at least locally. But then

$$\nabla p = -\rho\nabla\frac{1}{2}|u|^2, \tag{62}$$

whence

$$p = H(\rho), \qquad H'(s) = -\frac{s}{2} h'(s). \tag{63}$$

We point out however that we do not need that a barotropic equation of state be given *a priori.*

What does happen if the flow experiences a discontinuity, while being irrotational on one side ? The answer depends of course on the type of discontinuity. Along a vortex sheet V, the tangential velocity admits a nonzero jump $[u \times \nu]$ (this is the only discontinuous quantity there), meaning that ω has a non-trivial singular part $[u \times \nu] \otimes \delta_V$.

The situation is a little more interesting across a shock wave S. Our first observation is that because of (48), the vorticity does not have a singular part on S, but is an ordinary function away from slip lines, as long as the flow is piecewise smooth. In the simple case of a planar shock wave separating two constant states, the vorticity is thus identically zero, even in the distributional sense: the flow is irrotational. It turns out that this is not true once the shock is curved. For the sake of simplicity, the following result is stated in two space dimensions.

Theorem 2.6 *Let U be a two-dimensional steady flow, constant on one side Ω^- of a $\mathcal{C}^1$ shock curve S and of class $\mathcal{C}^1$ on the other side Ω^+. If ω vanishes on Ω^+, then S is a straight line.*

Proof

We proceed in two steps. We first show that if ρ has a regular point ($\nabla\rho \neq 0$) at some point $P \in S^+$, then the normal ν to S is constant in a neighbourhood of P. There remains the case where ρ is locally constant, and there we obtain the same conclusion.

We begin with the easier second case. Thus let $\rho \equiv \rho_+$ be constant in Ω^+ near $P \in S^+$, say in D_+. From (62), we have

$$p = p_0 - \frac{1}{2}\rho_+ |u|^2$$

in D_+ (p_0 a constant). Let us write the Rankine–Hugoniot conditions

$$\rho_+ u \cdot \nu = \rho_- u_- \cdot \nu, \quad u \times \nu = u_- \times \nu, \quad [p] = -j^2 \left[\frac{1}{\rho}\right].$$

Elementary calculations yield an equation for the normal (since the flow is constant in Ω^-, $p \equiv p_-$ is constant there)

$$N(\nu) := 2(p_- - p_0) + \rho_+ (u_- \times \nu)^2 + \left(\frac{2}{\rho_-} - \frac{1}{\rho_+}\right) \rho_-^2 (u_- \cdot \nu)^2 = 0.$$

Since N is a quadratic trigonometric polynomial, its roots are isolated. The only exception to this rule is when N is constant, but this means

$$\rho_+ = \left(\frac{2}{\rho_-} - \frac{1}{\rho_+}\right) \rho_-^2,$$

or in other words $[\rho] = 0$. But then we obtain $[u] = 0$ and there is no shock at all.

Since the roots of N are isolated and ν varies continuously along S, we deduce that ν is locally constant, hence S is locally straight.

We turn now to the case where $\nabla\rho \neq 0$ at P. Then we may write locally $|u|^2 = h(\rho)$ and $p = H(\rho)$ with H as in (63), and h, H are $\mathcal{C}^1$ functions. We denote $\theta = h^{-1}$. Let us introduce the sine $s := (u_- \times \nu)/|u_-|$ (with say $\sqrt{1-s^2} = (u_- \cdot \nu)/|u_-|$). We may choose a system of coordinates in which s does not vanish at P. We write the Rankine–Hugoniot equations in the form (recall that $\tau = 1/\rho$)

$$M\begin{pmatrix} j \\ \tau \\ s \end{pmatrix} := \begin{pmatrix} j^2(\tau - \tau_-) - H(\rho) + p_- \\ \tau\theta\left(\tau^2 j^2 + |u_-|s^2\right) - 1 \\ j - \rho_-|u_-|\sqrt{1-s^2} \end{pmatrix} = 0.$$

A lengthy computation gives the Jacobian of M at any root:

$$(M = 0) \Longrightarrow \left(\det(\mathrm{d}M) = s|u_-|^2[\rho]^2(1 + 2j^2\tau^3\theta'(\tau^2 j^2 + |u_-|s^2))\right).$$

The last parenthesis vanishes precisely when the flow on the + side is sonic with respect to the shock (see Section 2.1). When this happens, we find easily that

$$dM = \begin{pmatrix} 2j[\tau] & 0 & 0 \\ -j^{-1} & 0 & -sj^{-2}\rho^2|u_-|^2 \\ 1 & 0 & js/(1-s^2) \end{pmatrix}.$$

The rank of this matrix equals two. Its kernel is spanned by the vector $(0,1,0)^T$. Therefore the derivatives of j and s along S^+ vanish. In particular, ν is constant, so that S is a straight line. ∎

Remarks.

- If the flow is $\mathcal{C}^3$, we may continue the calculation above. Assuming that τ is non-constant along S^+, we find two other equations which involve j, τ, θ' and θ''. Eliminating j and τ, we conclude that
$$\theta\theta'' = 3\theta'^2.$$
This means exactly that the fluid behaves locally like a *Chaplygin gas*. In all other cases, a shock between a constant state and an irrotational flow separates two constant states.

- Since we do not prescribe *a priori* an equation of state, the proof above indicates that the generation of vorticity across curved shocks is, at least at the qualitative level, a phenomenon of kinematical nature. We may see Theorem 2.3 as a variant of Theorem 2.6, because a wedge formed by two shocks can be viewed as a unique shock with curvature concentrated at one point. Thus the impossibility of pure triple shock configuration expresses the fact that the vorticity generation at each of the three wedges cannot cancell.

The barotropic case. When a barotropic equation of state is prescribed, it becomes possible to predict in a quantitative way the generation of vorticity across a curved shock. We again assume that the flow is uniform in Ω^-.

Let τ be the unit tangent to the shock and κ be the curvature. In the following calculations, the dot is the derivative along S with respect to arc length:

$$\dot{\tau} = \kappa\nu.$$

Denoting by $B := i(\rho) + |u|^2/2$ (with $i'(s) = s^{-1}p'(s)$) the Bernoulli invariant, we have

$$\dot{B} = \tau \cdot \nabla B = u \cdot (\tau \cdot \nabla)u + \frac{1}{\rho}\tau \cdot \nabla p = (\nabla u) : (u \otimes \tau - \tau \otimes u) = \omega u \times \tau = j\frac{\omega}{\rho},$$

where we drop the subscript on the plus side. Since the identity

$$[B] = [i(\rho)] - \left\langle \frac{1}{\rho} \right\rangle [p(\rho)] =: F(\rho)$$

holds true, because of (47), this gives

$$F'(\rho)\dot{\rho} = j\frac{\omega}{\rho}. \tag{64}$$

On the other hand, the Rankine–Hugoniot conditions give

$$u_- \cdot \nu = \sqrt{\frac{\rho[p(\rho)]}{\rho_-[\rho]}} =: G(\rho).$$

Differentiating along the shock, we obtain

$$G'(\rho)\dot{\rho} = \kappa(u_- \times \nu),$$

from which we deduce the formula

$$G'(\rho)j\omega = \kappa(u_- \times \nu)\rho F'(\rho). \tag{65}$$

This identity shows that, given the states $\rho_-, \rho = \rho_+, j$ and the direction of the shock, the vorticity on the non-constant side is proportional to the curvature of the shock front. It is also proportional to the tangential component of the velocity, hence vanishes at points where the shock is normal. With Theorem 2.6, this shows that such points are isolated along a curved shock.

2.4 Diffraction for the full Euler system

We aim to give in this section a precise description of the planar steady shocks in which the state U_0 is prescribed on one side, and the unit normal ν pointing from the state U_0 to the state U is given. For convenience, we shall use coordinates (ρ, u, p) in state space.

The important point is that, whenever the shock front is not normal to the speed u_0 (that is, ν is not parallel to u_0, one speaks of an *oblique shock*), the direction of the speed must change across the shock, *i.e.* u_1 is not parallel to u_0. More precisely, the angle between the velocity and the normal to the front is larger downstream[7] than upstream. This is clear from the fact $[\rho u \cdot \nu] = 0$ while $\rho_+ > \rho_-$, hence $|u_+ \cdot \nu| < |u_- \cdot \nu|$, while $[u \times \nu] = 0$. This is the *diffraction* phenomenon, analogous to the one that arises in optics.

Since shock reflection in presence of walls imposes a direction to the velocity in the diffracted flow (because of the boundary condition $u \cdot \mathbf{n} = 0$), it is important to describe quantitatively the diffraction angle across a shock, in terms of the shock strength and of the incidence angle. This is not too hard since an oblique shock, that is a shock where $u \times \nu \neq 0$, is nothing but the superposition of a normal shock (namely one with $u \times \nu = 0$) with an arbitrary (though continuous) tangential component of the velocity $u - (u \cdot \nu)\nu$. Notice that, given U_0 *and* the direction of the shock front, we just have to determine a normal shock, ignoring the tangential part of the velocity ; this normal shock will usually be completely determined by $(\rho_0, u_0 \cdot \nu, p_0)$ and the fact that the shock is steady. The worst situation is when these data lie beyond some threshold, so that there does not exist any such normal shock. This usually happens when the incidence angle is too large, exactly as in optics, where a light ray cannot enter a medium of higher refraction index when the incident angle exceeds a critical angle. This phenomenon is one of the ingredients that are responsible for the transition from RR to MR.

The steady shock curve (full Euler system). Our first observation is that the thermodynamical variables $(\rho, p(\rho, e))$ on the other side of the shock front must belong to the curve (the *Hugoniot curve*) defined by (9), namely (recall that τ denotes the *specific volume* $1/\rho$)

$$e - e_0 + \frac{p(\rho, e) + p_0}{2}(\tau - \tau_0) = 0. \tag{66}$$

On an other hand, the mass flux j across the shock is $j = \rho_0 u_0 \cdot \nu$. Hence the Rankine–Hugoniot condition for the conservation of momentum is linear in (p, τ):

$$j^2(\tau - \tau_0) + p - p_0 = 0 \qquad (j := \rho_0 u_0 \cdot \nu). \tag{67}$$

Conversely, assume that (j, p, τ) satisfies (66, 67). We define u by

$$u \cdot \nu := j\tau, \qquad u \times \nu = u_0 \times \nu.$$

By definition, we have $[\rho u \cdot \nu] = 0$. From (67), we immediately have $[\rho(u \cdot \nu)^2 + p] = 0$. With $[u \times \nu] = 0$, this gives the Rankine–Hugoniot condition for the conservation of momentum. At last, we have

$$\begin{aligned} \left[\frac{1}{2}|u|^2 + e + p\tau\right] &= \left[\frac{j^2}{2}\tau^2 + e + p\tau\right] = [e + p\tau] - \frac{[p][\tau^2]}{2[\tau]} \\ &= [e + p\tau] - \langle\tau\rangle[p] = [e] + \langle p\rangle\,[\tau] = 0, \end{aligned}$$

[7] Notice that we do not specify whether U_0 is downstream or upstream.

which is the Rankine–Hugoniot equation for the conservation of energy.

In summary, a solution (j, p, τ) of (66,67) yields a piecewise constant solution of the full Euler model, where the front is normal to ν. This discontinuity is an admissible shock if j, or equivalently $u_0 \cdot \nu$, has the sign of $\tau_0 - \tau$. In conclusion, the set of admissible shocks with the state U_0 on one side is a curve parametrized by the angle of incidence.

Calculations. We consider the system (66,67) when the equation of state is that of an ideal gas, $p\tau = (\gamma - 1)e$. Then (66) reads $[p\tau] + (\gamma - 1)\langle p\rangle[\tau] = 0$. In other words, $[p]\langle\tau\rangle + \gamma\langle p\rangle[\tau] = 0$. Eliminating $[\tau]$ with (67), we obtain $[p](j^2\langle\tau\rangle - \gamma\langle p\rangle) = 0$. The case $[p] = 0$ corresponds to the trivial pattern where $U = U_0$. Hence non-trivial patterns obey the linear equation $j^2\langle\tau\rangle - \gamma\langle p\rangle = 0$. Using again (67), we obtain the unique value of p:

$$p = (1 - \mu^2)j^2\tau_0 - \mu^2 p_0. \tag{68}$$

We notice that in the limit of an infinitesimal shock, that is $p = p_0$, one has $(1 + \mu^2)p_0 = (1 - \mu^2)j^2\tau_0$, which yields $|u_0 \cdot \nu| = c_0$. In other words, the small shocks behave like sonic waves.

From (68), we derive the value of the specific volume

$$\tau = \mu^2\tau_0 + (1 + \mu^2)j^{-2}p_0. \tag{69}$$

Although (69) gives a positive value, this is not alwaystrue for (68). A value of p is relevant as a pressure only if being positive. Denoting by θ_0 the angle of incidence on the (0) side, namely the angle between the shock normal ν and the velocity u_0, this relevance is equivalent to

$$\sqrt{\frac{\gamma - 1}{2\gamma}} < M_0 \cos\theta_0, \qquad M_0 := \frac{|u_0|}{c_0}, \tag{70}$$

where M_0 is the *Mach number*[8] on the (0) side. We point out that this inequality imposes a lower bound for the Mach number on either side of a steady shock wave:

$$\sqrt{\frac{\gamma - 1}{2\gamma}} < M_0. \tag{71}$$

Notice that (71) is automatically satisfied when the state is supersonic ($M_0 > 1$). For a given state U_0, the Hugoniot curve is parametrized by the incidence angle $\theta_0 \in [0, \theta_{0*})$ where $\theta_{0*} < \pi/2$ is given by the equality in (70). This interval is non trivial when (71) is fulfilled, in particular when U_0 is supersonic. Notice that, contrary to the optics, the maximal reflected angle that is obtained when $\theta_0 = \theta_{0*}$ is smaller than $\pi/2$; it is given by

$$\tan\theta_* = \mu^2 \tan\theta_{0*}.$$

[8]This is an *absolute* Mach number, with respect to the reference frame, rather than a Mach number with respect to the shock, whose value is $M_0(\nu) := |u_0 \cdot \nu|/c_0$. In terms of the latter, the positivity of p amounts to saying that $M_0(\nu) > \sqrt{(\gamma - 1)/(2\gamma)}$. But since we shall let θ_0 vary while keeping u_0 fixed, it is better to make use of M_0, which remains fixed.

From $[u \times \nu] = 0$ and $[\rho u \cdot \nu] = 0$, we get $|u| \sin\theta = |u_0| \sin\theta_0$ and $\rho|u|\cos\theta = \rho_0|u_0|\cos\theta_0$. This gives $[\tau \tan\theta] = 0$. Together with (69), this yields

$$\tan\theta = \frac{\tan\theta_0}{\mu^2 + (1+\mu^2)j^{-2}\rho_0 p_0}.$$

Equivalently, we obtain the relation between the incident and reflected angles:

$$\tan\theta = \frac{(\gamma+1)M_0^2 \sin\theta_0 \cos\theta_0}{(\gamma-1)M_0^2\cos^2\theta_0 + 2}. \tag{72}$$

This yields a function $\theta_0 \mapsto \theta$ that is not necessarily monotonic at a fixed Mach number. However, it has at most one change of monotony.

It is more interesting to consider the *deviation angle* $[\theta] = \theta - \theta_0$. From the addition formula for tangents, and from (72), we obtain

$$\tan[\theta] = 2\frac{M_0^2 \sin\theta_0\cos\theta_0 - \tan\theta_0}{2 + (\gamma-1)M_0^2 + 2M_0^2\sin^2\theta_0} \qquad [\theta] := \theta - \theta_0. \tag{73}$$

Notice that if we use the *relative* (to the shock front) Mach number $m_0 = |u_0 \cdot \nu|/c_0 = M_0 \cos\theta_0$, then Formula (73) reads

$$\tan[\theta] = 2\frac{m_0^2 - 1}{2 + (\gamma+1)M_0^2 - 2m_0^2}\tan\theta_0.$$

This expression is interesting in that it shows that $[\theta]$ vanishes only for $\theta_0 = 0$ (that is for normal shocks) and for $m_0 = 1$. The latter case necessitates $M_0 \geq 1$, and then is always compatible with condition (70).

Variations of $\theta_0 \mapsto [\theta]$. The function $\theta_0 \mapsto [\theta]$ must be studied on the interval $[0, \theta_{0*})$. Its derivative vanishes when $P(m_0^2) = 0$, where P is the polynomial

$$P(X) := 2\gamma X^2 + (4 - (\gamma+1)M_0^2)X - 2 - (\gamma+1)M_0^2.$$

This polynomial has always one positive real root X_0, which corresponds to an admissible angle θ_0 if and only if

$$X_0 \in \left(\frac{\gamma-1}{2\gamma}, M_0^2\right].$$

This means precisely that

$$P\left(\frac{\gamma-1}{2\gamma}\right) < 0 \leq P(M_0^2).$$

These inequalities read $M_0^2 > (\gamma-3)/(3\gamma-1)$ and $M_0^2 \geq 1$ respectively, and the second one implies the first one. Hence the function $\theta_0 \mapsto [\theta]$ is monotonic decreasing when $M_0 \leq 1$ (the state U_0 is subsonic in the reference frame), but has exactly one maximum if $M_0 > 1$ (the state U_0 is supersonic). We recall that in the latter case, U_0 may be either subsonic or supersonic

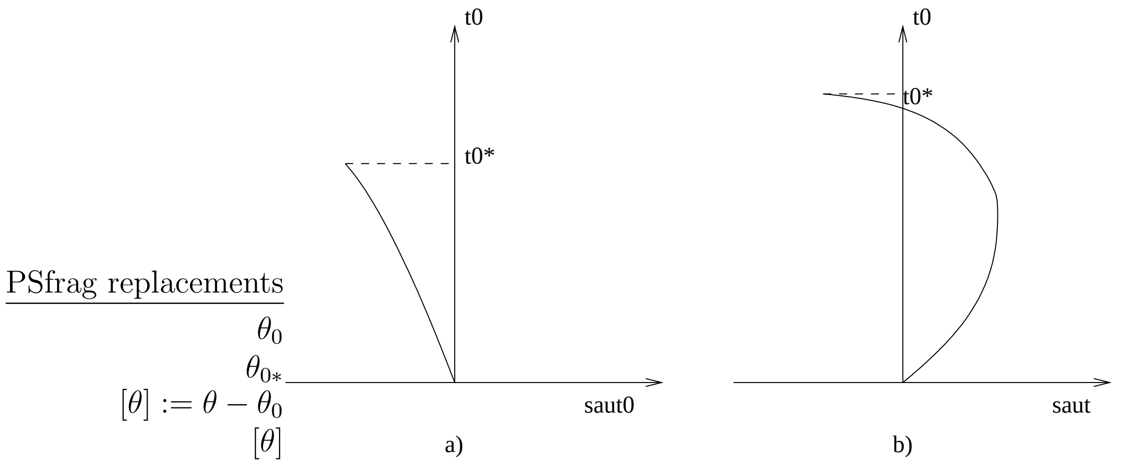

Figure 2: Deviation angle *vs* incidence angle. Left (a): $\sqrt{\frac{\gamma-1}{2\gamma}} < M_0 \leq 1$. Right (b): $M_0 > 1$.

with respect to the shock. Actually, $P(1) = 2(\gamma+1)(1-M_0^2)$ being negative, we see that $\theta_0 \mapsto [\theta]$ is monotonic increasing on the interval $[0, \theta_{0s})$ for which U_0 is supersonic with respect to the shock. In particular, we have

$$[\theta] \sim 2\frac{M_0^2 - 1}{2 + (\gamma-1)M_0^2}\theta_0$$

for small incident angles (almost normal shock).

Shock polar. We summarize the above calculations.

If $M_0 \leq \sqrt{\frac{\gamma-1}{2\gamma}}$, there does not exist any steady shock from U_0.

