Causal Inference: DAGs, reasoning and discovery

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Sommaire

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Causal hierarchy

Some epidemiological questions

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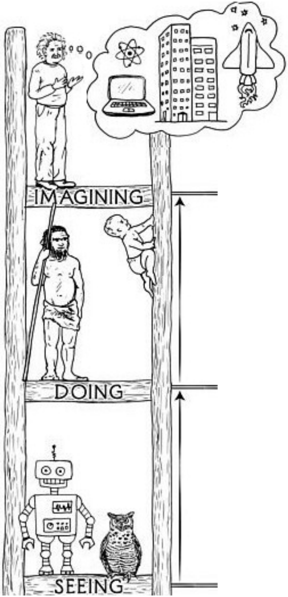
Pearl causal hierarchy



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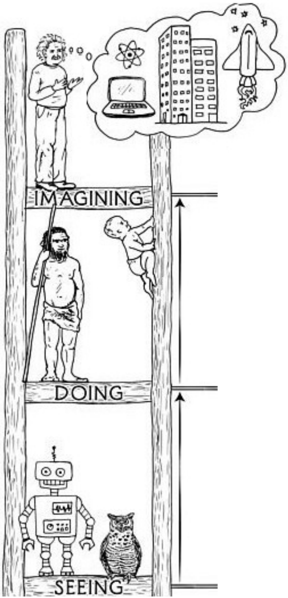

- Associations
  - Questions : What if I see ...?
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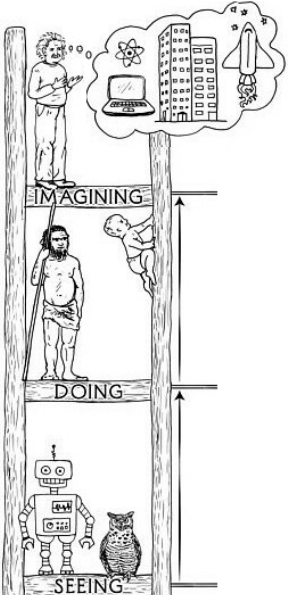

- Interventions
  - Questions: What if I do ...? How?
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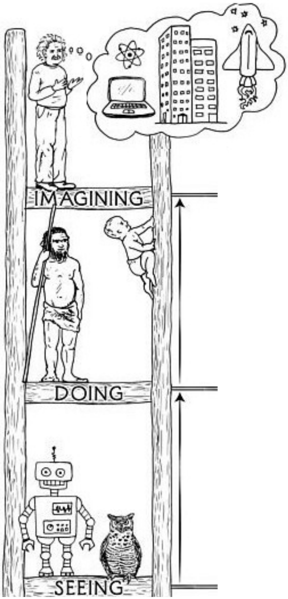

- Counterfactuals
  - Questions: What if I had done ...? Why?
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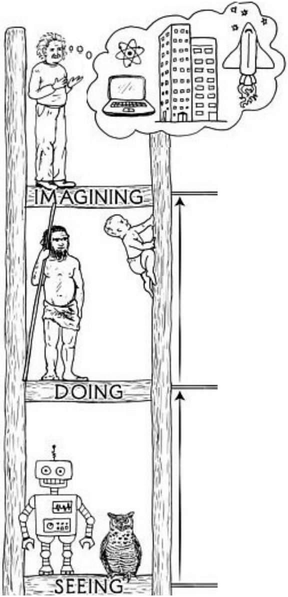

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Causal effect

The causal effect of A on Y

$$= \mathbb{E}(Y \mid do(A = a)) - \mathbb{E}(Y \mid do(A = a'))$$

The operator $do()$ represents interventions.

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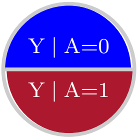
Sub-populations



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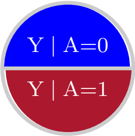

Conditioning

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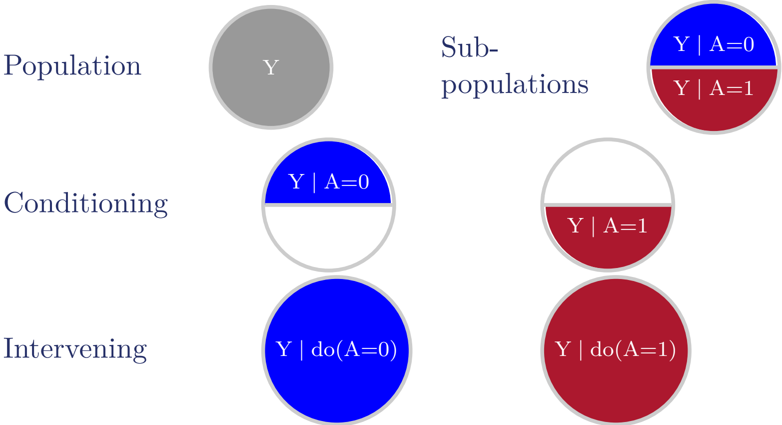
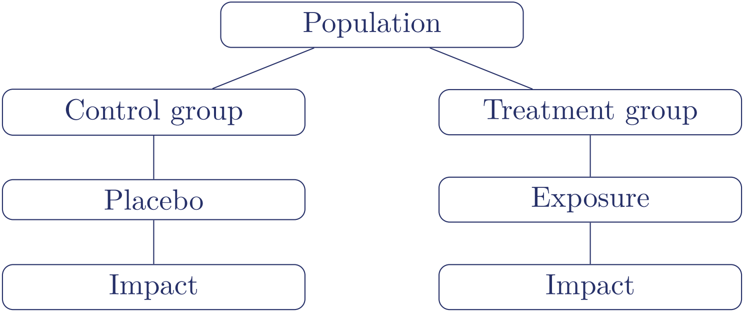


Gold standard: Randomized controlled trials



Control:

- Eliminate temporal bias

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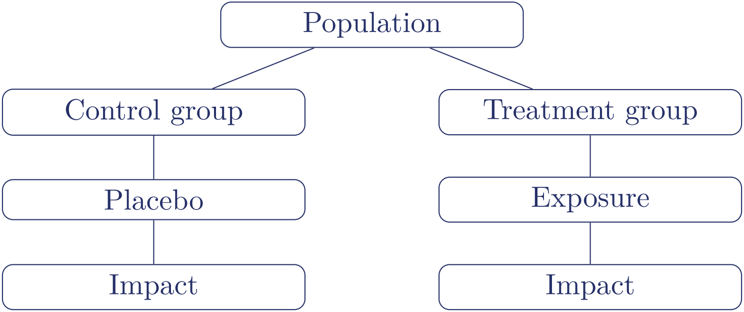

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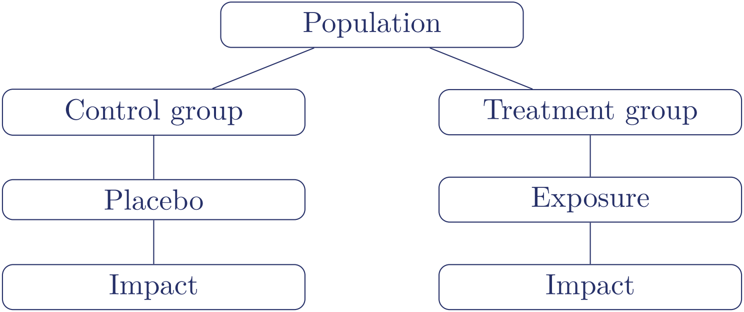

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Limitations:

- Costly
- Unethical
- infeasible

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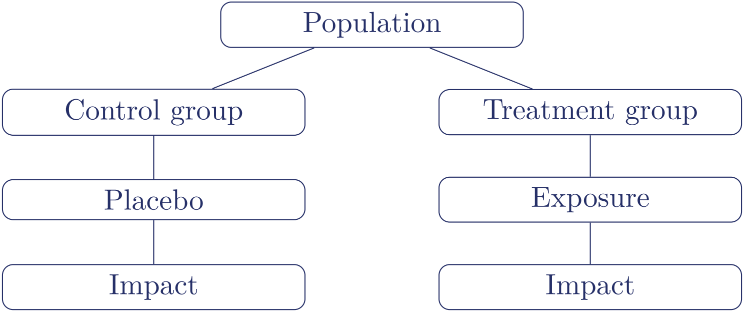

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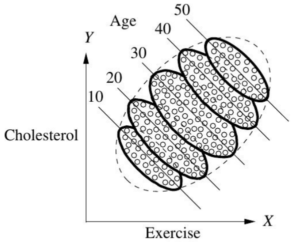
$\Longrightarrow$ Sometimes we have to rely on observational studies

Simpson paradox [3]

In a study, we measure weekly exercise and cholesterol levels for various age groups.