If $\sqrt{\frac{\gamma-1}{2\gamma}} < M_0 \leq 1$, the interval $[0, \theta_{0*})$ is non-trivial, and $\theta_0 \mapsto [\theta]$ is monotone decreasing. Both θ_{0*} and the corresponding reflected angle θ_* are less than $\pi/2$.

If $M_0 > 1$, the function $\theta_0 \mapsto [\theta]$ is monotone increasing until a critical angle, and then is decreasing towards a negative value.

Notice that in all non-trivial cases, the deviation is negative at the end point, meaning that $\theta_* < \theta_{0*}$. The patterns are represented in Figure 2, called shock polars.

Small strength. We say that a shock has small strength if $[U]$ is small, but the incidence θ_0 is not. In particular, $[\theta] \ll 1$, meaning that $m_0 \sim 1$. Such a shock is therefore almost sonic and is close to an acoustic wave[9]. We have $M \sim M_0$, and since one of the states must be supersonic, we see that *both states are absolutely supersonic*:

$$M > 1, \qquad M_0 > 1.$$

[9] A *acoustic wave* is a continuous, piecewise $\mathcal{C}^1$, solution of which the pressure is not $\mathcal{C}^1$.

On the other hand, both m and m_0 are close to one, but they are on opposite sides of the unity:

$$(m-1)(m_0-1)<1,$$

as it must be for every steady shock.

Small *vs* strong shocks. Small shocks correspond to the part of the curve around its intersection with the upper semi-axis in Figure 2 (b). The deviation angle of such a shock is small, as well as the jump of the state ; as mentioned above, the shock is approximately a sonic wave, meaning that $c \sim c_0 \sim u \cdot \nu \sim u_0 \cdot \nu$. More generally, we say that a shock is *small* if it corresponds to an angle θ where the slope of the shock polar is negative, while it is *strong* if this slope is positive. With this terminology, a normal shock is always strong, although we may built a one-parameter family of normal shocks whose strength tends to zero at some value of the parameter. The flow behind a strong shock is always absolutely subsonic (see [72]).

2.5 Diffraction for a barotropic gas

When the fluid is barotropic, we have only the conservation of mass and momentum, which yield the Rankine–Hugoniot conditions

$$[\rho u\cdot\nu]=0,\qquad [\rho(u\cdot\nu)u]+[p]\nu=0.$$

With the same notations as in the previous section, they read

$$j:=(\rho u\cdot\nu)_\pm,\qquad j[u]+[p]\nu=0.$$

Once again, we have $[u\times\nu]=0$, from which we derive easily

$$\tan\theta=\phi^{-1}\tan\theta_0,\qquad \phi:=\frac{\rho}{\rho_0}.$$

We determine the ratio ϕ through

$$j^2[\tau]+[p]=0,$$

which we rewrite as

$$\phi\frac{[p]}{[\rho]}=c_0^2M_0^2\cos^2\theta_0,\qquad M_0:=\frac{|u_0|}{c_0}. \tag{74}$$

For a perfect gas with $p(\rho)=\rho^\gamma$, this reads

$$F(\phi)=M_0^2\cos^2\theta_0,\qquad F(s):=\frac{s(s^\gamma-1)}{\gamma(s-1)}. \tag{75}$$

As before, it is more interesting to work with the angle of deviation $[\theta]$:

$$\tan[\theta]=\frac{(1-\phi)\tan\theta_0}{\phi+\tan^2\theta_0}. \tag{76}$$

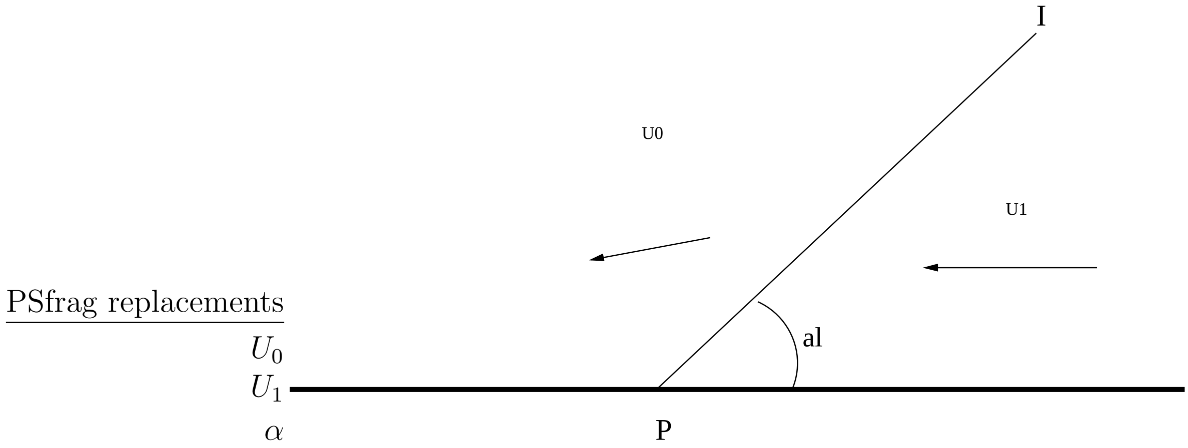

Figure 3: Incident shock along a planar wall (the reflected pattern is to be determined).

The same description as in the previous section holds. The deviation angle vanishes for $\theta_0 = 0$ (normal shock). It vanishes also if $\phi = 1$ (infinitesimal shock), a property that happens, when $M_0 > 1$, for some angle that solves

$$\cos\theta_0 = \frac{1}{M_0}.$$

The shock polar has the same general form (see Figure 2) as in the case of the full Euler system.

Notice that the relative Mach number $m_0 := M_0 \cos\theta_0$ is not limited here, contrary to the general case.

3 Reflection along a planar wall

Let us consider the case where the physical domain is a half-plane, bounded by a rigid wall, say the horizontal axis. We give ourselves (see Figure 3) an *incident* steady shock I. This means that the states U_0 and U_1, as well as the angle α, are given. By *incident*, we always mean that $\alpha \in (0, \pi/2)$ and U flows *into* the shock. With our convention, we have $(U_-, U_+) = (U_1, U_0)$ with respect to the incident shock.

Although we shall not make use of the data U_1 and α, it is worth saying that $(U_0, U_1; \alpha)$ satisfies the Rankine–Hugoniot relations and the entropy condition, together with the natural boundary condition that u_1 is parallel to the boundary: $u_1 \cdot \mathbf{n} = 0$.

In general[10], U_0 does not satisfy the slip boundary condition and therefore the picture given in Figure 3 is not complete. We are thus looking for a *reflected shock* R, that is a state U behind U_0, and an angle β, which satisfy the Rankine–Hugoniot relations and the entropy condition ; naturally, we require the boundary condition that u is parallel to the boundary (see Figure 4 for the simplest pattern).

[10]It may happen exceptionally that $u_0 \cdot \mathbf{n} = 0$: When $M_1 > 1$, there is a critical angle α_c for which the deviation angle is zero (see Figure 2 (b).)

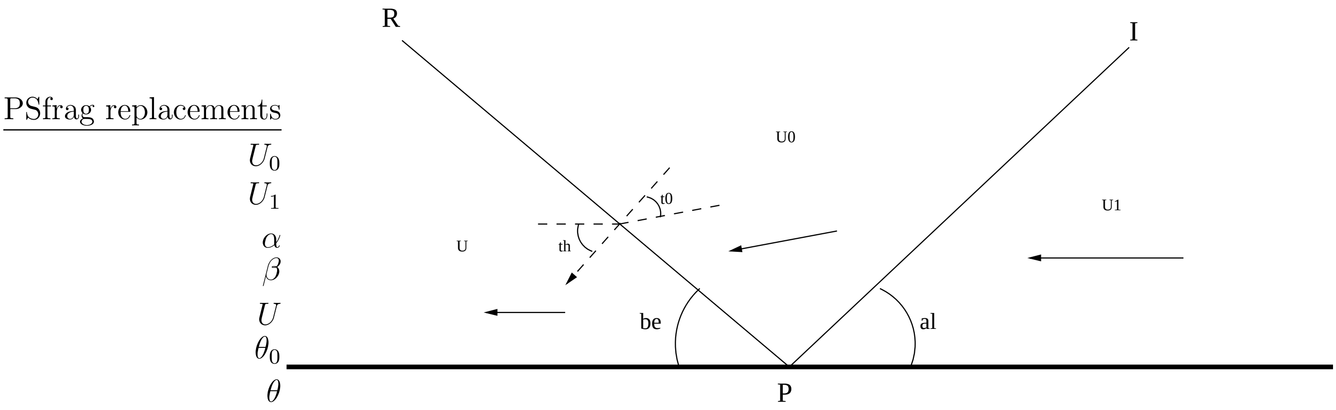

Figure 4: Regular Reflection (RR) along a planar wall.

Remark. The state U_0 will flow *into* the reflected shock ; thus U_0 will play the role of U_- with respect to the reflected shock, contrary to what held along the incident shock. We summarize this important remark in the formula

$$U_-^R = U_+^I. \tag{77}$$

We notice that, U_0 being supersonic with respect to the reflected shock, we need that M_0 be larger than one[11]. We have seen above that this requirement is automatically satisfied if the strength of the incident shock is weak enough.

3.1 Regular Reflection

Figure 4 displays the *Regular Reflection* (RR), where the exact solution of our problem is piecewise constant, the only discontinuities being the straight lines where the incident and reflected shocks take place. The shocks meet at a point located along the boundary, which can be taken as the origin.

The problem to solve: Since U_0 is given, and in particular its angle δ with respect to the wall, we only have to select a steady shock from U_0 to a state U, such that the deviation $[\theta]$ across the shock equals δ. Coming back to Formula[12] (73), we have to solve the equation

$$2\frac{M_0^2 \sin\theta_0 \cos\theta_0 - \tan\theta_0}{2 + (\gamma-1)M_0^2 + 2M_0^2 \sin^2\theta_0} = \tan\delta, \qquad \theta_0 \in [0, \theta_{0*}(M_0)), \tag{78}$$

where θ_0 is the (unknown) incidence angle of U_0 in the reflected shock. This amounts to taking the intersection of the shock polar (Figure 2) with the vertical line of abcissa δ. Then the

[11] Hence U_0 is absolutely supersonic, though subsonic relatively to the incident shock.

[12] For definiteness, we consider a perfect gas obeying the full Euler system. We leave the reader to carry out the calculations in the barotropic case. The results are qualitatively similar.

reflected angle β will be given by

$$\beta = \frac{\pi}{2} - \theta_0 - \delta. \tag{79}$$

We point out that, since U_0 is given as part of a steady shock (the incident one), (71) holds true. Therefore θ_{0*} is positive and the interval $[0, \theta_{0*})$ where the left-hand side of (78) is defined is non trivial.

Since the range of $\theta_0 \mapsto [\theta]$ is strictly contained in $(-\pi/2, \pi/2)$, we see that there is a critical angle δ_c beyond which (78) has no solution. In this regime, there is a need for a more complicated pattern, say a Mach Reflection (see Section 3.2). Actually, experiments suggest that the transition from RR to MR happens at some angle δ_t strictly smaller than δ_c. In this situation, a Mach Reflection occurs, while a Regular Reflection is still theoretically possible. This anticipated transition must be due to some instability, but has not been explained rigorously so far.

3.1.1 Weak and strong reflections.

Although small values of δ do not always correspond to a weak incident shock, this will be the case in RR, according to both physical and numerical experiments. Hence we shall restrict to this case. As seen above, we have $M_0 > 1$ and $M > 1$. The figure 2 (b) is thus relevant for both the incident and the reflected shocks. In the former, the angle of incidence is close to that at which the curve intersects the upper semi-axis. Hence α is not close to $\pi/2$: the incident shock is significantly not perpendicular to the wall. The deviation δ may be either negative or positive (the figure 3 displays a positive δ).

Concerning the reflected shock, the situation depends whether δ is positive or negative. In the former case, there are two solutions of Equation (78). One of them is small ($\theta_0 \ll 1$), meaning that β is close to $\pi/2$. We warn the reader that the corresponding $[U]$ is not small, and we call this a *strong reflection*. The other solution is associated to a weak shock and is called a *weak reflection*. For a negative δ, there is only one solution, which is the weak one.

The physical and numerical experiments suggest that a strong reflection, though perfectly defined at a mathematical level, does not happen in practice. This might be related to a Hadamard instability with respect to the evolutionary problem. Such instability is not that of one of the shock waves, these being individually stable, as shown by Majda [52]. It must be a property of the whole pattern formed by the initial-boundary value problem and both shocks. This phenomenon has been investigated by Teshukov [73], se Paragraph 3.3.1.

We therefore restrict in the sequel to the case of weak RR. Notice that if the incident shock strength is small, both incident and reflected shocks are approximately sonic waves. In particular, we have $u_0 \cdot \nu_R \sim c_0 \sim u_0 \cdot \nu_I$. Since δ is small, that is u_0 is approximately parallel to the wall, this reads $M_0 \sin\beta \sim M_0 \sin\alpha$. We conclude that a weak RR with small incident strength follows approximately the law of optical (specular) reflection:

$$\beta \sim \alpha. \tag{80}$$

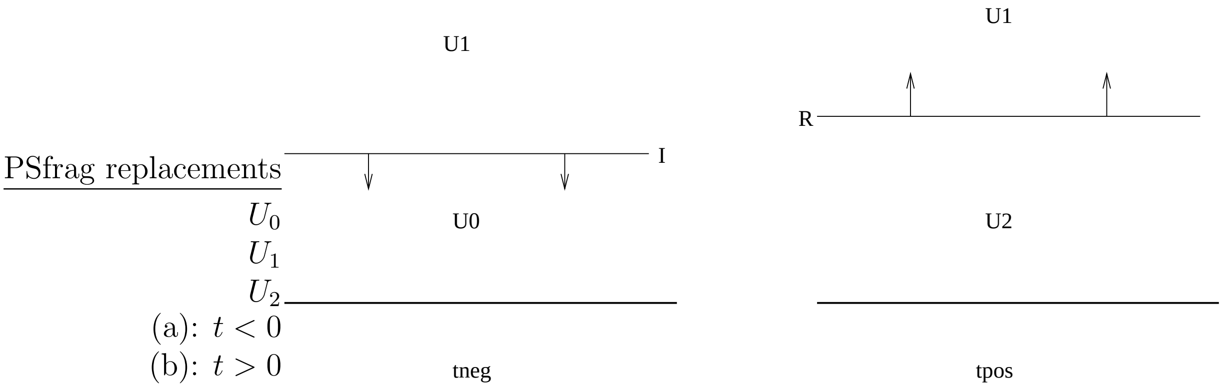

Figure 5: A normal reflection. The shock bounces on the wall at time $t = 0$.

Transonic *vs* supersonic RR. In a RR, we know that the state U_2 behind the reflected shock R is subsonic with respect to R. However, it may happen that this state be either subsonic (if $|u_2| < c_2$) or supersonic (if $|u_2| > c_2$) with respect to the reference frame. In the first case, we say that the reflection is *transonic*. In the latter case, we say that it is a *supersonic* reflection. Teshukov [72] has shown that under a mild assumption on the equation of state, a strong RR is always transonic.

3.1.2 Normal reflection

When we let the angle α tend to zero, a limit situation happens, where it is not possible any more to keep the reflection point at a finite distance : the incident shock becomes parallel to the wall and bounces against the wall at time $t = 0$. For this reason, the strong reflection does not happen (another insight that a strong reflection is not physical) and the reflected shock is parallel to the wall too. However, the incident shock is present only at negative times, while the reflected one is there only at positive times (Figure 5).

In this problem, the state U_1 is the only data. It defines a Hugoniot curve of normal shocks, parametrized by the shock velocity s. The rest states U_0 and U_1 are found by saying that their velocities vanish. For a reasonable equations of state, there is exactly one solution with a positive s (the incoming shock) and one with a negative s (the reflected shock). Let us see the calculations.

Barotropic gas. Let U be the rest state and j be the net flow across the shock. We identify the velocitiy vectors with their normal component. We have $j = \rho_1(u_1 - s) = -\rho s$ and $p(\rho) = p(\rho_1) + ju_1$. Eliminating j and s, we arrive at the equation

$$p(\rho) = p(\rho_1) + \frac{\rho\rho_1}{\rho - \rho_1} u_1^2. \tag{81}$$

Let $g(\rho)$ denote the right-hand side of (81). Apart from a ole at $\rho = \rho_1$, this is a decreasing function (see Diagram 6). When p is monotone increasing, Equation (81) admits precisely two

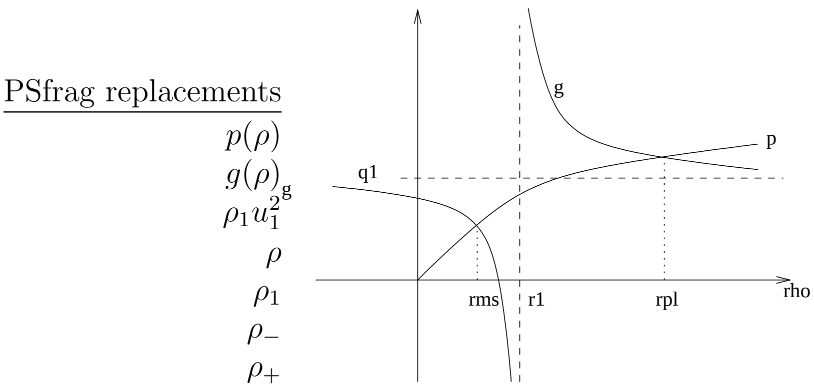

Figure 6: Normal reflection for a barotropic gas with $p' > 0$. The rest densities at positive (ρ_-) and negative (ρ_+) times.

solutions ρ_0 and ρ_2 with $\rho_0 < \rho_1 < \rho_2$. There correspond the rest states U_0 and U_2, with corresponding shock speeds $s_I > 0$ and $s_R < 0$.

Full Euler system. The identity (81) still hold true, except that now $p = p(\rho, e)$ and $p_1 = p(\rho_1, e_1)$. We have to solve the system made of (81) and the analogue of (66):

$$e - e_1 + \frac{p(\rho, e) + p_1}{2}(\tau - \tau_1) = 0. \tag{82}$$

Let us define $g(\rho, e) := p - p_1 - \frac{\rho\rho_1}{\rho - \rho_1}u_1^2$, which is a piecewise increasing function of ρ (provided $p_\rho > 0$, with a pole at $\rho = \rho_1$ and

$$g(0, e) = -p_1 < 0, \quad g(\rho_1 - 0, e) = +\infty, \quad g(\rho_1 + 0, e) = -\infty, \quad g(+\infty, e) = +\infty.$$

Equation (81) thus defines two functions $\rho_-(e) < \rho_1 < \rho_+(e)$, which are decreasing under the natural assumption that $p_e > 0$. In terms of the specific volume, we have increasing functions $\tau_+(e) < \tau_1 < \tau_-(e)$. Because of (81)), solving (82) amounts to solving

$$e + p_1\tau_\pm(e) = e_1 + p_1\tau_1 + \frac{1}{2}u_1^2. \tag{83}$$

Since the left-hand side is an increasing function of e, it is enough to verify that

$$p_1\tau_\pm(0) < e_1 + p_1\tau_1 + \frac{1}{2}u_1^2.$$

This is true provided

$$e_1 > \frac{1}{2}\frac{u_1^2}{p_1}, \tag{84}$$

because $\tau_-(0) = \tau_1 + u_1^2/p_1$ and $\tau_+(0) = 0$. Condition (84) is precisely that which ensures the existence of an incoming normal shock with downstream state U_1. As a matter of fact,

$$s = -\frac{\rho_1 u_1}{[\rho]}$$

is of the sign opposite to that of $[\rho]$. Hence ρ_- corresponds to $s > 0$, that is to the incoming shock, while ρ_+ correspond to the reflected shock.