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Bayesian networks

Notations

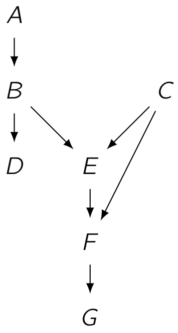
- Sets: $\mathbb{A} = \{X, Y, Z\}$
- Statistical independence: $\perp\!\!\!\perp_P$
- Statistical dependence: $\not\perp\!\!\!\perp_P$
- $P(Y = y \mid X = x) \equiv P(y \mid x)$

Directed graphs

A graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is said to be a directed graph iff

- $\mathbb{V}$ is the set of vertices (usually each vertex corresponds to a random variable),
- $\mathbb{E}$ is the set of edges,
- $\forall (X, Y) \in \mathbb{E},$ there is an arrow pointing from X to Y.

Consider the following directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$:



Directed graphs: basic concepts

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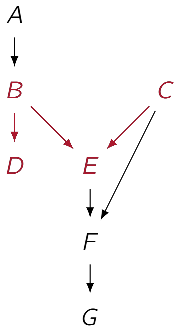

Path: $D \leftarrow B \rightarrow E \leftarrow C$

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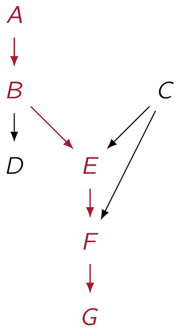

Path: $D \leftarrow B \rightarrow E \leftarrow C$
Directed path: $A \rightarrow B \rightarrow E \rightarrow F \rightarrow G$

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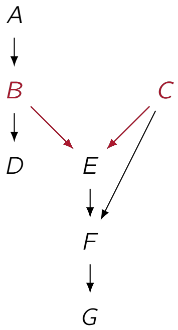

Path: $D \leftarrow B \rightarrow E \leftarrow C$
Directed path: $A \rightarrow B \rightarrow E \rightarrow F \rightarrow G$
Parents: $Pa(E) = \{B, C\}$

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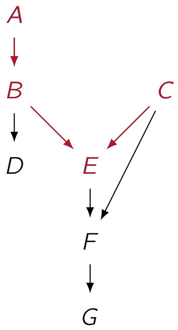

Path: $D \leftarrow B \rightarrow E \leftarrow C$
Directed path: $A \rightarrow B \rightarrow E \rightarrow F \rightarrow G$
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Ancestors: $An(E) = \{A, B, C, E\}$

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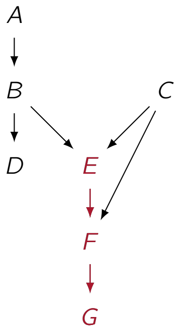

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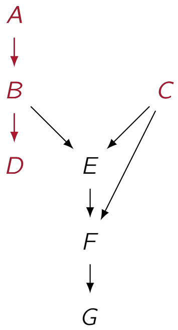

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Non-descendants: $Nd(E) = \{A, B, C, D\}$

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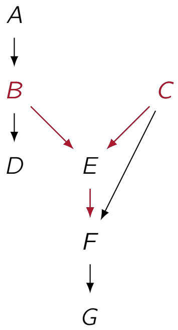

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Neighbors: $Ne(E) = \{B, C, E\}$

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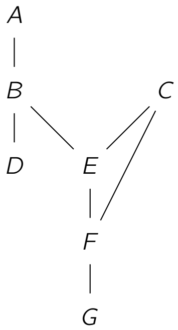

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Skeleton of $\mathcal{G}$

A directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is said to be a directed acyclic graph (DAG) iff

$$\forall X \in \mathbb{V}, An(X) \cap De(X) = \{X\},$$

i.e., there are no cycle in $\mathcal{G}$.

DAGs

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From now on, we will only deal with DAGs.

Python code: graphs using networkx package

```
#Load the networkx package
import networkx as nx
#Define the Directed graph
g = nx.DiGraph()
g.add_nodes_from(["A","B","C"])
g.add_edge("B", "A")
g.add_edge("B", "C")
g.add_edge("A", "C")
g.remove_edge("A", "C")
#Print nodes and edges
print(g.nodes)
print(g.edges)
#Print parents and children
print(g.predecessors("A"))
print(g.successors("B"))
```

Definition

A distribution $P(\mathbb{V})$ is compatible with a DAG $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ if

$$P(\mathbb{V}) = \prod_{X \in \mathbb{V}} P(X \mid Pa(X))$$

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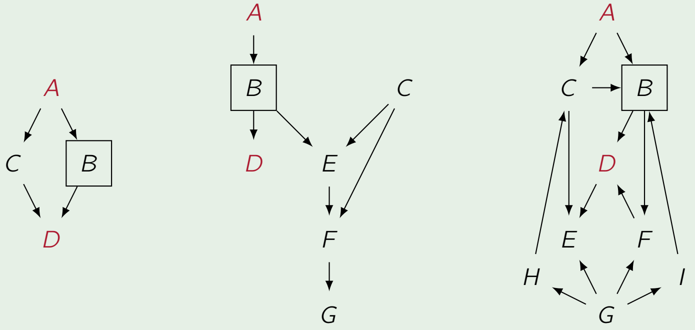
A DAG $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is a Bayesian network iff there exists a joint distribution $P(\mathbb{V})$ that is compatible with $\mathcal{G}$.

Example

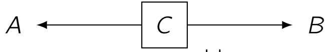
$A \overset{?}{\perp\!\!\!\perp}_P D \mid B$



Basic structures

Fork: contains a confounder

$A \overset{?}{\perp\!\!\!\perp}_P B$



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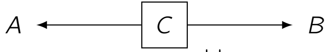
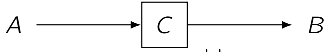


$A \perp\!\!\!\perp_P B \mid C$

Chain: contains a mediator or an intermediate cause

$A \overset{?}{\perp\!\!\!\perp}_P B$



$A \perp\!\!\!\perp_P B \mid C$

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Basic structures

Fork: contains a confounder

$$A \longleftarrow C \longrightarrow B \qquad\qquad A \longleftarrow \boxed{C} \longrightarrow B$$

$$A \overset{?}{\perp\!\!\!\perp}_P B \qquad\qquad A \perp\!\!\!\perp_P B \mid C$$

Chain: contains a mediator or an intermediate cause

$$A \longrightarrow C \longrightarrow B \qquad\qquad A \longrightarrow \boxed{C} \longrightarrow B$$

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Collider : contains a common effect

$$A \longrightarrow C \longleftarrow B \qquad\qquad A \longrightarrow \boxed{C} \longleftarrow B$$

$$A \perp\!\!\!\perp_P B \qquad\qquad A \overset{?}{\perp\!\!\!\perp}_P B \mid C$$

Basic structures

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Collider : contains a common effect

$A \rightarrow C \leftarrow B$

$A \perp\!\!\!\perp_P B$

$A \rightarrow C \leftarrow B$, $C \rightarrow \boxed{D}$

$A \overset{?}{\perp\!\!\!\perp}_P B \mid D$

Basic structures

Fork: contains a confounder

$A \leftarrow C \rightarrow B$ $\qquad$ $A \overset{?}{\perp\!\!\!\perp}_P B$

$A \leftarrow \boxed{C} \rightarrow B$ $\qquad$ $A \perp\!\!\!\perp_P B \mid C$

Chain: contains a mediator or an intermediate cause

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Collider : contains a common effect

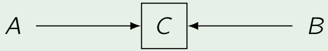
$A \rightarrow C \leftarrow B$ $\qquad$ $A \perp\!\!\!\perp_P B$

$A \rightarrow C \leftarrow B$, $C \rightarrow \boxed{D}$ $\qquad$ $A \overset{?}{\perp\!\!\!\perp}_P B \mid D$

A collider is unshielded if its extremities are not adjacent.