> In summary, for reasonable equations of state, a normal incident shock always result in a uniquely defined normal reflected shock. This one is the limit of the weak Regular Reflection as the angle α tends to zero.

3.1.3 Regular Reflection for a barotropic gas

For a barotropic gas, the equation to solve is

$$\frac{(1-\phi)\tan\theta_0}{\phi + \tan^2\theta_0} = \tan\delta,$$

where ϕ and θ_0 are related through (75). Recall that M_0 and δ are given, and that θ_0 is our unknown. This problem amounts, as above, to finding the intersection of a shock polar with a vertical line of abcissa δ, yielding a strong and a weak reflection. On an analytical level, $\tan\theta_0$ may be eliminated, thanks to the formula

$$1 + \tan^2\theta_0 = \frac{1}{\cos^2\theta_0} = \frac{M_0^2}{F(\phi)}.$$

There remains an "algebraic" equation in ϕ, of degree $2(\gamma + 1)$. For a polytropic gas with D degrees of freedom, this is a genuine algebraic equation in $\phi^{2/D}$, since $\gamma = 1 + 2/D$.

3.2 Mach Reflection

When the incident shock is too strong, or the angle α between the front and the wall is too large, the RR does not happen, either because Equation (78) does not have a solution, or because its solutions yield unstable patterns (see Section 3.3). Experiments suggest that the reflection point P is removed from the wall. There is a reflected shock R, but the point P where I and R meet is not any more along the boundary. Of course, the extreme states U_1 and U_2 are distinct (as they were in the simpler situation of the RR) and there is a need of a third shock to match them. Thus we expect a triple shock pattern at point P. Since we have proved in Section 2.2 that a pure triple shock structure does not exist, we also need a fourth wave at P. The simplest possibility is that of a slip line V (for *vortex sheet*), giving rise to a *Mach stem* shown in Figure 7. This terminology is due to von Neumann, after the experiments by Ernst Mach [51].

At a first glance, the Mach stem is a piecewise constant steady solution, thus has the advantage of being explicit (algebraic calculations). However, its flaw is that it cannot fit the boundary condition $u\cdot\mathbf{n} = 0$. Were U_3 to satisfy the boundary condition, V would be horizontal

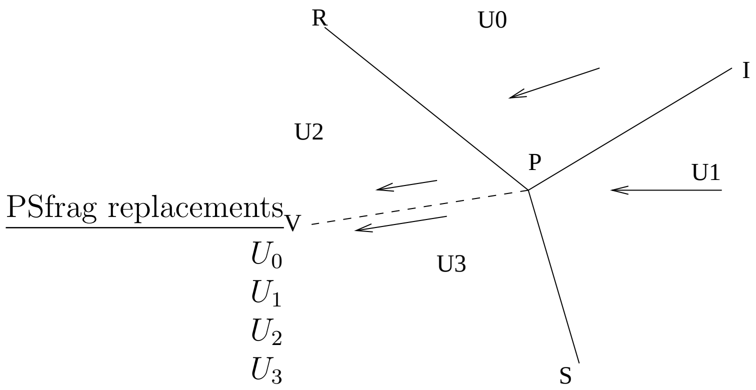

Figure 7: A Mach stem. The velocities u_2 and u_3 are parallel though not equal. The pressures p_2 and p_3 are equal, but the entropies S_2 and S_3 are not.

since it is parallel to the flow ; but then U_2 would be horizontal too so that I and R could be used to make a RR. Hence the third shock would be useless. This argument also shows that in a Mach stem, the slip line cannot be horizontal.

Another difficulty in the matching of the Mach stem with the boundary condition is that since U_3 and U_1 are parallel, the third shock S must be normal. If S is straight, it must be vertical, but this is a severe restriction that makes the construction overdetermined.

From this analysis, we must take the following conclusion. First of all, the shock reflection is not piecewise constant. Although the shock S and the slip line V may be straight near P when either of U_2 or U_3 is supersonic, they must be curved when approaching the boundary. In particular, the tangent to S at P can be determined algebraically using shock polars, while it is perpendicular to the wall. Next, the solution is unsteady. What we may expect in general is a self-similar solution generated, for instance, by the interaction of the incident shock with a very thin wedge. Then the state $U(x,t)$ will depend only on the self-similar variable $y = x/t$. As a matter of fact, it is hard to imagine a physical experiment yielding automatically a steady shock reflection. When a RR occurs, it is only as part of a self-similar solution, and it actually travels at constant speed in the laboratory frame.

3.3 Uniqueness of the downstream flow in supersonic RR

Let us consider a supersonic RR, meaning that we assume that the flow U_2 behind the reflected shock is supersonic ($|u_2| > c_2$). In particular, the RR is a weak one. For instance, in a near-to-normal weak RR, u_2 is very large (in the reference frame where P is fixed) and therefore we do have $|u_2| > c_2$.

Under this assumption, the steady Euler system is hyperbolic in the direction of the flow, that is in the direction of the horizontal axis. Thus we may view this system as an evolution system of PDEs, and consider the determination of the downstream state and of the reflected shock as a kind of Boundary Value problem (BVP). Actually, since R is a free boundary, this is a Free Boundary Value Problem (FBVP), with data U_1 and boundary conditions. The latter

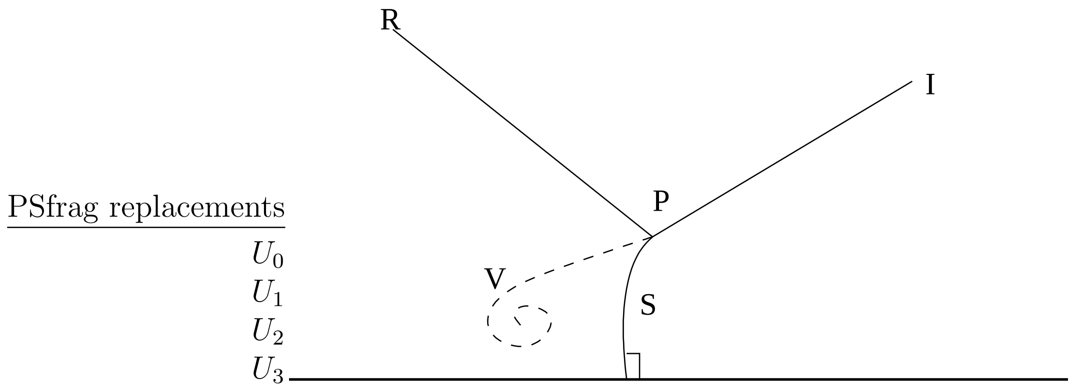

Figure 8: A Mach stem in presence of a boundary. The vortex sheet and the shock S are curved.

are the no-flow condition along the wall, and the Rankine–Hugoniot condition across R.

We constructed in Section 3.1 a solution (actually *two* solutions) of this FBVP by assuming *a priori* that R was straight and the downstream flow uniform. These assumptions were natural since the incident pattern is self-similar and the steady Euler system is invariant under space dilations. In particular, if a germ of solution defines uniquely the solution, then the latter must be self-similar, which means that R is straight and the downstream flow is uniform. The purpose of this paragraph is to get rid of these assumptions, by showing that in reasonable regimes, they are necessary. In other words, we are concerned with a uniqueness problem.

The FBVP has a rather special form, since the boundaries meet at initial "time". They form a wedge at P, so that there is no "initial data". We assume however that the solution is piecewise smooth ; in particular, the downstream state has a limit $U(P)$ and R has a tangent $\mathcal{R}$ at P. From the invariance of the Euler equations under dilation, we see that $U(0)$ and $\mathcal{R}$ do provide a uniform solution of the FBVP, that is a uniform RR. Thus $U_2 := U(0)$ must be one of the states computed in Section 3.1. The behaviour at P of any piecewise smooth solution is given therefore by either the weak RR or the strong RR. Our problem here is whether such a limit value *must* extend downstream. This is a uniqueness question concerning the FBVP, related to the well-posedness. The general theory for that kind of problem has been developed by Li & Yu [65].

Since the FBVP is quasilinear and first-order, we shall analyze first the linearized problem at the state (weak or strong RR) $U_2 = (\rho_2, u_2, e_2)$. It consists of linearized PDEs. For the sake of simplicity, we restrict to the barotropic case. The PDEs are

$$u_2 \cdot \nabla\rho + \rho_2 \mathrm{div} u = 0, \qquad \rho_2(u_2 \cdot \nabla)u + c_2^2 \nabla\rho = 0. \tag{85}$$

Along the ramp, we just write

$$u \cdot \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = 0,$$

while along R we set the linearized Rankine–Hugoniot condition. After elimination of the increment $\delta\nu$ of the normal, this reads $MU = 0$ with M a 2×3 matrix. Using characteristic components (U_-, U_0, U_+), these boundary conditions are respectively[13]

$$U_+ = c_0 U_0 + c_- U_-$$

along the ramp, and

$$U_0 = b_0 U_+, \quad U_- = b_- U_+$$

along R.

We describe now the analysis of [65]. The solution in the interior is completely determined thanks to (85), once we know the trace of U_- on R and of U_0, U_+ on the ramp. Denoting by $t > 0$ a time-like variable ($t = 0$ at P), these traces $f_-(t), f_0(t), f_+(t)$ are to be determined by a kind of difference equations:

$$f_+(t) = c_0 f_0(\sigma_0 t) + c_- f_-(\sigma_- t), \quad f_0(t) = b_0 f_+(\theta_0 t), \quad f_-(t) = b_- f_+(\theta_+ t), \tag{86}$$

where the numbers $0 \leq \sigma_0, \ldots, \theta_+ < 1$ involve the characteristic velocities and the slopes of the boundaries, while the coeffcients $b_0, \ldots, c_-$ come from the boundary conditions. There remains to check the local well-posedness of the linear problem (86). The appropriate tool is the Laplace transform in terms of the logarithm of t. A natural space is $L^2(dt/t)$, though we may also consider Sobolev spaces associated to the same weight, and we need them when considering the non-linear well-posedness. The linear BVP is strongly well-posed in these spaces if and only if the modulus of the following *Evans function* is bounded away from zero in the right complex half-plane ($\Re z \geq 0$):

$$\Delta(z) := 1 - b_0 c_0 (\sigma_0 \theta_0)^z - b_- c_- (\sigma_- \theta_+)^z.$$

It is strongly ill-posed in Hadamard's sense (lack of uniqueness) if Δ vanishes somewhere in the open half-space ($\Re z > 0$). Remark that $\Delta(z)$ vanishes precisely when (86) admits a non-trivial solution homogeneous of (complex) degree z.

For general coefficients b, c, σ and θ, it is not easy to determine whether Δ vanishes in the closed right-half plane. Since we clearly have

$$|\Delta(z)| \geq 1 - |b_0 c_0| - |b_- c_-|,$$

one often contents oneself with the sufficient condition $|b_0 c_0| + |b_- c_-| < 1$ for uniform well-posedness. However, in the present problem, a simplification arises, because $c_0 = 0$, while $\sigma_- \theta_+ > 0$. Then $\Delta(z) = 1 - b_- c_- (\sigma_- \theta_+)^z$. Therefore it becomes clear that a necessary and sufficient condition for strong well-posedness is

$$|b_- c_-| < 1. \tag{87}$$

Remark that although b_- and c_- depend on the normalization of the eigenmodes of System (85), the amplifying ratio $b_- c_-$ does not.

[13]The fact that U_+ can be taken as the output on the ramp, or the input on the shock, amounts to the normality of the boundary conditions.

Eigenmodes. We begin by computing the eigenmodes and characteristic directions of (85). This amounts to solving

$$(u_2\cdot\xi)\rho+\rho_2 u\cdot\xi=0,\qquad \rho_2(u_2\cdot\xi)u+c_2^2\rho\xi=0.$$

The characteristics are either parallel to the flow ($u_2\cdot\xi=0$) or normal to the vectors $\xi_\pm$ given by $u_2\cdot\xi=c_2|\xi|$. We normalize $|\xi_\pm|=1$, so that $u_2\cdot\xi_\pm=c_2$. Remark that this equation has two solutions because of our assumption $c_2<|u_2|$. The corresponding modes are, in (ρ,u) variables,

$$r_\pm=\begin{pmatrix}-\rho_2\\ c_2\xi_\pm\end{pmatrix},\qquad r_0=\begin{pmatrix}0\\ u_2\end{pmatrix}.$$

We point out that this eigenbasis becomes ill-conditioned as we approach a sonic shock, since we have $r_+=r_-$ in the limit.

Calculation of c_0 and c_-. Let us write the boundary condition $u\cdot\nu=0$ along the wall, with $U=U_-r_-+U_0r_0+U_+r_+$:

$$c_2(U_+\xi_++U_-\xi_-)\cdot\nu+U_0u_2\cdot\nu=0.$$

Since we already have $u_2\cdot\nu=0$, this reduces to

$$U_+\xi_+\cdot\nu+U_-\xi_-\cdot\nu=0,$$

confirming that $c_0=0$. We obtain additionally that $c_-=-(\xi_-\cdot\nu)/(\xi_+\cdot\nu)$. Since ξ_+ and ξ_- are symmetric to each other with respect to u_2, that is to the wall, we deduce

$$c_-:=1. \tag{88}$$

Calculation of b_-. We now perform the calculation of b_-. We start with the linearized boundary condition

$$\begin{aligned}\rho u_2\cdot\nu+\rho_2u\cdot\nu+[\rho u]\cdot\delta\nu&=0, &(89)\\ \nu\cdot\delta\nu&=0, &(90)\\ \rho_1u_1\cdot\delta\nu[u]+ju+[p]\delta\nu+c_2^2\rho\nu&=0. &(91)\end{aligned}$$

We have used the following notations: $[p]=p(\rho_2)-p(\rho_1)$ and so on, while $\delta\nu$ is the increment of the unit normal vector ν. We point out that since $\delta\nu$ is a tangent vector, we have $[u]\cdot\delta\nu=0$. In particular, the last term in (89) can be written $[\rho]u_2\cdot\delta\nu$, while in (91), we may replace u_1 by u_2.

Let us investigate first the case of an *infinitesimal incident shock*. Recall that there are two cases: either the reflected shock is strong, or it is weak. Since the the downstream flow is subsonic in the former case[14], we concentrate to the latter case. Then the reflected shock is infinitesimal too, hence is characteristic. This makes our FBVP in a corner a little bit harder

[14]The reflected shock in a strong RR is almost normal, thus $|u_2|\sim|u_2\cdot\nu|<c_2$.

to analyze. However, we can use the fact that the background state is constant everywhere. Therefore, our stability problem is equivalent to the stability of the constant state $(\rho_2, u_2) = (\rho_1, u_1) = (\rho_0, u_0)$ in a standard IBVP, where the "time variable" is $-x_1$ and the boundary is $x_2 = 0$. This 1-d IBVP needs exactly one boundary condition, a normal one, a fact that is easy to check. In conclusion, our FBVP is well-posed in this limit case.

We now pass to the general case. To understand what is going on, it is easier to concentrate first on the problems which are transitional between the strongly well-posed ones and the strongly ill-posed. This means $b_- = \pm 1$.

If $b_- = -1$**,** the solution $(U, \delta\nu)$ of (89,90,91) satisfies $U = U_+(r_+ - r_-) + U_0 r_0$, meaning that $\rho = 0$. We easily obtain that $u_2 \parallel \nu$; that is, the shock is normal. This is not the case, for any weak or strong RR. We point out that this situation occurs only in the limit of a strong RR, when the incident shock strength tends to zero.

If $b = 1$**,** we have $U = U_+(r_+ + r_-) + U_0 r_0$, meaning that $u \parallel u_2$. After some calculations that we leave to the reader, we find that a transition occurs when the following quantity vanishes

$$G := ((u_2 \cdot \nu)^2 + c_2^2)(u_2 \times \nu)^2 + ((u_2 \cdot \nu)^2 - c_2^2)(u_2 \cdot \nu)(u_1 \cdot \nu). \qquad (92)$$

The conclusion of this analysis is that Property (87) depends only on the sign of G. Notice that, since $|u_2 \cdot \nu| < c_2 < |u_2|$, the sign of G is not obvious. We have shown above that the FBVP is well-posed for the weak reflection of an infinitesimal incident shock. Hence we have to determine sign$G$ in this limit case, in order to decide which sign of G gives well-posedness : we have $|u_2 \cdot \nu| = c_2$ (an infinitesimal shock is characteristic) and $u_1 = u_2$. Whence $\lim G = 2c_2^2(u_2 \times \nu)^2$ is positive. Thus we obtain

Theorem 3.1 *Consider a supersonic RR for a barotropic gas. Then the FBVP (96,97), in the wedge defined by P, the wall and the reflected shock, is strongly well-posed locally at P whenever G, defined by (92), is positive.*

This FBVP is strongly ill-posed if $G < 0$.

Applications. We have seen above that at fixed angle α, the weak reflection is strongly stable as the incident strength goes to zero. Consider now the weak reflection for a quasi-normal shock. In the limit $\alpha \to 0$, we find that P escapes to infinity, while the reflected shock tends to that of the perfectly normal reflection, thus remains at finite distance of the origin. This shows that $|v_2 \times \nu| \to \infty$, while $|v_2 \cdot \nu| = O(1)$. Therefore

$$G \sim ((v_2 \cdot \nu)^2 + c_2^2)(v_2 \times \nu)^2$$

is positive. We conclude that, given an incident shock, the FBVP associated to the weak shock reflection is strongly well-posed when the angle α is small enough.

Non-linear stability. The passage from linear stability to the nonlinear one is performed by a fixed point argument and integration along charcteristics. See [65] for details. Strong linear well-posedness implies nonlinear well-posedness. The limit case when $b_- c_- = 1$ is unclear. Strong linear instability is likely to imply ill-posedness.

Uniqueness and constancy of the downstream flow. From the non-linear stability, we deduce that, under the assumption that G is positive at (U_2, ν), there is only one stationary solution downstream the state U_1, achieving the limit U_2 at P. By a solution, we mean a flow U of the Euler system, together with a reflected shock (a free boundary in this problem) between U and U_1. This unique solution is made of the constant state U_2 and the straight reflected shock with normal ν.

3.3.1 Dynamical stability

Teshukov has considered in [73] the most important *dynamical stability* of the RR as a stationary solution of the (full) Euler system. For this purpose, he linearizes the system about the RR. This resembles our analysis above, but with an extra $\partial_t U$. A Laplace transform in time replaces this term by τU, and one is reduced to the study of a stationary PDE system, where additional terms come from the fact that the shocks are free boundaries originally. The problem is to get an estimate in $L^2(H)$ (H the physical half-plane), uniform with respect to τ when $\Re\tau > 0$. Notice that the limit case where $\tau = 0$ yields exactly the same system as the one we studied in Section 3.3.

Since the background state is constant along rays, it makes sense to use polar coordinates (r, θ), at the expense that the new problem has variable coefficients. Teshukov chooses to perform a second Laplace transform in r (instead of $\log r$ as above), though this does not reduce the complexity of the problem, since the r-dependency of the coefficients turns into partial derivatives in the new Laplace frequency.

After a rather much elaborated analysis, which involves the solvability of a Riemann–Hilbert problem, Teshukov obtains the result that for realistic gases, in particular for a perfect gas, the steady RR is dynamically stable if and only if it is a weak RR. We emphasize that this result is independent of ours, since we dealt instead with a *static* property, of the steady *supersonic* RR only. Teshukov's analysis applies to the transonic case also, which is relevant in particular for every strong RR.

3.3.2 The analysis for the UTSD model

The UTSD model is an approximation near the reflection point, valid for small strength and small angle (thin wedge), with the strength of the order of the square of the angle. With these restrictions in mind, it is interesting to consider RR for this model, as well as stability properties. The characterization of the RR regime was done in the seminal paper by Hunter [42].

In moving coordinates associated to the sound wave of the state U_0, the UTSD reads (see Paragraph 1.6)

$$v_t + \left(\frac{v^2}{2}\right)_x + w_y = 0, \qquad v_y - w_x = 0. \tag{93}$$

The coordinate x is parallel to the wall and y is normal. The unknowns v and w do not represent a velocity, but leading coefficients in an asymptotic expansion of U in terms of $\alpha^2/|U_1 - U_0|$.

System (93) is Galilean invariant with respect to (x,v), but not in terms of (y,w). Therefore, we may look for a steady state, where the incident shock is given only in terms of w_0, w_1 and the jump $v_1 - v_0$. Typically, $v_1 > v_0$, because the shock is compressive. Since the system is invariant under the scaling $(t,x,y,v,w) \mapsto (\delta^{-1}t, \delta x, y, \delta^2 v, \delta^3 w)$, we may also fix this jump to $[v] = 1$. Then the strength of the incident shock wave is small when $[w]$ is large. Using the Rankine–Hugoniot conditions, the slope of I is

$$\frac{dx}{dy} = a := \sqrt{\langle v \rangle} = \sqrt{-\frac{v_0 + v_1}{2}}.$$

Notice in particular that we need $\langle v \rangle < 0$. Because of $[w] = -a[v] = -a$, the parameter a plays the role of an inverse of the shock strength.

The boundary condition along $y = 0$ is written $w = 0$. Therefore the data satisfies $w_0 = 0$. This implies $w_1 = -a < 0$. Looking for a RR, we have to solve

$$w_1 = b(v_1 - v_2), \qquad b = \sqrt{-\frac{v_1 + v_2}{2}}, \tag{94}$$

where $-b$ is the slope of R, and we have used the boundary condition $w_2 = 0$. This yields an algebraic equation of degree 3, $P(v_3) = 0$, where

$$P(v) := (v + v_1)(v - v_1)^2 + 2w_1^2,$$

and the restriction that $v_1 < v_2 < -v_1$. It is easy to see that a solution exists if and only if $a \geq \sqrt{2}$. Actually, when $a > \sqrt{2}$, there are *two* admissible solutions $v^* \in (v_1, -v_1/3)$ (strong RR) and $u_* \in (-v_1/3, -v_1)$ (weak RR).

The state U_2 is supersonic provided that v_2 is negative and subsonic otherwise. It is immediate that it is subsonic in a strong RR, while in a weak RR it can be either supersonic, if $a > a_s := \sqrt{1 + \sqrt{5}/2}$, or subsonic, if $a \in (\sqrt{2}, a_s)$.

Let us assume that U_2 is supersonic (in particular, the RR is weak). We perform the same stability analysis as in Section 3.3., by looking at the stationary model

$$\left(\frac{v^2}{2}\right)_x + w_y = 0, \qquad v_y - w_x = 0.$$

as a hyperbolic system in the direction of negative x's. The boundary conditions along the wall ($w = 0$) and the reflected shock (Rankine–Hugoniot) are clearly normal. There remains to check an Evans condition. Once again, the latter reduces to $|\rho| < 1$, where ρ is an amplifying factor. We compute easily

$$\rho = \frac{4\alpha w_1 - P'(v_2)}{4\alpha w_1 + P'(v_2)}, \qquad \alpha := \sqrt{-v_2}$$

where α is the sound speed. On the one hand, we have $P'(v_2) < 0$ because the RR is weak. On the other hand, we already know that $w_1 < 0$. Therefore $|\rho| < 1$. We conclude that this

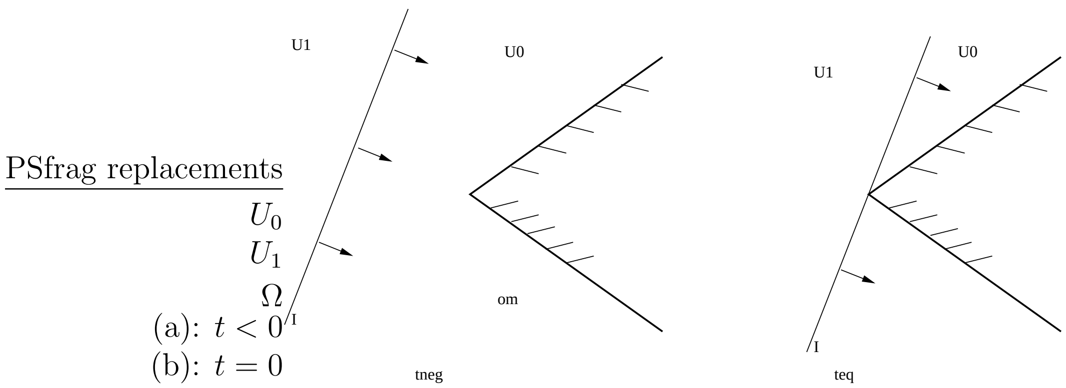

Figure 9: An incoming shock at a wedge. The upstream flow is at rest: $u_0 = 0$. At time $t = 0$ the pattern is self-similar.

stationary problem in the wedge between the wall and the reflected shock is strongly well-posed. In particular, the state remains constant and R remains straight. This remains true in the self-similar problem

$$\left(\frac{v^2}{2}\right)_x + w_y = xv_x + yv_y, \qquad v_y - w_x = 0,$$

until one reaches the sonic locus. Since $U \equiv U_2$ on one side of the sonic curve, the latter is (an arc of) the parabola given by the equation

$$y^2 = 4(v_2 - x).$$

4 Reflection at a wedge

We now consider a more complex as well as interesting geometry, presented in Figure 9. A solid wedge is surrounded by the gas. For $t < 0$, a planar shock wave is approaching the wedge and the state is piecewise constant, the gas being at rest near the wedge. This is clearly an admissible solution of the initial-boundary value problem. At time $t = 0$, the shock wave hits the wedge and something happens, since the flow downstream does not satisfy the slip boundary condition. A kind of reflection develops, which is plainly two-dimensional. As a matter of fact, the incoming shock plays the role of an incident shock on both sides of the wedge, yielding two reflected shocks, and these latter interact in a complicated way.

4.1 A 2-D Riemann problem

At first glance, the problem is genuinely time dependent. However, considering the state at time $t = 0$ as an initial data, we observe that it is invariant under space dilations. The same is true for the spatial domain Ω (the complement of the wedge). Since the system to solve is quasilinear of first order, we expect therefore that the solution (if there is any) be self-similar:

$$U(x_1, x_2, , t) = \tilde{U}\left(\frac{x_1}{t}, \frac{x_2}{t}\right) \qquad (x \in \Omega,\, t > 0).$$

Denoting the self-similar variable by $y := x/t \in \Omega$, we may rewrite the Euler equations as a stationary-like system. Generally speaking, a fist-order system of conservation laws

$$\partial_t u + \sum_\alpha \partial_\alpha f^\alpha(u) = 0 \tag{95}$$

yields

$$\sum_\alpha \partial_\alpha f^\alpha(\tilde{u}) = (y \cdot \nabla_y)\tilde{u}$$

for self-similar solutions (mind that now ∂_α is the derivative with respect to y_α). Such a system has the bad feature of having variable coefficients on its right-hand side. It is a remarkable property of the Euler equations, a consequence of Galilean invariance, that these variable coefficients may be removed, at the price of the introduction of a *pseudo-velocity* v. This vector field is defined by

$$v(y) := \tilde{u}(y) - y = u(x,t) - \frac{x}{t},$$

where as usual, u is the fluid velocity.

The full Euler equations (3,4,5) in self-similar variables then can be rewritten

$$\begin{aligned}
\operatorname{div}(\rho v) + 2\rho &= 0, && (96)\\
\operatorname{div}(\rho v \otimes v) + 3\rho v + \nabla p &= 0, && (97)\\
\operatorname{div}\left(\left(\frac{1}{2}\rho|v|^2 + \rho e + p\right) v\right) + 2(\rho|v|^2 + \rho e + p) &= 0. && (98)
\end{aligned}$$

Likewise, the entropy inequalities (31) yield

$$\operatorname{div}(\rho f(S) v) + 2\rho f(S) \le 0 \tag{99}$$

for the same set of functions f.

Away from discontinuities, Equation (97) may be combined with (96) to give

$$\rho(v \cdot \nabla)v + \rho v + \nabla p = 0, \tag{100}$$

while (98) may be rewritten as

$$v \cdot \nabla \left(\frac{1}{2}|v|^2 + e + \frac{p}{\rho}\right) + |v|^2 = 0. \tag{101}$$

This equation generalizes (16). Recall that when v is replaced by the genuine velocity u, the expression in parenthesis above has been called the Bernoulli invariant:

$$B := \frac{1}{2}|u|^2 + e + \frac{p}{\rho}.$$

In the self-similar case, we should merely speak of the *pseudo-Bernoulli invariant*:

$$B := \frac{1}{2}|v|^2 + e + \frac{p}{\rho}.$$

Of course, in a barotropic flow (isentropic or isothermal), we have $p = P(\rho)$. Equation (99) must be dropped, while (98) becomes an inequality:

$$\text{div}\left(\left(\frac{1}{2}\rho|v|^2 + \rho e + p\right) v\right) + 2(\rho|v|^2 + \rho e + p) \leq 0. \tag{102}$$

Boundary conditions. In a self-similar problem, the initial data is equivalent to a boundary condition at infinity, since $t \to 0+$ amounts to $|y| \to +\infty$. Hence we search a solution of (96,97,98) (only the first two equations in the barotropic case), such that $U(ry) \to V(y)$ as $r \to +\infty$, where V is the shock data at time $t = 0$. However, we point out that a hyperbolic system has the property of finite speed of propagation. This means that given a (small) domain ω, our solution U must equal, at time $t = 1$ in ω, any other solution U' such that $U'(y, 0) = U(y, 0)$ for y in a large enough domain ω_0, called the dependence domain of ω. Typically, $\omega_0 \subset \omega + B(0; \Lambda)$, where Λ is the maximum of the propagation velocities. Of course Λ depends on the solution itself, and it is therefore important to establish pointwise bounds on the solution, in order to have an explicit bound of Λ, justifying *a posteriori* our qualitative assumptions.

Since our initial data $U(\cdot, 0)$ and our physical domain are piecewise one-dimensional data, we first apply the above idea by choosing $\Omega' = \mathbb{R}^2$ and extending $U(\cdot, 0)$ by U_0 in the wedge. The corresponding solution U' is just a travelling wave $U(x - s_I t\nu_I)$ ($s_I\nu_I$ the velocity of the incident shock), and the equality $U = U'$ is valid away from the influence domain of the wall. Thus U is piecewise constant for $d(y : \partial\Omega) > \Lambda$.

A more involved choice allows to treat points in the strip $d(y : \partial\Omega) < \Lambda$ that are not in the influence domain of the wedge tip: $|y| > \Lambda$. If there exists a regular reflection (in the sense of Section 3.1) when we extend infinitely one of the walls of the wedge, we may take for U' the corresponding exact RR and Ω' the corresponding half-plane. Mind that U' will be a travelling wave of constant speed parallel to the ramp, since our incident shock is moving. If its velocity is larger than Λ, we deduce that in our true solution U, the incident shock remains straight until it meets the ramp, where it reflects according to the calculations of Section 3.1. Additionally, the reflected shock is straight and the state behind it is constant, as long as one stays away from $B(0; \Lambda)$. This pattern is again called Regular Reflection. See Figure 10 for the case where the incident shock propagates in the direction of the symmetry axis of the wedge.

Self-similar *vs* steady flows. It is worth to notice that the system (96,97,98) for self-similar flows differs from (3,4,5) for steady flows, only by low order terms 2ρ,... We see two important consequences. The first is that the type may be determined in exactly the same terms (though with v replacing u) as in Section 1.4. The second is that the discontinuities (shocks, contacts) obey the same set of Rankine–Hugoniot conditions and admissibility criteria. In particular, a shock must separate a supersonic state from a subsonic one (in the pseudo sense, see immediately below). Mind however that the low order terms play a crucial role in the sequel, as they allow to get *a priori* estimates that would be false in the stationary case. See Section 6.2.

Sub- or supersonic self-similar flows. In a time-dependent problem, the notions of subsonic or supersonic flows are not well-defined, because the Galilean invariance makes us free to add a constant to the velocity. When some external relation between time and space is given, these notions become meaningful. This is the case for steady flows, as we have discussed in Section 1.4. This is also relevant in self-similar flows. In this latter case, sonicity refers to the pseudo-velocity v. We say that the flow is pseudo-subsonic (respectively pseudo-supersonic) when $|v|$ is smaller (respectively larger) than the sound speed $|c|$. Finally, the flow is sonic where $|v| = c$. Thus it makes sense to introduce the *pseudo-Mach number*

$$M := \frac{|v|}{c}.$$

When $M < 1$, the self-similar system is not hyperbolic in any direction, while if $M > 1$, it is hyperbolic in directions $\xi \in \mathcal{S}^1$ such that $|v \cdot \xi| > c$. In the former case, the principal part defines only one family of characteristic lines through the equation $\dot{y} = v$. In the latter case, there are two other families of characteristic lines, which obey $|v \times \dot{y}| = c$, using arclength parametrization.

It is worth noticing that if the real flow U is constant in some open set (as it will be in applications), $v = u - y$ and M are non constant. In particular, the sonic line defined by $M = 1$ will be an arc of the circle centered at u, with radius c. For a weak incident shock, u vanishes upstream and is small downstream for weak reflection, so that the center of this circle is close to the origin.

A note about figures. For a self-similar flow, $\tilde{U}$ is identical to U at time $t = 1$. Thus we always display the flow at time one. In practice, only the geometrical features are shown: incident (I), reflected (R) and diffracted (Δ) shocks, vortex sheet (if any), sonic line (S) and the walls. Special points are denoted by P (reflection point), O (wedge tip), and Q (pseudo-sonic point on R). We point out that α is now the angle between the ramp and the horizontal axis. Since the incident shock is vertical, it makes an angle $\pi/2 - \alpha$ with the wall, instead of α in Section 3.

Flow and pseudo-flow. The pseudo-velocity turns out to have a physical meaning. For let $t \mapsto x(t)$ be a particle trajectory of the flow:

$$\frac{dx}{dt} = u(x,t) = u\left(\frac{x}{t}\right).$$

For $y := x/t$, we have

$$\frac{dy}{dt} = \frac{1}{t} v(y),$$

or in other words

$$\frac{dy}{d\tau} = v(y), \qquad \tau := \log t.$$

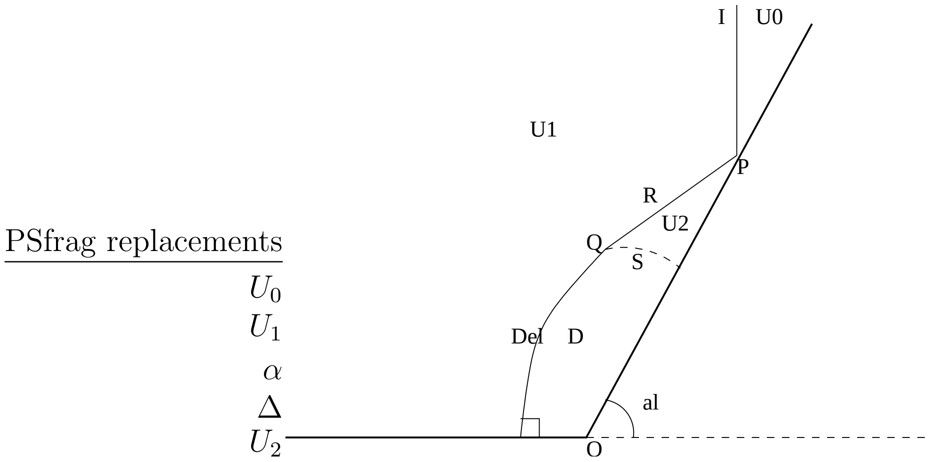

Figure 10: The symmetric RR. Incident (I), reflected (R) and diffracted (Δ) shocks, sonic line (S) and subsonic domain D. Upstream (U_0) and downstream (U_1) flows are given.

This shows that the physical flow is identical, up to the parametrization of the trajectories, to the *pseudo-steady flow* generated by the pseudo-velocity field.

4.2 Patterns

There are several possible patterns, depending on the strength of the incident shock and on its angles with the walls of the wedge. Ordered in increasing complexity, we mention Regular Reflection (RR), the Mach Reflection (MR), the Double Mach Reflection (DMR) and the Complex Mach reflexion (CMR). Though realistic phenomena often involve an MR or more shaky patterns, we shall address mainly the RR in the sequel, since there are more mathematical results available in this case.

4.2.1 The symmetric RR

We say that the reflection is *symmetric* when the incident shock is normal to the axis of the wedge. Then we expect (if uniqueness holds) that our solution is symmetric:

$$\begin{pmatrix} \rho \\ v_1 \\ v_2 \\ e \end{pmatrix} (y_1, -y_2) = \begin{pmatrix} \rho \\ v_1 \\ -v_2 \\ e \end{pmatrix} (y_1, y_2).$$

In particular, the vertical velocity vanishes along the vertical axis $y_2 = 0$. This amounts to considering the half-domain Ω_+ bounded below by the wedge and the horizontal axis. The latter plays the role of another rigid wall. Figure 10 displays the symmetric regular reflection.

This reduction may seem useless at first glance. Actually, it is of great importance in *a priori* estimates, because the reduced domain Ω_+ is *convex*. This will allow us to establish a minimum

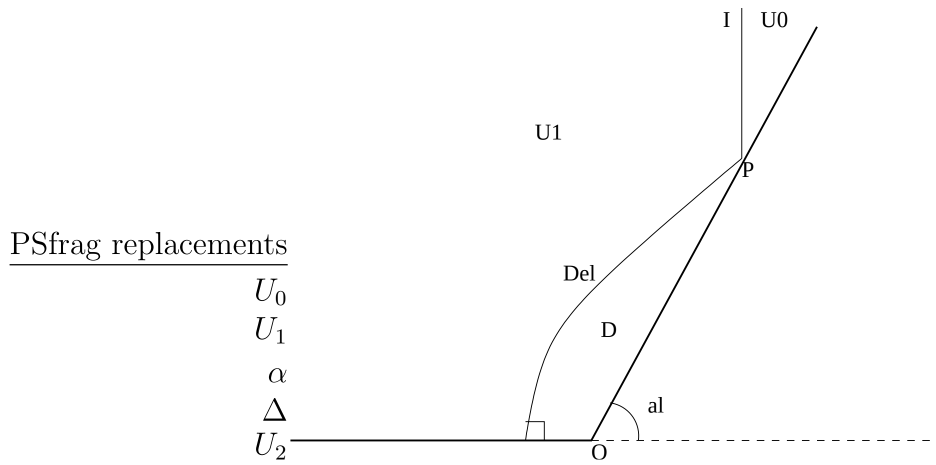

Figure 11: The symmetric RR when $|y_P - u_2| \leq c_2$. Incident (I) and diffracted (Δ) shocks, subsonic domain D. Upstream (U_0) and downstream (U_1) flows are given.

principle for the pressure ; the possibility of such a property is unclear in the non-symmetric case.

Both numerical and physical experiments suggest that, if the incident shock strength $\epsilon := \Delta\rho/\rho_0$ is small, and if the ratio $\epsilon\alpha^{-2}$ is small too, then the domain of influence of the wedge tip does not contain the intersection point P. Then the reflected shock is a straight line and the state (ρ, u, e) remains constant, until the sonic line at point Q (this will be discussed in Section 4.3). The interaction with the sonic line gives rise to the *diffracted shock*, which bends until it reaches the symmetry axis. The pattern consisting of the states U_0, U_1 and U_2, separated by the incident and reflected shock waves can be computed explicitly with the help of shock polars (see Sections 3.1 and 4.1). Since the state equals the constant U_2 on one side of the sonic line, this must be a circle (or an arc of the circle) with equation $|y - u_2| = c_2$. The present picture is characterized by a weak RR at point P, which is supersonic in the pseudo sense ($|u_2 - y_P| > c_2$).