Artificial correlation in Colliders: example 1

Example



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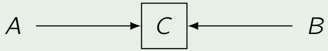

$$A = \begin{cases} \text{Mother carrier} \\ \text{Mother not carrier} \end{cases} \qquad B = \begin{cases} \text{Father carrier} \\ \text{Father not carrier} \end{cases}$$

$$C = (A \text{ or } B) = \begin{cases} \text{Child carrier} \\ \text{Child not carrier} \end{cases}$$

Artificial correlation in Colliders: example 1

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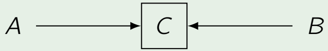

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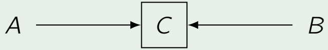
$$C = (A \text{ or } B) = \begin{cases} \text{Child carrier} \\ \text{Child not carrier} \end{cases}$$

If $C = \text{Child carrier} \implies$

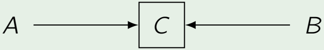
$$\begin{cases} \text{If } A = \text{Mother not carrier then } B = \text{Father carrier} \\ \text{If } B = \text{Father not carrier then } A = \text{Mother carrier} \end{cases}$$

Example



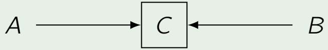
Artificial correlation in Colliders: example 2

Example



$A, B \sim U(-1, 1)$ $\xi_c \sim N(0, \frac{1}{2})$ $C = 2AB + \xi_c$

Example



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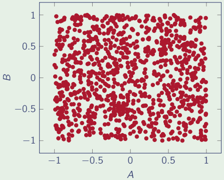
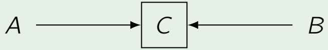


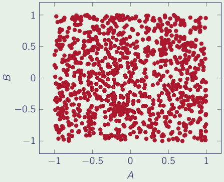
$Corr(A; B) = 0.002$

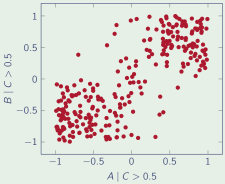
Artificial correlation in Colliders: example 2

Example



$A, B \sim U(-1, 1)$ $\xi_c \sim N(0, \frac{1}{2})$ $C = 2AB + \xi_c$





$Corr(A; B) = 0.002$ $Corr(A; B \mid C > 0.5) = 0.8$

Blocked paths

A path is said to be blocked by a set of vertices $\mathbb{Z} \in \mathbb{V}$ if:

- it contains a chain $A \to B \to C$ or a fork $A \leftarrow B \to C$ and $B \in \mathbb{Z}$; or
- it contains a collider $A \to B \leftarrow C$ such that no descendant of B is in $\mathbb{Z}$.

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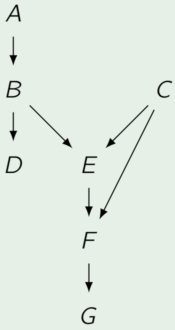
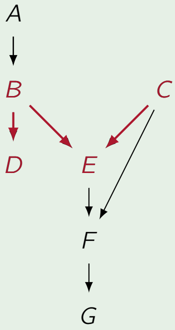
A path that is not blocked is active.

Example



Blocked path: examples

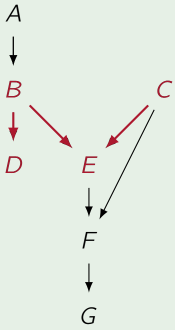
Example



Is the path $< D, B, E, C >$ blocked?

Blocked path: examples

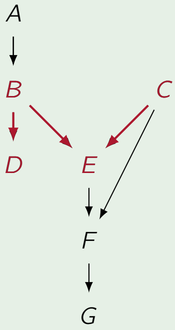
Example



Is the path $< D, B, E, C >$ blocked? Yes

Blocked path: examples

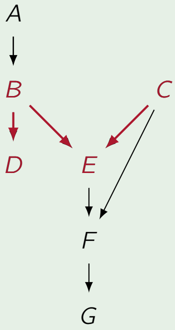
Example



Is the path $< D, B, E, C >$ blocked? Yes

Is the path $< D, B, E, C >$ blocked by E?