There is another possibility for a RR, when the state U_2 given by shock polar analysis gives a subsonic pseudo-state $y_P - u_2$ (transonic RR). Then we do not exclude that the incident shock still reaches the ramp and the diffracted shock begins at P. In this case, there is no sonic line, and the subsonic zone fills the domain between Δ and the wall. See Figure 11.

4.2.2 Mach Reflections

When the incident shock strength increases, or when α decays, there happens a transition from a RR to a MR. The interaction between the incident shock and the ramp has been described in Section 3.2. The rest of the picture resembles much the RR for moderate data (SMR), but becomes increasingly complex (DMR and CMR) as the angle α decreases or the shock strength increases. If the pseudo-state U_2 behind P is subsonic, there occurs a simple Mach Reflection. Otherwise, the pattern depends on whether the slip line reflects against the ramp before crossing the sonic circle attached to U_2, or not. In one case, a fourth shock will interact

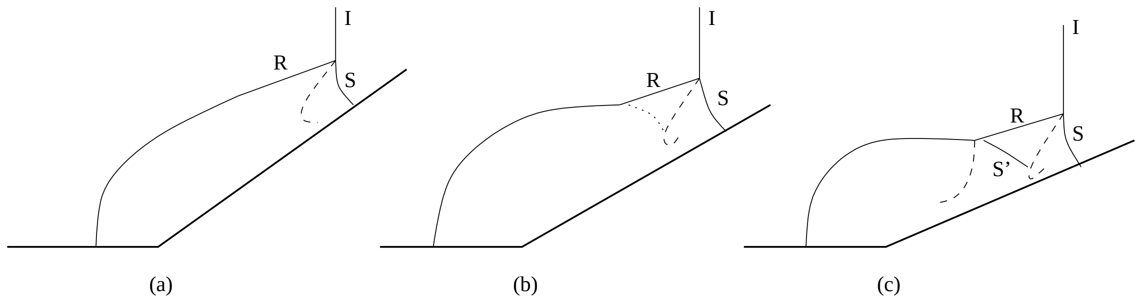

Figure 12: The three kinds of Mach Reflection. Left (a) simple (SMR) ; center (b) complex (CMR), right (c) double (DMR).

again with the reflected shock, giving rise to a secondary Mach stem typical of a DMR. In the other case, the reflected shock suffers only a kink, which characterizes a CMR, intermediate between SMR and DMR. A detailed analysis based on shock polars can be found in [4, 5, 41] and [18], while [26, 60, 61] provide convincing numerical experiments. We point out however that an existence theory remains to be worked out.

For the convenience of the reader, we display in Figure 12 the various kinds of Mach Reflection, although we do not intend to attack them at a mathematical level. Notice that experiments suggest that a transition occurs between every pair of types RR, SMR, CMR, DMR, but the pair (SMR,DMR).

4.3 The supersonic domain behind the reflected shock

In the case of a supersonic RR, it is usually taken from granted that the state U (the physical one, but not the pseudo-state) is constant in the supersonic region D_2 surrounded by the reflected shock R, the ramp and the pseudo-sonic line S defined by $|v| = c$. In particular, the sonic line obeys the equation

$$|u_2 - y| = c_2. \tag{103}$$

The sonic line is therefore an arc of the circle $\mathcal{C}$ of radius c_2, centered at u_2. Additionally, the reflected shock, separating two constant states U_1 and U_2, remains straight until it reaches the sonic line.

This constancy, though a little bit intuitive, deserves some mathematical investigation. In the sequel, we assume only that the state U is smooth in D_2 and tends to U_2 as $y \to y_P$ within D_2. Our task is to prove that U remains constant in D_2. The idea is to apply a uniqueness result for a boundary value problem (refered to as BVP in the sequel) associated with an evolution system of PDEs. The system is of course given by the self-similar Euler equations. The assumption that the reflected state U_2 be pseudo-supersonic at P tells that this system is hyperbolic in the direction of the flow. Defining a time-like variable in the direction of the ramp, from P to O, we see on the one hand that the system is (at least locally) evolutionary and hyperbolic in this direction, and on the other hand that D_2 is locally in the "future" of

P. These properties will remain valid as long as U stays close enough to U_2. We point out that the domain of this BVP is wedge-like. The boundary value problem is completed by the slip condition $u \cdot \nu$ along the ramp and the Rankine–Hugoniot condition along R. Notice that R is a free boundary. A particular solution of our BVP consists of $U \equiv U_2$, together with the straight reflected shock given by the shock polar analysis, and the circular sonic line defined by (103). Hence a uniqueness result will solve the question.

We decompose the uniqueness problem in three parts:

- A local uniqueness for the BVP in the wedge-like domain at P. This is essentially the problem addressed in Section 3.3, with v instead of u. The lower order terms play no role in the strong well-posedness or in the strong instability, though we do not exclude that they be important in the marginal case of weak linear well-posedness.
- Local uniqueness for standard BVPs, either along the ramp, or along the free boundary R.

Following Section 3.3, we thus assume that the quantity

$$G := ((v_2 \cdot \nu)^2 + c_2^2)(v_2 \times \nu)^2 + ((v_2 \cdot \nu)^2 - c_2^2)(v_2 \cdot \nu)(v_1 \cdot \nu) \tag{104}$$

is positive. For instance, this is true for the weak RR when either the incident strength is small, or the angle α is close to $\pi/2$ (near normal reflection).

Since the state is pseudo-supersonic, the system is hyperbolic in D_2. It has three characteristics, a backward one, a forward one and a third one in between. The words *backward* and *forward* refer to an evolution, with the coordinate along the boundary as a pseudo-time ; recall that this coordinate increases from the reflection point P to the tip of the wedge. Forward characteristics emerge from the wall when the pseudo-time increases, that is when one escapes from P ; on the contrary, backward characteristics approach the ramp as the pseudo-time increases (see Figure 13). The remaining characteristics are tangent to the boundary, because of the no-flow boundary condition ; they are flow line ($\dot{y} = v$), with multiplicity two in the case of the full Euler system or one in the barotropic case. Because of hyperbolicity, there is a propagation property, implying that the region K where $U \equiv U_2$ is bounded either by boundaries of D_2, or by pseudo-characteristics. Since U is smooth, these characteristics are either the tangents to the pseudo-sonic circle $\mathcal{C}$ associated to the state U_2:

$$(y - w) \cdot N = 0, \qquad N := \frac{w - u_2}{c_2}, \tag{105}$$

where w is some point of the circle, or flow lines. We point out that each tangent consists in *two* characteristic lines, a forward one and a backward one, both separated by the point of tangency ; see Figure 13.

Because of local uniqueness at P, K is non void. Assume also that K is strictly contained in D_2. Then the boundary of K contains at least one straight characteristic described above, which meets either the ramp (if it is forward) or R (if it is backward or a flow line). Thus we have to verify the uniqueness property for two special BVPs, associated with the following domains. Such BVPs have been studied thoroughly by Li & Yu [65].

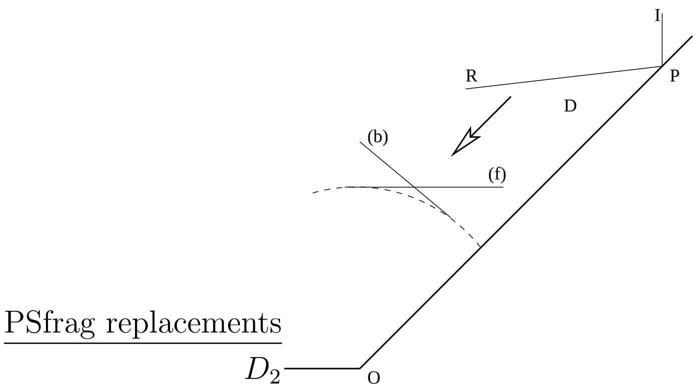

Figure 13: A forward (f) and backward (b) characteristics. The arrow shows the time-like direction in the supersonic domain D_2.

The domain of the first BVP is bounded by the ramp and a characteristic Γ, which is either backward or forward. On the latter, we have a Goursat problem, on which the state is prescribed in a consistent way. Along the ramp, one characteristic is incoming, one is glancing and one is outgoing. Thus we need precisely one boundary condition. Since there are $1+1$ dimensions, the BVP is well-posed provided the boundary condition, here $u_1 \cos\alpha + u_2 \sin\alpha = 0$ is *normal* and Γ is forward. Normality means that the unique incoming mode does not satisfy the boundary condition, a fact that can be verified easily. We conclude that either K extends along the ramp up to the sonic line, or it is bounded by a backward charateristic Γ. We point out that such a Γ cannot meet the circle within the physical domain ; the tangency point lies below the ramp.

If K is bounded by a backward characteristic Γ, we face the second BVP, which is actually a FBVP (free boundary value problem). Its domain is bounded by Γ and the shock R. Along the former, we still have a Goursat problem and the state is prescribed in a consistent way. On the reflected shock, two characteristics[15]are incoming, because the flow is pseudo-incoming to D_2. The forward characteristic is outgoing, because the flow is subsonic, relatively to the shock. Was the boundary fixed, we should need precisely two (barotropic case) or three (full Euler case) scalar boundary conditions. Since the shock is a free boundary, we actually need one more boundary condition, that is as many as the number of equations in the system. The Rankine–Hugoniot conditions give exactly the right number of boundary conditions. The verification that they form a normal boundary condition at U_2 is straightforward. Therefore the FBVP is locally well-posed, whence the uniqueness property. This proves that K extends along the ramp up to the sonic line.

In conclusion, K extends between the ramp and R, till S. Thus we have proved that

Theorem 4.1 *Consider a symmetric RR past a wedge. Assume that this reflection is supersonic: $|u_2 - P| > c_2$, in particular, the reflection is a weak one. Assume in addition that $G > 0$ (with G defined in (104)). Then*

[15]One being double in the full Euler case.

1. *the reflected shock R is straight between P and the circle $\mathcal{C}$ (of equation $|u_2 - y| = c_2$),*

2. *the state U remains constant, equal to U_2, within the domain bounded by the ramp, the reflected shock and the circle $\mathcal{C}$.*

When the incident shock strength increases, G may become negative even though the reflection remains supersonic at P. In such a case, one does not expect either that R remains straight, or U remains constant, till $\mathcal{C}$. In particular, $\mathcal{C}$ does not any more coincide with the sonic line of the pseudo-flow. The instability of the accepted pattern could explain why the RR gives way to Mach Reflection in cases where the planar RR still exists.

4.4 Mathematical difficulties

The following list displays the mathematical difficulties that are encountered in the mathematical treatment of reflection past a wedge. All of them but the slip line arise already in the simplest RR pattern.

Vertex at the origin. The boundary of the physical domain has a singularity at the origin. It will be responsible for a lack of smoothness of the solution. In view of Equation (118), it is unclear whether the pressure has a singularity, or the velocity vanishes.

Sonic line. This difficulty arises in a supersonic RR. Across the sonic line S, the type of the Euler system changes from hyperbolic (in the supersonic zone labelled 2) to hyperbolic-elliptic (in the subsonic domain D). In D, the elliptic part of the system degenerates as one approaches the boundary. When the strength of the incident shock is small, we are tempted to use linear of weakly non-linear geometrical optics (see next Paragraph), which suggest that the flow is Hölder continuous across the sonic line, with a singularity of order $\sqrt{d(y;S)}$ where $d(\cdot;S)$ is the distance to S in D ; see for instance the acoustic approximation of Keller & Blank [47]. This picture turns out to be incorrect. It has been uncovered recently by G.-Q. Chen & M. Feldman [21] that the nonlinearity in the system yields Lipschitz continuity, at least in the potential case, where the system reduces to a nonlinear wave equation.

Vertex at P. Alternatively, in a transonic RR, the subsonic zone extends till P, and we have to solve a nonlinear system of PDEs in a domain that has a vertex at P.

Mixed type. In the subsonic domain D, the system is of mixed type hyperbolic-elliptic, meaning that one characteristic field is real, while the other ones are complex. This makes the analysis very hard, since it is not possible either to apply standard techniques of hyperbolic problems or to employ ideas from elliptic theory.

Vortical singularity. We shall see in Section 6.6 that the vorticity $\nabla \times u$ cannot be square integrable in D, at least in the barotropic case. A singularity is expected at some point along the ramp. It is unclear whether this singularity is present in full gas dynamics.

Slip line In Mach Reflection, the interaction point P (defined as the point where the reflected and the incident shocks meet) is detached, and a secondary shock called Mach stem connects P to the ramp. Since the Rankine–Hugoniot relations are the same as in the steady case, a pure three-shocks pattern is not possible (Theorem 2.3). Hence a fourth wave must originate from P. Experiments suggest that it is a slip line, or in other words a vortex sheet. According to Artola & Majda [2], such jumps are known to be dynamically unstable unless the jump of the tangential velocity exceeds $2\sqrt{2}\,c$, which is unlikely[16]. Finally, one observes that the slip line rolls up endlessly.

Diffracted shock. The solution is known everywhere but in the subsonic zone D. However, the part of the boundary of D formed by the diffracted shock is a free boundary. For an incident shock of small strength, this curve is approximately a circle, the continuation of the sonic line.

Triple point. The sonic line and the reflected shock form a corner at their meeting point Q. This is a singularity of the boundary of the subsonic domain. This singularity is non uniform in terms of the shock strength, as both lines tend to become tangent when the strength vanishes.

We point out that the vortical singularity and the slip line are obviously not present in irrotational flows. Additionally, the system becomes purely elliptic in D. This makes the irrotational RR much more tractable, with only the difficulties of non-uniform ellipticity, boundary vertex, triple point and free boundary. In particular, one may expect that the flow be of class H^1 within D. Y. Zheng [84], as well as G.-Q. Chen & M. Feldman [21] anounced recently an existence result in this case, when the incident shock is almost normal.

Another simplification occurs in the transonic case ($|u_2 - P| < c_2$). Then we avoid the degeneracy problem across the sonic line and the triple point Q. In conclusion, the simplest situation for a RR is that of an irrotational flow for which the subsonic zone reaches the point P. Then the mathematical problem is to solve a scalar second order nonlinear elliptic equation in terms of the potential. The remaining difficulties are the free boundary and two geometrical singularities, at O and P. We point out on the one hand that this problem does not seem to follow from the minimisation of some action. On the other hand, it happens in some intermediate (very narrow) range of parameters and thus is not a perturbation of some trivial configuration. Therefore it cannot be attacked by pertubative tools.

Finally, let us mention the work by S.-X. Chen [23], who proves the existence of a local solution for the reflexion of a shock against a *smooth* convex obstacle. Of course, this result is sensitive to the curvature of the boundary, and does not survive when the shape of the obstacle becomes sharp. S.-X. Chen also proved a local stability result, near the triple point, of a Mach configuration ; this result is in the spirit of Paragraph 3.3.

[16] In a three dimensional setting, vortex sheets are unconditionally unstable.

5 Reflection at a wedge: Qualitative aspects

5.1 Weak incident shock

When the strength of the incident shock tends to zero, the initial data (at $t = 0$) of the evolutionary Euler system tends to a constant state, say U_0. This constant being a rest state, it is a solution of the IBVP. Since it is a smooth one, we may apply stability results (see for instance Dafermos [29], Chapter 5.2)[17]. Thus the expected solution must be close[18] to U_0. Although this has not been proved yet, we shall assume that this proximity holds uniformly:

$$\sup_y |U(y) - U_0| \ll 1. \tag{106}$$

Assuming also that the solution has the piecewise smooth structure described in Figure 10, it should be described, at first order, by weakly nonlinear geometric optics, applied to the background state U_0. This approach has been introduced in [47] and developed further in [45].

Linear and nonlinear geometrical optics. We begin with *linear* geometrical optics, which gives the leading order for the evolution of the singularity locus. It was worked out by Keller & Blank [47]. There are two types of initial disturbances. First the incident shock itself, and then the vertex, where the downstream state is not compatible with the boundary condition. In general, every pointwise disturbance generates a front that travels at one of the velocities $\pm c_0$ or 0 associated to U_0. The zero speed means that a part of the singularity stays fixed, while the velocity $\pm c_0$ concerns every direction: a pointwise singularity surrounded by a constant state generates a circular front. In particular, we must have a direct wave along a (approximate) circle S whose center is located at the wedge tip. The situation is a bit different when the disturbance is localized along a line. In general, the initial front splits into three fronts, of which two move apart in opposite directions at normal velocity c_0, and one stays fixed. In our problem, the pattern is simpler since the initial disturbance is compatible with the Rankine–Hugoniot conditions: the front that originates from the incident shock remains the part of this shock that is not influenced by the vertex.

Besides these direct waves, we must keep track of secondary waves (as well as ternary,... if any) that are generated by the interactions between them, and between one of them and the wall. We have already seen the effect of an incident shock along a wall, which gives rise to a reflected shock with (asymptotically) specular reflection. It is immediate that, in the limit of zero strength, this reflected wave is tangent to the circular wave S. Hence the direct and secondary waves do not produce ternary waves. On an other hand, the wave S is already compatible with the boundary condition (its velocity is normal to the wall) and does not yield a secondary wave. More importantly, because of compatibility at the tangency point Q, there is no need to continue the straight reflected shock beyond this point ; the proper continuation is the circle itself ! One may say that the reflected shock is bent once it meets the sonic line.

[17] Recall that stability results use relative entropy estimates, and they are closely linked to weak-strong uniqueness.

[18] In terms of the relative entropy, or of the relative energy in the barotropic case.

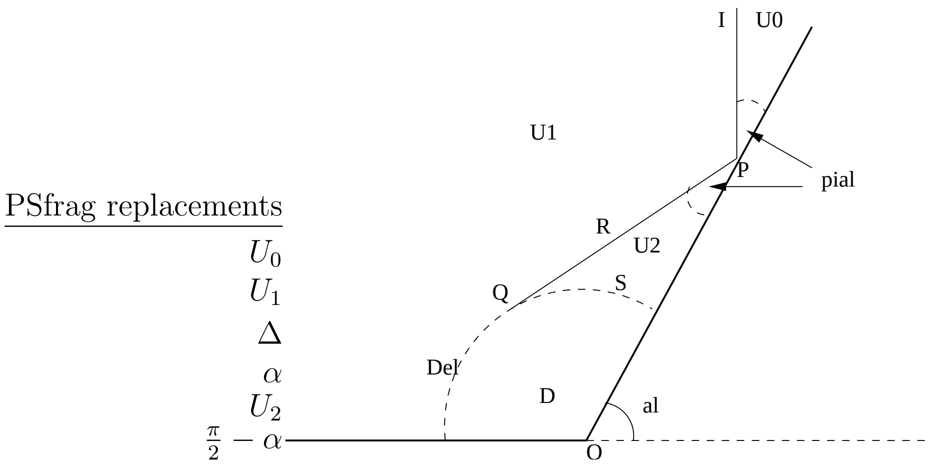

Figure 14: The limit pattern as the incidence strength tends to zero.

Finally, it is clear that S does not meet the incident shock, because the point P in Figure 10 travels at speed $c_0/\cos\alpha$, hence faster than S.

We summarize the previous analysis: In the limit of zero incident strength, the solution exhibits three singularities along curves. Direct ones are the incident shock I moving forward and touching the wall at point P, and a circular curve S emanating from the vertex. The reflected shock is made of a part Δ of this circle (the *diffracted* wave) and a secondary wave R along the tangent to this circle between P and Q. The remaining part of the circle is the sonic line. See Figure 14.

At the analytical level, the linear theory yields a second-order equation of the form

$$r\partial_r((1-r^2)\partial_r\bar{\rho}) + r\partial_r\bar{\rho} + \partial_\theta^2\bar{\rho} = 0, \tag{107}$$

where $\bar{\rho} := \Delta\rho/\rho_0$ is the relative density disturbance, and (r,θ) is a polar coordinate system, rescaled in such a way that the circle $|y| = c_0$ has equation $r = 1$. Equation (107) is elliptic for $r < 1$. The boundary condition is of Dirichlet type along the circle

$$\bar{\rho}(1,\theta) = \rho_b(\theta) := \begin{cases} 1 & \text{along} \quad \Delta \\ 2, & \text{along} \quad S, \end{cases} \tag{108}$$

and of Neumann type

$$\partial\bar{\rho}/\partial\nu = 0$$

along the wall. The solution was found in explicit form by Keller & Blank, thanks to the Busemann transformation

$$R := \frac{r}{1+\sqrt{1-r^2}}.$$

They found that $\rho - \rho_b$ behaves like $h(\theta)\sqrt{1-r}$ for some function $h \neq 0$, everywhere but in a neighbourhood of the triple point ($\theta = 2\alpha$).

A more elaborate analysis carried out by Hunter & Keller [44, 45] uses *weakly nonlinear* geometrical optics. It yields a refined description of the flow in a neighbourhood of the diffracted shock. There, the leading-order approximation satisfies a one-dimensional cylindrical Burgers equation. The asymptotic expansion at the triple point Q was described by Harabetian [35].

When α is small together with I, the sonic circle and the triple point approach the interaction point P. Then the description of the flow near P needs the more involved UTSD approximation of [42], which combines a multi-dimensional context with a nonlinearity.

More about the supersonic domain. Let $\delta \ll 1$ denote the quantity $\sup_y |U(y) - U_0|$. Since the wave velocities are bounded by $c = c_0 + O(\delta)$, we can prove (see again [29]) that the solution U at (x,t) depends only on the data and the geometry of the domain in a ball $D(x;ct)$. At time $t = 1$, this means that the solution, outside of the disk $D(0;c)$, is not influenced by the vertex. In particular, it coincides with solutions that are known explicitly:

- For $|y| > c$, $y_2 \cos\alpha - y_1 \sin\alpha > c$ and $y_1 < c$, one has $U \equiv U_1$ (downstream data),
- For $|y| > c$, $y_2 \cos\alpha - y_1 \sin\alpha > c$ and $y_1 > 0$, one has the incident shock, namely $U \equiv U_1$ for $y_1 < s_I$ (s_I the incident shock speed) and $U \equiv U_0$ for $y_1 > s_I$,
- For $|y| > c$ and $0 < x_2 \cos\alpha - x_1 \sin\alpha < c$, we have the piecewise constant reflection described in Section 3. In particular, the reflected shock is straight near the reflection point P and the solution $U \equiv U_2$ is constant in some wedge bounded by P, the rigid wall and the reflected shock. We warn the reader that a constant u means a non-constant v ! We shall see below that the constancy of U and the straightness of R extend up to the sonic line.

Remarks.

1. Perhaps the most important point is that, using the qualitative assumptions made above about the structure of RR, we shall be able (see Section 6) to make *quantitative* pointwise estimates, and therefore to justify the smallness assumption (106) made at the begining of this paragraph.
2. Since the reflected shock is of small strength $O(\delta)$, the reflection at point P is a weak one, and the reflection angle equals approximately the angle of incidence. Such a reflection follows approximately the laws of optics.
3. If the wedge angle α is small, it may happen that the sonic line travels faster than the point P along the ramp. A sufficient condition for this to happen is

   $$\left| u_2 - \begin{pmatrix} s_I \\ s_I \tan\alpha \end{pmatrix} \right| < c_2. \tag{109}$$

   In this case, we might have either a Regular Reflection (as in Figure 11), or a Mach Reflection with interaction point remote from the wall. Of course, we do not claim that equality in (109) is the criterion for the transition from RR to MR.

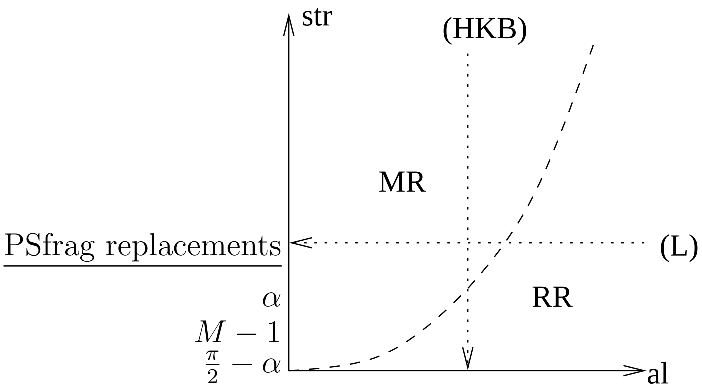

Figure 15: Various limits of shock reflection past a wedge. The vertical coordinate (here $M := \sigma_I/c_0 > 1$ is the Mach number of I) represents the strength of the incident shock. The dashed curve separates the RR regime from the MR. The limits studied by Lighthill (L) and by Hunter, Keller and Blank (HKB) occur along horizontal and vertical lines. The parabolic scaling at the origin yields the UTSD model of Brio and Hunter.

4. In some regimes, an RR solution is technically possible in a neihbourhood of P, but is not observed in numerical and/or physical experiments, presumably for instability reasons. This does happen for incident shocks of moderate strength. In this case, one observes a MR instead.

5.2 Small and large angle

Another asymptotic limit occurs when one keeps the incident shock strength constant, while making $\alpha \to 0+$. It was considered by Lighthill [50], with the idea that in the limit problem ($\alpha = 0$), there is no reflection at all since the shock travels parallel to the wall. Therefore, in spite of the fact that a Mach Reflection must occur, we expect that the reflected pattern consists only of weak waves. Hence the problem can be studied through linearization behind the incident shock. The resulting system is essentially the same as that of Keller & Blank[19]. It reduces to a second order equation in the pressure, which can be solved thanks to Busemann's transformation. Remark that the reflected pattern fills approximately the acoustic disk D of center O and radius c_1, the sound speed behind I. Since the incident shock is of Lax type, c_1 is larger than the shock speed σ_I, so that the disk is truncated by the shock locus (as well as by the wall). This makes clear the relevance of a Mach Reflection.

The opposite case, when α tends to $\pi/2$, yields in the limit the normal reflection described in Paragraph 3.1.2. Since it has coordinates $(\sigma_I, \sigma_I \tan\alpha)$, the point P escapes to infinity. In particular, this kind of reflection is supersonic. The rest of the pattern in the weak RR has a limit which is piecewise constant. In this limit, we have $u_2 = 0$ and the reflected shock is straight and vertical. The subsonic domain coincides with the part of the disk $|y| < c_2$ to the right of R. The velocity in D vanishes identically. Remark that σ_R is less than c_2, so that the

[19] It is more chronologically correct to say that the analysis by Keller & Blank has similarities with Lighthill's.

sonic line is transverse to the reflected shock, contrary to the other limits considered above. This happens because R has a non zero limit strength, hence is not sonic. The pattern is nonlinear in essence at the leading order.

5.3 Entropy-type inequalities

This section is not specific to gas dynamics and may be skipped in a first reading. However, we believe that it has some interest for numerical purposes.

We consider self-similar solutions of a first-order system of conservation laws (95), endowed with an entropy inequality

$$\partial_t \eta(u) + \mathrm{div}_x \vec{q}(u) \leq 0,$$

η being strictly convex ($\mathrm{D}^2\eta > 0_n$). The corresponding PDEs

$$\mathrm{div}_y \vec{f_j}(u) = (y \cdot \nabla_y) u_j, \quad (j = 1, ..., n), \qquad \mathrm{div}_y \vec{q}(u) \leq (y \cdot \nabla_y)\eta(u) \tag{110}$$

are understood in the distributional sense, and $u(y)$ is locally bounded. As pointed out in [37], these equations can be combined to give inequalities in conservative form, though involving variable coefficients. Given any open subdomain ω with smooth boundary, we denote by $\mathbf{n}$ the outgoing unit normal to ω, by $|\omega|$ the volume and by d the space dimension ($d = 2$ in our reflection problems). Then every self-similar solution satisfies

$$\eta\left(\frac{1}{d|\omega|}\int_{\partial\omega}((\mathbf{n}\cdot y)u - f(u;\mathbf{n}))\,ds(y)\right) \leq \frac{1}{d|\omega|}\int_{\partial\omega}\mathbf{n}\cdot(\eta(u)y - \vec{q}(u))\,ds(y), \tag{111}$$

where $f(u;\mathbf{n}) := \sum_\alpha n_\alpha f^\alpha(u)$.

It is remarkable that, as ω runs over all subdomains, (111) is equivalent to (110). Therefore it can be used to give *a posteriori* estimates in numerical simulations. Given a finite volume method where fluxes across control volumes are practically computable, one should either make a correction on those volumes where (111) fails, or refine the grid there. This idea has not yet been implemented to our knowledge, though it looks promising.

6 Regular Reflection at a wedge: Quantitative aspects

The purpose of this section is mainly to establish pointwise estimates for Regular Reflection. Both the barotropic and the full Euler cases are considered, except in Paragraphs 6.1 (non-barotropic model only) and 6.6 (barotropic model only). We assume throughout the section that the solution (if any), is piecewise smooth and obeys the qualitative description given in Section 4 for a supersonic RR[20]. The only places where U is singular are

- the incident and reflected shocks, where we have discontinuities,
- the sonic line, where U is likely to be at most Lipschitz continuous,

[20]However, most of our estimates are valid for either transonic RR or MR, as long as the flow is piecewise smooth.

- possible vortex sheets (slip lines) in the interior of the subsonic domain D. Notice that, according to Theorem 2.2, steady shocks are forbidden between two subsonic states,
- the origin, where we shall see that the pressure gradient experiences a Dirac mass,
- the likely vortical singularity, located at some point of the ramp, where the vorticity lacks square integrability.

6.1 Minimum principle for the entropy

This paragraph is devoted to the case of full Euler system. The easiest pointwise estimate has been known for two decades [67] and follows directly from (31) or from (99). As stated in Theorem 1.1, the minimum of the entropy is a non-decreasing function of time. Since the initial data experiences two states U_0 (upstream) and U_1 (downstream), we deduce

$$S(y) \geq \min\{S_0, S_1\}.$$

It turns out that in a shock (here, the incident one), the entropy is lower upstream than downstream, so that $S_0 < S_1$, whence the estimate

$$\boxed{S(y) \geq S_0.} \tag{112}$$

We emphasize that this is a sharp estimate, since $S(y)$ equals S_0 for $y_1 \geq s_I/\cos\alpha$. However, the better estimate

$$\boxed{S(y) \geq S_1.} \tag{113}$$

holds true whenever $y_1 \leq s_I/\cos\alpha$. This can be proved from the transport equation $v\cdot\nabla S = 0$, and the fact that S increases across shocks when following the pseudo-flow.

6.2 Minimum principle for the pressure

The next estimate is more involved, but still sharp. We begin by defining the angle $\theta(y)$ of the flow :

$$v = |v| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Of course, this definition makes sense only away from stagnation points, and θ can be chosen in a smooth way in every simply connected domain where v is smooth and does not vanish. The important point is that the gradients of p and θ are related by a linear identity (see [63])

$$\rho|v|^2\nabla\theta^\perp + \rho v = (I_2 - c^{-2} v \otimes v)\nabla p, \tag{114}$$

where $X \mapsto X^\perp$ is a rotation by 90°. Notice that (114) is valid even across contacts, because the pressure and the angle are continuous across a vortex sheet[21].

[21] Actually, the angle θ equals that between the tangent of the contact curve with the horizontal axis.

After dividing by $\rho|v|^2$, one eliminates θ from (114) by taking the curl. A kind of miracle happens here because the low order term yields again the gradient of the pressure:

$$\text{div}\left(\frac{1}{\rho|v|^2}(I_2 - c^{-2}v \times v)\nabla p\right) + \frac{|v|^2c^{-2} - 2}{\rho|v|^4}v \cdot \nabla p = 0. \tag{115}$$

We notice that for the moment, we do not know whether ρ may vanish, but we shall see soon that it does not, because of an explicit lower bound. This expresses the fact that the wedge has an obvious compressive effect.

We interpret (115) as a *linear second order* equation $Np = 0$ in the pressure, with variable coefficients. For cosmetic reasons, our operator N is given by the left-hand side of (115), multiplied by $\rho|v|^2$. We have

$$N = \sum_{i,=1}^{2} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i,$$

where the second order terms are precisely

$$\sum_{i,j}(\delta_i^j - q_iq_j)\partial_i\partial_j, \qquad \left(q := \frac{v}{c}\right).$$

We point out that the first order terms b_i in N have singularities at stagnation points, namely when v vanishes. There, (115) becomes a first order equation.

The operator N is elliptic precisely in the subsonic domain $|q| < 1$, because of the inequality between symmetric matrices:

$$v \otimes v < c^2 I_2.$$

We thus have an extremum principle for the pressure: it cannot achieve a local minimum or a local maximum at an interior point, unless v vanishes simultaneously.

Of course, the steady problem yields an equation similar to (115) (the right-hand side being replaced by zero) and the same conclusion can be stated. Even the incompressible case is relevant, with equation

$$\text{div}\left(\frac{1}{|u|^2}\nabla p\right) = 0,$$

as pointed out by H. Weinberger (private communication).

For the moment, we have shown that the pressure cannot achieve an local extremum in the interior of the subsonic domain, except at stagnation points. However, it is known that in the steady case, p does achieve local extrema a stagnation points. Typically, p may be maximal at saddle points of the flow, and minimal at foci (see Figure 16). Thus it seems hopeless to get an *a priori* bound for the pressure in general. It is therefore remarkable that the lower order terms present in the self-similar problem help in establishing a minimum principle for the pressure !

Stagnation points. Let y_0 be a stagnation point ($v(y_0) = 0$) and assume that p achieves a local minimum at y_0. Let M denote $\nabla v(y_0)$. From (96), we have $\text{Tr}M = -2$. Hence the spectrum of M consists in eigenvalues $-1 \pm \lambda$ where λ is either real or purely imaginary, because

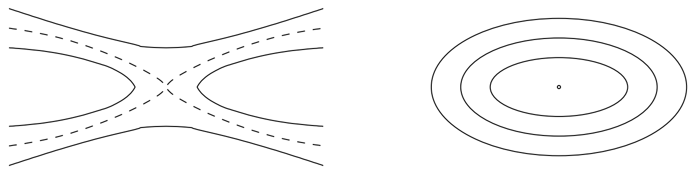

Figure 16: Steady flows with a stagnation point. Left (a): saddle point, the pressure achieves a local maximum. Right (b): a focus, the pressure achieves a local minimum.

M has real entries. Differentiating (100) yields $D^2p(y_0) = -\rho(M^2 + M)$. Since the spectrum of a symmetric matrix is real, we deduce that λ is real. Then one of the eigenvalues $-\rho\lambda(\lambda \pm 1)$ is non-positive. Since $D^2p(y_0)$ must be non-negative, we deduce that $\lambda = 0$ and therefore

$$D^2p(y_0) = 0_2. \tag{116}$$

Concerning M, we already have two conclusions. On the one hand, its spectrum reduces to $\{-1\}$, while on the other hand we have $M^2 + M = 0_2$. Since the polynomial $X^2 + X$ has simple roots, M is semi-simple and therefore equals $-I_2$. We conclude that[22]

$$Dv(y_0) = -I_2. \tag{117}$$

The analysis of [63] stopped there, so that the minimum principle for the pressure remained a formal result, leaving aside the marginal case where p is flat at y_0. We complete it here with a rigorous proof, assuming only that U is locally smooth. There remains to treat the case of a stagnation point with (116,117). To begin with, we remark that, thanks to (117), y_0 is an isolated stagnation point and that $v(y) = -z + O(|z|^2)$ with $z := y - y_0$. Our key observation is that, because of (117), N has bounded coefficients at y_0 ! More precisely, we have

$$N \sim \sum_{i,=1}^{2} a_{ij}\partial_i\partial_j + \sum_i b_i^0\partial_i,$$

where

$$b_i^0 = -\frac{\partial_i\rho}{\rho} + \frac{2}{|z|^2}\sum_{k,l} z_k z_l \frac{\partial^2 v_i}{\partial y_k \partial y_l}.$$

The boundedness of the coefficients and the uniform ellipticity of N allows (see [58], Chapter 2, Theorem 5) us to apply the maximum principle at y_0. Therefore the pressure cannot reach a local minimum at some interior point.

Various components of the boundary. Since we have shown that the minimum of the pressure over D must be achieved somewhere on the boundary ∂D, we now investigate on which part it occurs.

[22]This equality means that $u = y$ holds at second order at the point y_0.

The wall. This minimum cannot occur along the wall, because of the following consequence of (114) and of the boundary condition:

$$\frac{\partial p}{\partial \nu} = 0,$$

where the left-hand side is the normal derivative. This, together with the elliptic equation (115) tells that p does not achieve a local minimum at such a point. There are however two subtleties:

- At the corner of the wedge, which is a geometrical singularity. Remark that our domain Ω is convex and could be approximated by smooth convex domains by smoothing out the vertex at the origin. Then the boundary condition is written, in terms of the pressure,

$$\frac{\partial p}{\partial \nu} = \rho |v|^2 \kappa, \tag{118}$$

  where κ is the curvature of the boundary, positive in this case. Hence $\partial p/\partial \nu$ is positive (treat the stagnation points as above) and p cannot be minimal along the wall.
- Once again, the maximum principle does not apply directly at a boundary point where v vanishes. At such a point, $\partial p/\partial \nu$ vanishes because of (118). The minimality along the boundary tells also that the tangential derivative vanishes, whence $\nabla p(y_0) = 0$. But then the same arguments as in the case of an interior stagnation point are valid. We find that $\nabla v(y_0)$ and the coefficients of N are locally bounded. Therefore the maximum principle holds at such boundary points.

We point out that the same argument works in the steady case and has a natural interpretation. If the ramp is compressive ($\alpha > 0$ as in our case), the pressure is likely to increase ahead, while if the corner is expansive ($\alpha < 0$), we have the opposite situation ; the pressure cannot be maximal at a boundary point and it is likely to decrease ahead.

The sonic line. Along the sonic line, the solution is continuous, thus $p \equiv p_2$. Since U_2 is the subsonic state in the reflected shock between U_1 and U_2, we have $p_2 > p_1$ and therefore $p > p_1$ along the sonic line.

The diffracted shock. The last component of ∂D is the part of the reflected shock between the point Q and the symmetry axis. Since the flow is subsonic on the inner side and supersonic in the outer, we have that the inner trace of the pressure is higher than p_1.

In conclusion, the following minimum principle is expected for the pressure:

$$\boxed{p(y) \geq p_1, \qquad \forall y \in D.} \tag{119}$$

We point out that this justifies *a posteriori* the fact that the density does not vanish, provided the internal energy remains bounded.

6.3 Estimate for the diffracted shock

We still follow [63] and assume that the reflection agrees with Figure 10, and use the fact that the state U_1 is (pseudo-)supersonic relative to the reflected shock:

$$|v_1 \cdot \nu| > c_1. \tag{120}$$

Here, $v_1 = u_1 - y$ and u_1, c_1 are constants. Hence this reads

$$|(y - u_1) \times \dot{y}^\perp| > c_1,$$

when parametrizing the shock by arclength. In particular, the vector product does not vanish. By continuity, it must keep a constant sign, which is positive if the arc length is measured from Q. We have therefore

$$(y - u_1) \times \dot{y}^\perp > c_1, \tag{121}$$

For the moment, let us study the curves passing through Q, which are defined by equality in (121):

$$(Y - u_1) \times \dot{Y}^\perp = c_1, \qquad |\dot{Y}| = 1. \tag{122}$$

The system (122) defines two well-posed differential equations for $|Y - u_1| > c_1$, where Y determines two values $\dot{Y}$ as intersection points of a circle and a straight line. It degenerates on the circle C_1 of equation $|Y - u_1| = c_1$, where $\dot{Y}$ must be the unit tangent to C_1. If $|Y - u_1| < c_1$, there is no solution $\dot{Y}$. We infer that the integral curves are made of an arc of C_1, followed at each extremity by the tangents to the circle[23]. One or two parts (a tangent and/or the arc) may be omitted. Since $|y_Q - u_1| > c_1$ (because the downstream flow is supersonic at Q), we see that an integral curve originating at Q must follow the tangent to C_1, the one that approaches C_1 counterclockwise. Then it is free to follow C_1 until it leaves it along another tangent. Several integral curves are shown on Figure 17.