Example

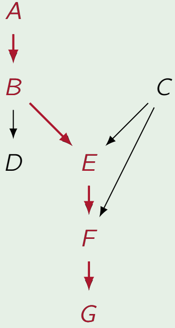


Is the path $< D, B, E, C >$ blocked? Yes

Is the path $< D, B, E, C >$ blocked by E? No

Blocked path: examples

Example



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Is the path $< D, B, E, C >$ blocked by E? No
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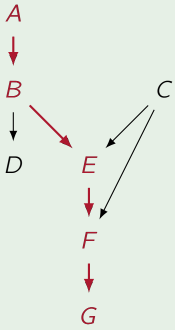
Blocked path: examples

Example



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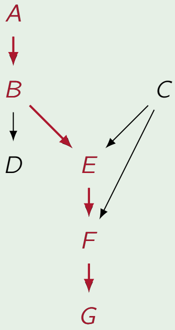

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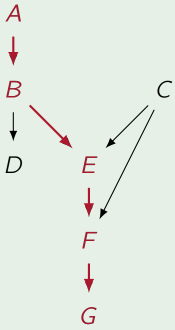

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d-separation

Given disjoint sets $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \subseteq \mathbb{V}$, we say that $\mathbb{X}$ and $\mathbb{Y}$ are d-separated by $\mathbb{Z}$ if every path between a vertex in $\mathbb{X}$ and a vertex in $\mathbb{Y}$ is blocked by $\mathbb{Z}$ and we write $\mathbb{X} \perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z}$.

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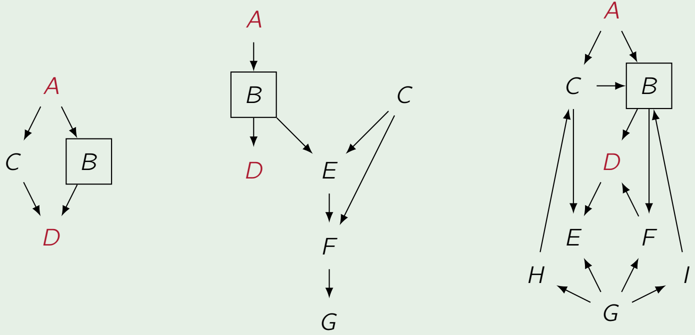
$$\mathbb{X} \perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z} \Rightarrow \mathbb{X} \perp\!\!\!\perp_{P} \mathbb{Y} \mid \mathbb{Z}$$

$$\mathbb{X} \not\perp\!\!\!\perp_{\mathcal{G}} \mathbb{Y} \mid \mathbb{Z} \Rightarrow \mathbb{X} \overset{?}{\perp\!\!\!\perp}_{P} \mathbb{Y} \mid \mathbb{Z}$$

Example

Is $A \perp\!\!\!\perp_P D \mid B$?



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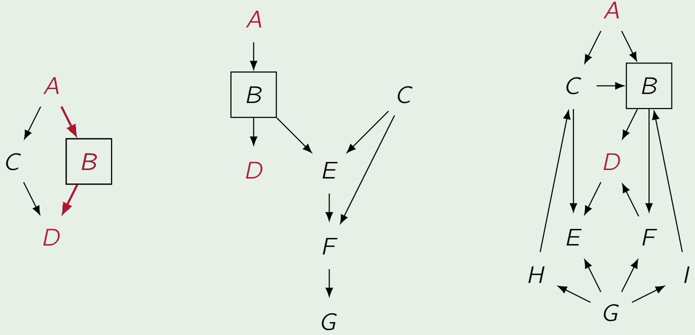

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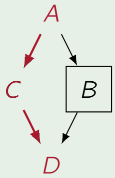

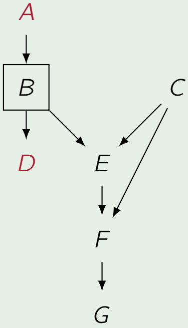

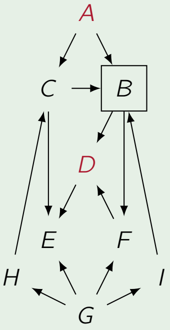

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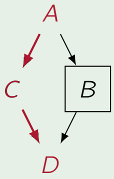
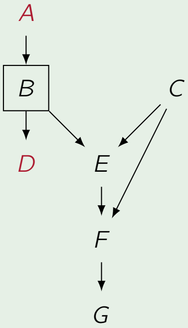
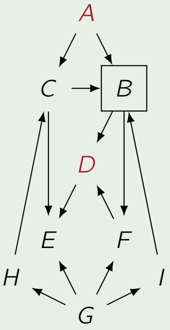


$< A, C, D >$ is not blocked

$\overset{?}{\Longrightarrow} A \perp\!\!\!\perp_P D \mid B$

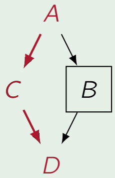




Reading conditional independencies using d-separation

Example

Is $A \perp\!\!\!\perp_P D \mid B$?



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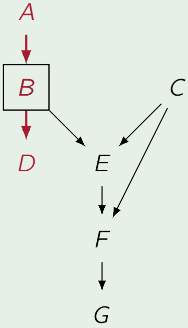

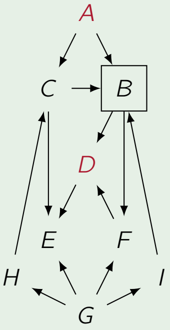

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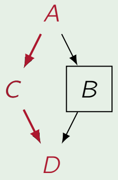

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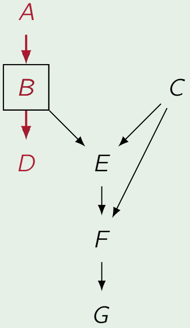
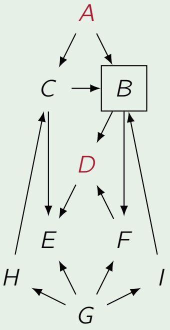


All paths are blocked

$\implies A \perp\!\!\!\perp_P D \mid B$



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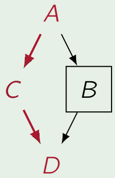

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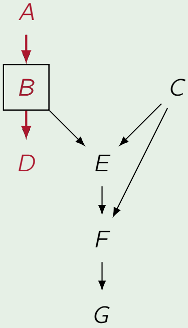

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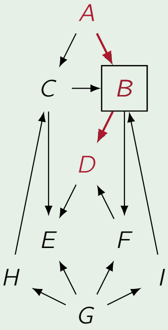

Reading conditional independencies using d-separation

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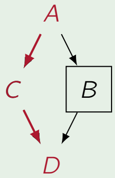

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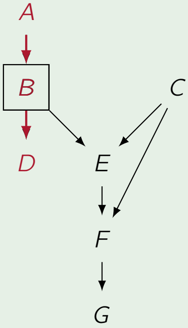

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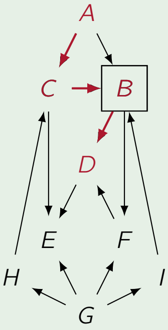

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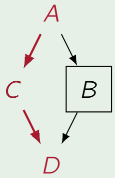

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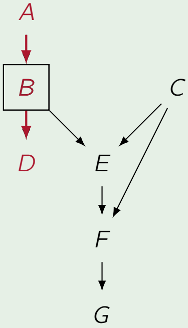

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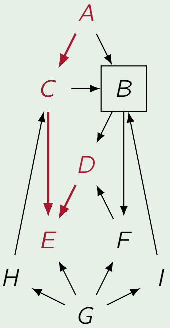

Reading conditional independencies using d-separation

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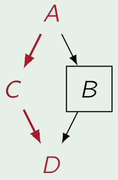

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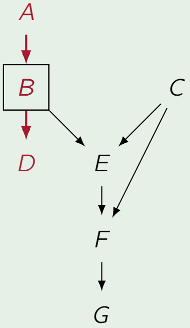