If the inequality in (121) held in the weak sense ($\geq$ instead of $>$), we could only say that the reflected shock stays below the upper extreme integral curve of (122), at least the one going towards C_1. This was the result obtained in [63], which gives a rather poor information, both in terms of sufficient condition for the boundedness of D, and in terms of estimates. But since the inequality in (121) is strict, we know that the reflected shock stays actually below every integral curve of (121) (because it is below every one near Q). Therefore, it is bounded by the extreme integral curve defined as the tangent from Q towards C_1, followed by the circle till the horizontal axis.

As a conclusion, we have the following estimate of the subsonic domain, which is illustrated in Figure 18:

> The subsonic domain D is contained in the convex set whose boundary is made of a part of the wall (horizontal axis and ramp), a part of the circle C_1 of equation $|Y - u_1| = c_1$, a part of the tangent to C_1 passing through Q, and the part of the sonic circle $|Y - u_2| = c_2$ between the ramp and the point Q.

[23]The conclusion in [63] was erroneous, because we did not pay attention to the degeneracy of (122) along the circle.

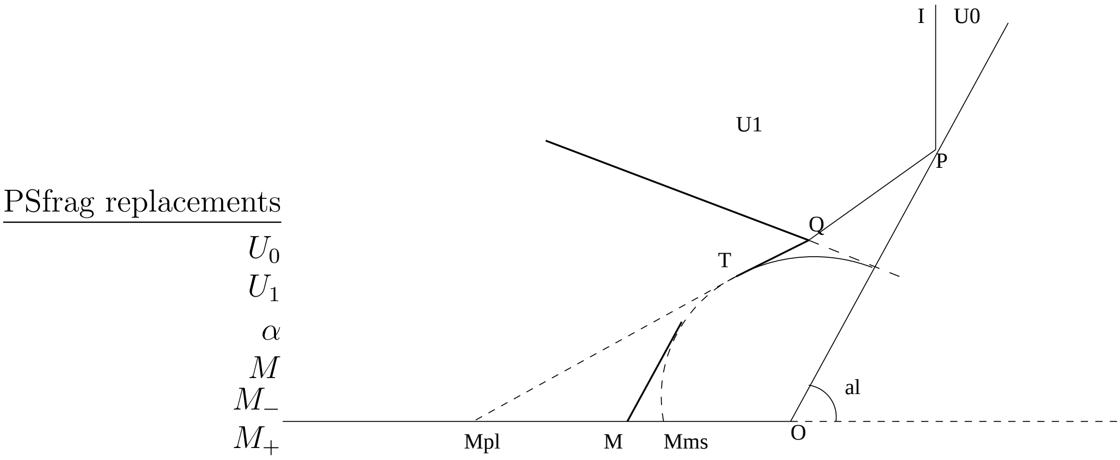

Figure 17: Integral curves for $(y - u_1) \times \dot{y} = c_1$. Boldface lines are tangents to the circle $C(u_1; c_1)$. One of them goes upper to the left. The other one meets the circle at T, from which every integral curve follows the dashed part of the circle, until it leaves it along a tangent. One extreme curve ends at point M_-. The other one is the continuation of the tangent passing through Q ; M_+ might not exist.

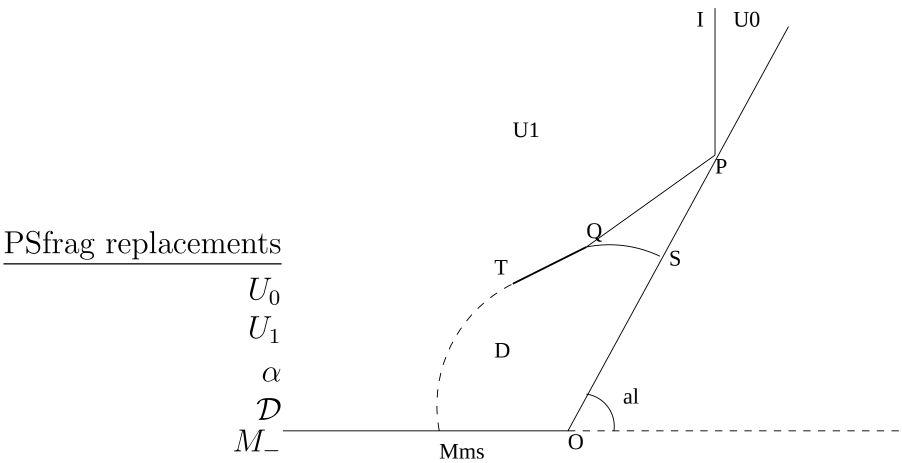

Figure 18: The convex set $\mathcal{D}$ contains the subsonic domain. It is bounded by the arc SQ of the sonic circle of state U_2, the arc TM_- of the sonic circle of state U_1, plus its tangent at T through Q, and finally the part of the wall M_-OS.

6.4 Using the Bernoulli invariant

We still follow [63]. We know that, away from the discontinuities, the pseudo-Bernoulli invariant

$$B := \frac{1}{2}|v|^2 + e + \frac{p}{\rho} = \frac{1}{2}|v|^2 + \gamma e$$

satisfies the damped transport equation

$$v \cdot \nabla B + |v|^2 = 0.$$

We recall that the damping term $|v|^2$ is due to self-similarity. Given a smooth function f, we deduce

$$v \cdot \nabla f(B) + |v|^2 f'(B) = 0.$$

Multiplying by ρ and using (96), we obtain

$$\text{div}(\rho f(B) v) + 2\rho f(B) + \rho |v|^2 f'(B) = 0. \tag{123}$$

Restricting our attention to D, (123) is valid everywhere in the sense of distributions, because there is no shock, and the slip condition $(v \cdot \nu)_\pm = 0$ along contacts is harmless. Therefore we may integrate over D and find

$$\int_{\partial D} \rho f(B) v \cdot \nu \, dl + \int_D \rho(2f(B) + |v|^2 f'(B)) \, dy = 0. \tag{124}$$

Let us discuss the boundary integral in (124). On the one hand, the integrand $\rho f(B) v \cdot \nu$ is a trace taken from the subsonic side. On another hand, it vanishes along the wall because $v \cdot \nu = 0$. Hence the integral can be taken on the upper part ∂D^+ only, made of the sonic line SQ and the diffracted shock.

Let us denote by $\bar{B}$ the supremum of the inner trace of B along ∂D^+. We now choose a non-decreasing and smooth function, such that f vanishes all along $(-\infty, \bar{B}]$ and is positive elsewhere. Then the boundary integral in (124) vanishes, and the last term is non-negative, whence

$$\int_D \rho f(B) \, dy \leq 0,$$

from which we deduce a *maximum principle*:

$$B(y) \leq \bar{B}, \qquad \forall y \in D. \tag{125}$$

We then need an explicit bound for $\bar{B}$. Since U is continuous across the sonic line, we have on the one hand

$$B = \frac{1}{2}|u_2 - y|^2 + \gamma e_2 \equiv \frac{c_2^2}{2} + \gamma e_2 = \frac{\gamma(\gamma+1)}{2} e_2$$

along QS. We denote this constant by B_2.

On the other hand, because of (102) (which is even an equality for the full Euler system), we have $j[B] \leq 0$ across a shock, say the reflected shock. However, we know that the pseudo-flow enters[24] D, meaning that $j < 0$ when we orient the normal to the exterior of D. Therefore the inner trace is less than the outer trace, the latter being

$$B_1(y) := \frac{1}{2}|y - u_1|^2 + \gamma e_1.$$

Of course, $B_1(y)$ is not explicit along the reflected shock, because this curve is not known with precision. However, the previous paragraph gives us a rather good estimate for D: the diffracted shock is bounded by the extreme integral curve QM_- of Figure 17. Therefore it is not difficult to bound B_1 on this part by the supremum of B_1 along QM_-, that is by $B_1(Q)$ (outer trace). Remarking as before that the outer trace $B_1(Q)$ dominates the inner trace $B(Q) = B_2$, we deduce at last an explicit upper bound $\bar{B} \le B_1(Q)$. Finally, we have

$$\boxed{B(y) \le B_1(Q), \qquad \forall y \in D.} \tag{126}$$

This is an accurate bound for the pseudo-Bernoulli invariant, except for the fact that $B_1(Q)$ is the largest value of the outer trace, which is strictly larger than the corresponding inner trace ; B_2 is likely to be a more accurate upper bound, though we do not have a satisfactory argument for this claim.

6.5 Conclusion ; pointwise estimates

Let us begin with the **full Euler** system. We use estimates (113,119,126), which imply

$$e\rho^{1-\gamma} \ge \exp(S_1), \qquad \rho e \ge \rho_1 e_1, \qquad \gamma e \le B_1(Q).$$

Altogether, these inequalities give us explicit lower and upper bounds for the density and specific energy (hence for the pressure and the temperature):

$$\boxed{(\rho_1 e_1)^{1-1/\gamma} \exp \tfrac{S_1}{\gamma} \le e \le \tfrac{B_1(Q)}{\gamma}, \qquad \tfrac{\gamma \rho_1 e_1}{B_1(Q)} \le \rho \le \left(\tfrac{B_1(Q)}{\gamma}\right)^{1/(\gamma-1)} \exp \tfrac{S_1}{1-\gamma}.} \tag{127}$$

Additionally, (126) gives $|v|^2 \le 2B_1(Q)$, which, together with the estimate of the size of D, say $D \subset B(0;R)$ for an explicit R, yields

$$\boxed{|u| \le R + \sqrt{2B_1(Q)}.} \tag{128}$$

We now turn to **barotropic flow**. The difference with the previous case is that there is no entropy. Hence we use only estimates (119) and (126):

$$\rho^\gamma \ge \rho_1^\gamma, \qquad \frac{\gamma}{\gamma - 1}\rho^{\gamma-1} \le B_1(Q).$$

We deduce the bounds

$$\boxed{\rho_1 \le \rho \le \left(\tfrac{\gamma-1}{\gamma} B_1(Q)\right)^{1/(\gamma-1)}} \tag{129}$$

Then the same estimate as in (128) holds true.

[24]This might not agree with common sense, but we must keep in mind that the pseudo-flow satisfies $\mathrm{div}(\rho v) < 0$ and thus is somehow convergent.

6.6 The vortical singularity

We show here that the vorticity cannot be square integrable. Our computation is valid for piecewise smooth solutions in the barotropic case. More precisely, we show that if the solution is (piecewise) smooth enough, then

$$(2-p)\int_D \rho \left|\frac{\omega}{\rho}\right|^p dy \qquad (\omega := \partial_1 v_2 - \partial_2 v_1)$$

admits a nonzero limit as $p \to 2^-$; in particular, it is not smooth. We recall that D denotes the subsonic zone, the domain bounded by the wall, the reflected shock and the sonic line (actually, ω vanishes everywhere else).

Up to Equation (131) below, our calculations are valid in D_r, the complement in D of discontinuities[25]. We start with Equation (100), which we divide by ρ (we have seen that the density does not vanish). Remark that $\rho^{-1}\nabla p$ is the gradient of enthalpy $\nabla i(\rho)$, where $i'(s) = s^{-1}p'(s)$. Taking the curl, we obtain

$$(v \cdot \nabla)\omega + (1 + \mathrm{div} v)\omega = 0. \tag{130}$$

Combining with (96), we deduce the transport equation

$$(v \cdot \nabla)\frac{\omega}{\rho} = \frac{\omega}{\rho}. \tag{131}$$

Let f be a Lipschitz continuous function of one variable. Multiplying (131) by $f'(\omega/\rho)$, we get

$$(v \cdot \nabla) f\left(\frac{\omega}{\rho}\right) = \frac{\omega}{\rho} f'\left(\frac{\omega}{\rho}\right). \tag{132}$$

We now recombine with (96) and get

$$\mathrm{div}\left(f\left(\frac{\omega}{\rho}\right)\rho v\right) = \rho g\left(\frac{\omega}{\rho}\right), \qquad g(s) := sf'(s) - 2f(s). \tag{133}$$

We emphasize that (133) is valid off vortex sheets. However, since the normal component of v along vortex sheets vanishes, the divergence in (133) does not present any singular part. Hence this identity holds in the distributional sense in D. We now integrate over D and obtain

$$\int_{\partial D} f\left(\frac{\omega}{\rho}\right)\rho v \cdot \nu \, dl = \int_D \rho g\left(\frac{\omega}{\rho}\right) dy,$$

where we warn the reader that the boundary integral involves inner traces. Choosing $f(s) := |s|^p$, this yields

$$(2-p)\int_D \rho \left|\frac{\omega}{\rho}\right|^p dy = -\int_{\partial D} \left|\frac{\omega}{\rho}\right|^p \rho v \cdot \nu \, dl. \tag{134}$$

[25]We recall that these discontinuities are only slip lines, because steady shocks are forbidden between two subsonic states, according to Theorem 2.2.

Let us examine the limit of the right-hand side as $p \to 2$:

$$F := -\int_{\partial D} \left|\frac{\omega}{\rho}\right|^2 \rho v \cdot \nu \, dl. \tag{135}$$

The integral over ∂D decomposes as a sum over the wall, and the rest of the boundary ∂D_-. The first part vanishes because of the boundary condition. It vanishes along the sonic line too, because the irrotationality of U_2 propagates across this line[26]. Along the reflected shock, we know that ω is non zero[27] and that the normal pseudo-velocity $v \cdot \nu$ is negative. In conclusion the limit F is positive. We summarize our results in the following statement

Theorem 6.1 *Let U be a piecewise smooth symmetric RR for a barotropic gas, with qualitative features as described in Figure 10. Then the vorticity $\omega := \partial_1 u_2 - \partial_2 u_1$ cannot be square integrable.*

We point out that the vortical singularity is located at a stagnation point. As a matter of fact, (131) is an ODE that transports and amplifies ω/ρ. The amplification remains finite as long as v does not vanish. Only at stagnation points can ω/ρ blow up. Of course, since ρ and $1/\rho$ are uniformly bounded, the singularity concerns only ω.

A reasonable expectation is that the vortical singularity arises somewhere along the ramp. As a matter of fact, there must exist a stagnation point along the boundary, since $v \cdot \nu = 0$ holds and the pseudo-velocity field is incoming at both ends. This does not rule out the possibility that several stagnation points exist. But, assuming the generic property that they are non-degenerate, there must be $N + 1$ attractors and N saddle points for some $N \geq 0$, since the total degree of the pseudo-flow in the subsonic zone is $+1$. Remark that Equation (96) forbids repellors. If $N \geq 1$, then the stable manifolds of the saddle points come from the exterior of the subsonic zone and divide D into $N + 1$ regions, each one containing precisely one attractor.

6.6.1 A formal description of the vortical singularity

We wish to describe qualitatively the behaviour of the flow near the vortical singularity located at some point y_0, using Equation (131) and the conservation of mass (96). Because of $v(y_0) = 0$, this point is critical for the pressure, thus for ρ, since the equation of state is barotropic. For this reason, we make the approximation that ρ is locally constant. Therefore, we base our analysis upon the simplified system

$$\operatorname{div} v = -2, \qquad v \cdot \nabla \omega = \omega, \qquad \omega = \operatorname{curl} v. \tag{136}$$

Because of the evidence given by Theorem 6.1, the solution must be such that the integral of ω^2 diverges precisely at y_0.

We begin with the simpler case (maybe not realistic) that y_0 is an interior point. Then system (136) admits a one-parameter family of rotationally invariant solution. Using polar

[26]This means somehow that the vorticity is continuous across the sonic line, in spite of the fact that the rest of ∇u is unbounded.

[27]Were it to vanish, the reflected shock would remain straight, which is false. As a matter of fact, the proof of Theorem 2.6 is easily adapted, since (130) still propagates irrotationality.

ccordinates around y_0, we have $v_r = -r$ and $v_\theta = v_\theta(r)$. There comes $\omega = v_\theta' + \frac{1}{r}v_\theta$. Since ω depends only upon r, we have

$$v \cdot \nabla\omega = -r\partial_r\omega.$$

Thus ω must be of the form ω_0/r, meaning that $(rv_\theta)' = \omega_0$. Since the pseudo-velocity is locally bounded (see Section 6.5), we conclude that

$$v \sim -r\vec{e_r} + \omega_0\vec{e_\theta}, \qquad \omega \sim \frac{\omega_0}{r}. \tag{137}$$

We point out that ω_0 is the amplitude of the singularity. This amplitude must be somehow proportional to the vorticity generated across the diffracted shock.

We now turn to the situation where the singularity occurs at a boundary point. Then the description given by (137) is not satisfactory, because the trajectories of this flow are transversal to the boundary. A refined analysis seems unable to give a solution where ω is of order $1/r$ at the singularity. Thus we look for a slightly more singular solution of (136). We begin by introducing a potential ϕ such that $v = -r\vec{e_r} + \mathrm{curl}\phi$, thanks to the first of (136) ; we have $\omega = \Delta\phi$. Then we search for a ϕ of the form

$$\phi \sim m(r)a(\theta).$$

Several choices of m are possible *a priori*, with the only constraint that

$$\int_0 (\Delta\phi(r))^2 \frac{dr}{r} = +\infty,$$

because of Theorem 6.1. Let us first try $m(r) = r^\alpha$, where we need $\alpha < 1$. This gives respectively

$$v \cdot \nabla\omega \sim r^{2\alpha-4}\left[\alpha a(\alpha^2 a + a'')' + (2-\alpha)a'(\alpha^2 a + a'')\right] \tag{138}$$

and

$$\omega \sim r^{\alpha-2}(\alpha^2 a + a'').$$

Since $r^{2\alpha-4}$ dominates $r^{\alpha-2}$, the dominant term in (138) must vanish. This means that

$$a^\gamma(\alpha^2 a + a'') = \mathrm{cst} =: \kappa, \qquad \gamma := \frac{2-\alpha}{\alpha}.$$

This autonomous equation of the form $a'' = F(a)$ is integrable by quadrature, as is well known. We want $a(0) = a(\pi/2) = 0$, in order that the trajectories be tangent to the boundary[28]. If $\kappa = 0$, then $a(\theta) = \mathrm{cst}\sin(\alpha\theta)$, with α an integer. With the constraint, we obtain $\alpha = 1$, but then ω is smaller than $1/r$ and is unlikely to satisfy Theorem 6.1.

If $\kappa \neq 0$, a remarkable phenomenon happens. The solution of $a'' = F(a)$ with $a(0) = 0$ is monotonic on some interval $(0, \theta_0)$ until a point where $a'(0) = 0$. Thus we need that π be

[28]We choose the angle θ such that the boundary is given by $\theta \equiv 0 \,(\mod \pi)$.

a mutiple of $2\theta_0$. It turns out that $\theta_0 \geq \pi/2$, with equality in the case where a satisfies the integrated equation

$$a'^2 + \alpha^2 a^2 = \kappa_1 a^{1-\gamma} \qquad (\kappa_1(1-\gamma) = \kappa). \tag{139}$$

Thus a must be precisely one of the solutions of (139). However, these solutions behave like θ^α near $\theta = 0$. Since $\alpha < 1$ and $v_\theta \sim m'(r)a(\theta)$, we find that the trajectories are still transversal to the boundary (because a is not Lipschitzian, or quasi-Lipschitz, at $\theta = 0$).

Thus let us try a slight modification of the interior case, with $m(r) = r \log r$. We then find

$$v \cdot \nabla\omega \sim \left(\frac{\log r}{r}\right)^2 (a(a + a''))'.$$

Once again, this dominant term must vanish, giving

$$a(a + a'') = \text{cst} = \kappa.$$

It is unlikely that κ vanishes. If it does not, a rescaling yields

$$a'' + a = -\frac{1}{2a},$$

and therefore

$$a'^2 + a^2 = \log\frac{\delta}{a},$$

δ a positive constant. An analysis as above yields a solution for which $a(0) = 0$ and $a'(\theta_0) = 0$, where

$$\theta_0 := \int_0^{a_*} \frac{da}{\sqrt{\log\frac{\delta}{a} - a^2}},$$

and $a_*(\delta)$ is the root of the denominator in $(0, \delta)$. It turns out that θ_0 is *always less than* $\pi/2$, and approaches this value when $\delta \to +\infty$. It also tends to zero when $\delta \to 0+$.

Taking such a solution on $(0, \theta_0(\delta))$ and extending by parity, we obtain a solution with $a(0) = a(2\theta_0) = 0$. We think that it is reasonable to match it with $a \equiv 0$ on $(2\theta_0, \pi)$, which still cancels the dominant term in $v \cdot \nabla\omega$. One advantage of this construction is that we do expect a zone where the vorticity vanishes identically: this is the influence domain of the sonic line ; see Figure 19. An other convenience is that the corresponding v_θ is quasi-Lipschitz at $\theta = 0$ and therefore the streamlines all converge to the stagnation point, instead of crossing the material boundary.

In conclusion, the singularity of ω may not be significantly larger than $1/r$, otherwise the streamlines rotate too much and do cross the material boundary. On the other hand, it cannot be significantly smaller, because of Theorem 6.1. Thus the real singularity must be of order $1/r$, up to say a logarithmic correction.

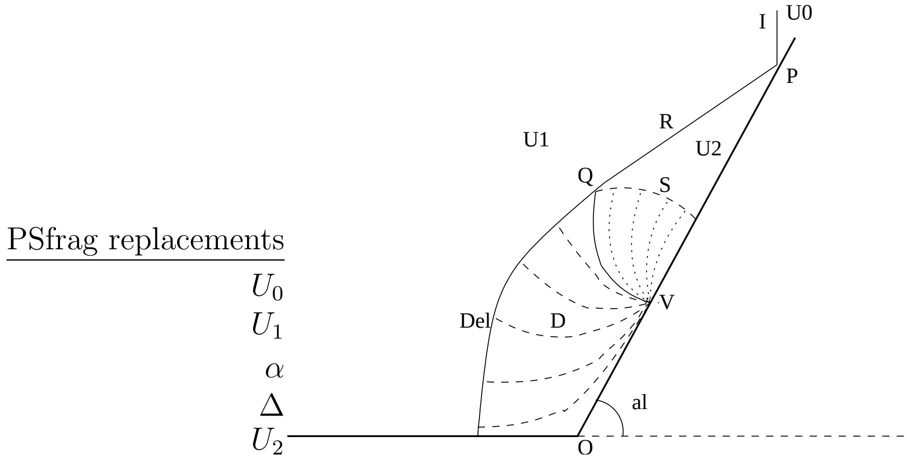

Figure 19: A vortical singularity at a boundary point. The streamlines converge to V. The dotted ones come from the sonic line. The dashed ones come from the diffracted shock. The vorticity vanishes identically in the doted zone.

Remark. At first glance, the identity (134), and the fact that the limit F given by (135) is finite, suggest a singularity of order exactly $1/r$. This is in contradiction with the impossibility to build such a vortical singularity at a boundary point. However, the derivation of (134) assumed an amount of regularity that is not valid. Being more carefull, we should have integrated (133) over $D \setminus B_\epsilon$, with B_ϵ a small (half-)disk around y_0. Then there is a contribution from ∂B_ϵ in (134), which does not vanish when $\epsilon \to 0$, and may even tend to infinity. With this correction, a singularity of order higher than $1/r$ becomes coherent.

Open problems. We conclude this discussion with two important questions that remain to be solved:

- Describe in more detail and with higher rigor the vortical singularity.
- Is there a vortical singularity for the full Euler model ?

This is unclear. It could be that the vortical singularity is just a consequence of the barotropic assumption. Whatever the answer to this question, we keep in mind that the solution *must* be singular, even in the full Euler model, because the transport equation $v \cdot \nabla S = 0$ that follows from the self-similar equations, and the fact that the entropy may not be constant along the diffracted shock, imply a discontinuity of S at the center V of the pseudo-flow (see Figure 19). If moreover the pressure is continuous, as suggested by the elliptic equation (115), then the discontinuity of S induces a discontinuity of the density ρ.

References

[1] R. K. Agarwal and D. W. Halt. A modified CUSP scheme in wave/particle split form for unstructured grid Euler flows. In D. A. Caughey and M. M. Hafez, editors, *Frontiers in Computational Fluid Dynamics*, pages 155–168, New York, 1994. John Wiley and Sons.

[2] M. Artola and A. Majda. Nonlinear development of instabilities in supersonic vortex sheet I. The basic kink modes. *Physica*, D28:253–281, 1987.

[3] V. Bargmann and D. Montgomery. Prandtl–Meyer zones in Mach reflection. OSRD 5011, Office of Scientific Research and Development, 1950 (?).

[4] G. Ben-Dor. *Shock wave reflection phenomena*. Springer-Verlag, New York, 1992.

[5] G. Ben-Dor. Oblique shock wave reflections. In G. Ben-Dor, O. Igra, and T. Elperin, editors, *Handbook of shock waves*, volume 2, pages 67–179. Academic Press, San Diego, 2001.

[6] W. Bleakney and A. H. Taub. Interaction of shock waves. *Rev. Mod. Phys.*, 21:584–605, 1949.

[7] M. Brio and J. K. Hunter. Mach reflection for the two-dimensional Burgers equation. *Physica D*, 60:194–207, 1992.

[8] S. Čanić and B. L. Keyfitz. Oblique shock interaction and the von Neumann paradox. In B. Sturtevant, J. Schepherd, and H. Hornung, editors, *Proc. 20th Int. Conf. on shock waves*, River Edge, NJ, 1996. World Scientific Pub.

[9] S. Čanić and B. L. Keyfitz. Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problem. *Arch. Rational Mech. Anal.*, 144:233–258, 1998.

[10] S. Čanić and B. L. Keyfitz. Riemann problems for the two-dimensional unsteady transonic small disturbance equation. *SIAM J. Appl. Math.*, 58(2):636–665, 1998.

[11] S. Čanić, B. L. Keyfitz, and Eun Heui Kim. Free boundary problem for the unsteady transonic small disturbance equation: Transonic regular reflection. *Methods and Appl. of Anal.*, 7(2):313–336, 2000.

[12] S. Čanić, B. L. Keyfitz, and Eun Heui Kim. Mixed hyperbolic-elliptic systems in self-similar flows. *Boll. Sociedade Brasileira Mat.*, 32(3):377–399, 2001.

[13] S. Čanić, B. L. Keyfitz, and Eun Heui Kim. A free boundary problem for a quasi-linear degenerate elliptic equation: Regular reflection of weak shocks. *Comm. Pure and Applied Math.*, LV:71–92, 2002.

[14] S. Čanić, B. L. Keyfitz, and Eun Heui Kim. Free boundary problems for nonlinear wave systems: Mach stems for interacting shocks. Preprint, Univ. of Houston (TX), 2004.

[15] S. Čanić, B. L. Keyfitz, and G. Lieberman. A proof of existence of perturbed steady transonic shocks *via* a free boundary problem. *Comm. Pure and Applied Math.*, LIII:484–511, 2000.

[16] S. Čanić, B. L. Keyfitz, and D. H. Wagner. A bifurcation diagram for oblique shock interactions in the unsteady transonic small disturbance equation. In J. Glimm, M. Graham, J. Grove, and B. Plohr, editors, *Hyperbolic problems, Theory, Numerics and Applications (Stony Brook, NY 1994)*, pages 178–187, River Edge, NJ, 1996. World Scientific Pub.

[17] S. Čanić and D. Mirković. A numerical study of Riemann problems for the two-dimensional unsteady transonic small disturbance equation. *SIAM J. Appl. Math.*, 58(5):1365–1393, 1998.

[18] Tong Chang and Gui-Qiang Chen. Diffraction of planar shock along compressive corner. *Acta Math. Scienta*, 6(3):241–257, 1986.

[19] Tong Chang and Gui-Qiang Chen. Some fundamental concepts about systems of two spatial dimensional conservation laws. *Acta Math. Scienta*, 6(4):463–474, 1986.

[20] Tong Chang and Ling Hsiao. *The Riemann problem and interaction of waves in gas dynamics*, volume 41 of *Pitman Surv. in Pure and Appl. Maths.* Longman, 1989.

[21] Gui-Qiang Chen and Misha Feldman. Existence of global solutions to regular shock reflection for potential flow. FRG Workshop on Multi-dimensional Hyperbolic Conservation Laws, Madison (WI), 2005.

[22] Gui-Qiang Chen and Dehua Wang. The Cauchy problem for the Euler equations for compressible fluids. In S. Friedlander and D. Serre, editors, *Handbook of mathematical dynamics*, pages 421–543. North-Holland, Elsevier, Amsterdam, 2002.

[23] Shuxing Chen. On reflection of multidimensional shock front. *J. Differential Equ.*, 80:199–236, 1989.

[24] Shuxing Chen. Linear approximation of shock reflection at a wedge with large angle. *Comm. Partial Diff. Equ.*, 21:1103–1118, 1996.

[25] Shuxing Chen. Linearization of shock reflection by almost perpendicular ramp. In *Conference on Nonlinear Evolution Equations and Infinite-Dimensionnal Dynamical Systems (Shanghai, 1995)*, pages 13–23, River Edge, NJ, 1997. World Scientific Pub.

[26] P. Collela and L. F. Henderson. The von Neumann paradox for the diffraction of weak shock waves. *J. Fluid Mech.*, 213:71–94, 1990.

[27] R. Courant and K. O. Friedrichs. *Supersonic flow and shock waves.* Springer-Verlag, New York, 1976. Reprinting of the 1948 original, Applied Mathematical Sciences, Vol. 21.

[28] C. Dafermos. Estimates for conservation laws with little viscosity. *SIAM J. Math. Anal.*, 18:409–421, 1987.

[29] C. Dafermos. *Hyperbolic conservation laws in continuum mechanics*, volume 325 of *Grundlehren der mathematischen Wissenschaften.* Springer, Heidelberg, 2000.

[30] I. M. Gamba, R. R. Rosales, and E. G. Tabak. Constraints on possible singularities for the unsteady transonic small disturbance (UTSD) equations. *Comm. Pure and Appl. Math.*, LII:763–779, 1999.

[31] I. M. Gel'Fand. Some questions of analysis and differential equations. *Amer. Math. Soc. Transl. (2)*, 26:201–219, 1963.

[32] H. M. Glaz. Self-similar shock reflection in two space dimensions. In J. Glimm and A. Majda, editors, *Multidimensional hyperbolic problems and computations*, volume 29 of *IMA volumes in Mathematics and applications*, pages 70–88, New York, 1991. Springer.

[33] K. G. Guderley. Considerations on the structure of mixed subsonic-supersonic flow patterns. Air Material Command technical report F-TR-2168-ND, ATI No. 22780, GS-AAF-Wright Field No. 39, U.S. Wright-Patterson Air Force Base, Dayton, OH, 1947.

[34] K. G. Guderley. *The theory of transonic flows.* Pergamon Press, Oxford, 1962.

[35] E. Harabetian. Diffraction of a weak shock by a wedge. *Comm. Pure and Appl. Math.*, XL:849–863, 1987.

[36] A. Harten, P. D. Lax, C. D. Levermore, and W. Morokoff. Convex entropies and hyperbolicity for general Euler equations. *SIAM J. Numer. Anal.*, 35:2117–2127, 1998.

[37] A. Heibig and D. Serre. Étude variationnelle du problème de Riemann. *J. Differential Equ.*, 96:56–88, 1992.

[38] L. F. Henderson. Regions and boundaries fro diffracting shock waves. *Z. Ang. Math. Mech.*, 67(2):73–86, 1987.

[39] L. F. Henderson. General laws for propagation of shock waves through matter. In G. Ben-Dor, O. Igra, and T. Elperin, editors, *Handbook of shock waves*, volume 1. Academic Press, San Diego, 2001.

[40] L. F. Henderson and R. Menikoff. Triple-shock entropy theorem and its consequences. *J. Fluid Mech.*, 366:179–210, 1998.

[41] H. Hornung. Regular and Mach reflection of shock waves. *Ann. Rev. Fluid Mech.*, 18:33–58, 1986.

[42] J. K. Hunter. Nonlinear geometrical optics. In J. Glimm and A. Majda, editors, *Multidimensional hyperbolic problems and computations*, volume 29 of *IMA volumes in Mathematics and applications*, pages 179–197, New York, 1991. Springer.

[43] J. K. Hunter and M. Brio. Weak shock reflection. *J. Fluid Mech.*, 410:235–261, 2000.

[44] J. K. Hunter and J. B. Keller. Weakly nonlinear high frequency waves. *Comm. Pure and Applied Math.*, 36:547–569, 1983.

[45] J. K. Hunter and J. B. Keller. Weak shock diffraction. *Wave motion*, 6:79–89, 1984.

[46] E. Jouguet. Sur la propagation des discontinuités dans les fluides. *C. R. Acad. Sci. Paris*, 132:673–676, 1901.

[47] J. B. Keller and A. Blank. Diffraction and reflection of pulses by wedges and corners. *Comm. Pure and Applied Math.*, 4:75–94, 1951.

[48] S. Kruzkov. First-order quasilinear equations with several space variables. *Math. USSR Sbornik*, 10:217–273, 1970.

[49] P. Lax. Shock waves and entropy. In E. A. Zarantonello, editor, *Contributions to Functional Analysis*, pages 603–634. Academic Press, New York, 1971.

[50] M. J. Lighthill. The diffraction of blast. I. *Proc. Roy. Soc. Lond. Ser. A*, 198:454–470, 1949.

[51] E. Mach. Über dev Verlauf von Funkenwellen in der Ebene und im Raume. *Sitzungsbr. Akad. Wiss. Wien*, 77:819–838, 1878.

[52] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* Springer-Verlag, New York, 1984.

[53] A. Majda. One perspective on open problems in multi-dimensional conservation laws. In J. Glimm and A. Majda, editors, *Multidimensional hyperbolic problems and computations*, volume 29 of *IMA volumes in Mathematics and applications*, pages 217–238, New York, 1991. Springer.

[54] C. Morawetz. Potential theory for regular and Mach reflection of a shock at a wedge. *Comm. Pure and Applied Math.*, XLVII:593–624, 1994.

[55] J. von Neumann. The main effect and height of burst. In *Collected works of John von Neumann*, volume VI, pages 309–347. Pergamon Press, New York, 1963.

[56] J. von Neumann. Oblique reflection of shocks. In *Collected works of John von Neumann*, volume VI, pages 238–299. Pergamon Press, New York, 1963.

[57] J. von Neumann. Refraction, intersection and reflection of shock waves. In *Collected works of John von Neumann*, volume VI, pages 300–308. Pergamon Press, New York, 1963.

[58] M. H. Protter and H. Weinberger. *Maximum principles in differential equations.* Prentice Hall, New York, 1967.

[59] A. Sakurai. On the problem of weak Mach reflection. *J. Phys. Soc. Japan*, 19:1440–1450, 1964.

[60] A. Sakurai, L. F. Henderson, K. Takayama, Z. Walenta, and P. Collela. On the von Neumann paradox of weak Mach reflection. *Fluid Dyn. Research*, 4:333–345, 1989.

[61] C. W. Schultz-Rinne, J. P. Collins, and H. M. Glaz. Numerical solution of the Riemann problem for two-dimensional gas dynamics. *SIAM J. Sci. Comput.*, 14(6):1394–1414, 1993.

[62] D. Serre. Domaines invariants pour les systèmes hyperboliques de lois de conservation. *J. Differential Equ.*, 69:46–62, 1987.

[63] D. Serre. Ecoulements de fluides parfaits en deux variables indépendantes de type espace. Réflexion d'un choc plan par un dièdre compressif. *Arch. Rational Mech. Anal.*, 132:15–36, 1995.

[64] D. Serre. *Systems of conservation laws. Vol. I.* Cambridge Univ. Press, Cambridge, 2000.

[65] Li Ta-tsien and Yu Wen-ci. *Boundary value problems for quasilinear hyperbolic systems.* Duke University, Durham, NC, 1987.

[66] E. G. Tabak and R. R. Rosales. Focusing of weak shock waves and the von Neumann paradox for oblique shock reflection. *Phys. Fluids. A*, 6:1874–1892, 1994.

[67] E. Tadmor. A minimum entropy principle in the gas dynamics equations. *Appl. Numer. Math.*, 2:211–219, 1986.

[68] Dechun Tan and Tong Zhang. Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws, I. Four-J cases. *J. Differential Equ.*, 111:203–254, 1994.

[69] Dechun Tan and Tong Zhang. Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws, II. Initial data involving some rarefaction waves. *J. Differential Equ.*, 111:255–282, 1994.

[70] L. Tartar. Une introduction à la théorie mathématique des systèmes de lois de conservation. Publ. Inst. Anal. Num. 682, Univ. Pavia, 1989.

[71] A. M. Tesdall and J. K. Hunter. Self-similar solutions for weak shock reflection. *SIAM J. Appl. Math.*, 63(1):42–61, 2002.

[72] V. M. Teshukov. Shock polars in a gas with general equation of state. *Prikl. Mat. Mekh.*, 50(4):98–103, 1986. In Russian.

[73] V. M. Teshukov. Stability of regular shock wave reflection. *J. Appl. Mech. Tech. Phys.*, 30(2):189–196, 1989.

[74] F. Ting and H. F. Ludloff. Aerodynamics of blasts. *J. Aero. Sci.*, 19:317–328, 1952.

[75] H. Weyl. Shock waves in arbitrary fluids. *Comm. Pure Appl. Math.*, 2:103–122, 1949.

[76] G. B. Whitham. *Linear and nonlinear waves.* John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

[77] Shuli Yang and Tong Zhang. The MmB difference solutions of 2-d Riemann problems for a 2×2 hyperbolic system of conservation laws. *Impact of computing in science and engineering*, 3:146–180, 1991.

[78] Tong Zhang, Gui-Qiang Chen, and Shuli Yang. On the 2-d Riemann problem for the compressible Euler equations I. Interaction of shocks and rarefaction waves. *Disc. Cont. Dyn. Sys.*, 1(4):555–584, 1995.

[79] Tong Zhang, Gui-Qiang Chen, and Shuli Yang. On the 2-d Riemann problem for the compressible Euler equations II. Interaction of contact discontinuities. *Disc. Cont. Dyn. Sys.*, 6(2):419–430, 2000.

[80] Tong Zhang and Yuxi Zheng. Two-dimensional Riemann problem for a single conservation law. *Trans. Amer. Math. Soc.*, 312(2):589–619, 1989.

[81] Yuxi Zheng. Existence of solutions to the transonic pressure-gradient equations of the compressible Euler equations in elliptic regions. *Comm. Partial Diff. Equ.*, 22(11–12):1849–1868, 1997.

[82] Yuxi Zheng. *Systems of conservation laws. Two-dimensional Riemann problems*, volume 38 of *Progress in Nonlinear Differential Equations and Their Applications.* Birkhäuser, Boston, 2001.

[83] Yuxi Zheng. A global solution of a two-dimensional Riemann problem involving shocks as free boundaries. *Acta Math. Appl. Sinica*, 19:559–572, 2003.

[84] Yuxi Zheng. Shock reflection for the Euler system. In Kawashima & al., editor, *Hyperbolic problems X (Osaka 2004). Theory, numerics and applications*, 2005.

[85] Yuxi Zheng. Two-dimensional regular shock reflection for the pressure gradient system of conservation laws. Preprint, 2005.

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