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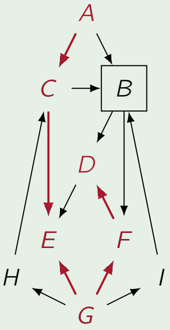

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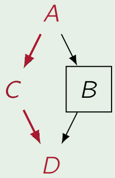

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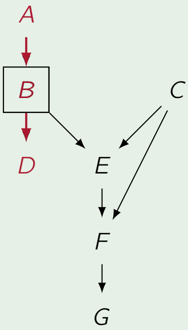

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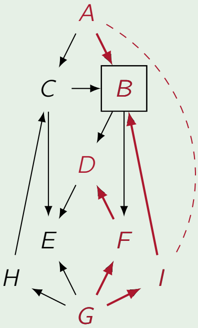

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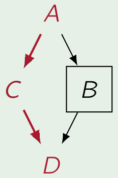

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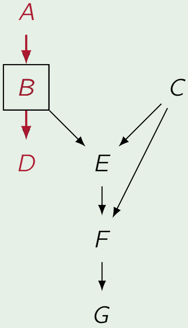

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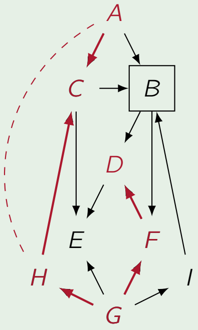

Reading conditional independencies using d-separation

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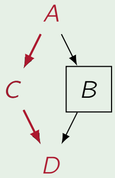

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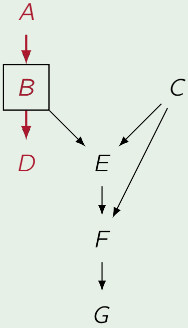

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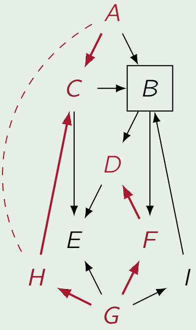


$< A, B, I, G, F, D >$ is not blocked

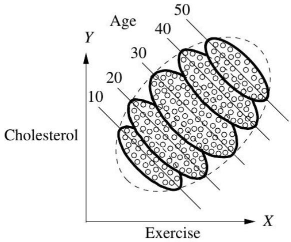
$\Longrightarrow A \overset{?}{\perp\!\!\!\perp}_P D \mid B$

Python code: d-separation using networkx package

```
#Load the networkx package
import networkx as nx
#Define the Directed graph
g = nx.DiGraph()
g.add_nodes_from(["A","B","C"])
g.add_edge("B", "A")
g.add_edge("B", "C")^^I^^I
#Check if B d-separates A and C
dsep = nx.is_d_separator(g, "A", "B", "C")
print(dsep)
```

Simpson paradox [3]

In a study, we measure weekly exercise and cholesterol levels for various age groups.



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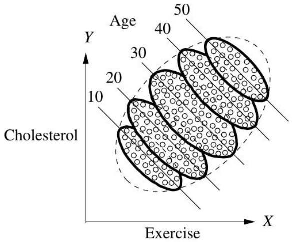

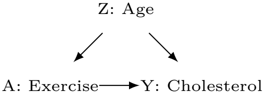

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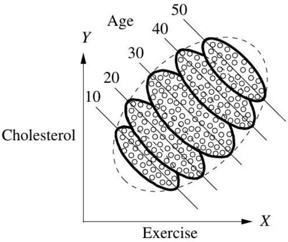

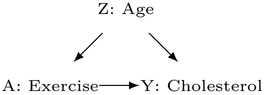

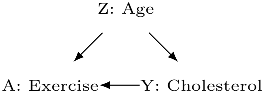

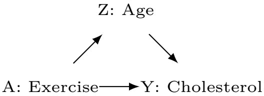

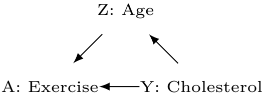

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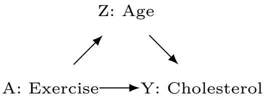
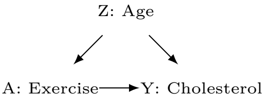
3

Causal DAGs

Bayesian networks vs causal DAGs





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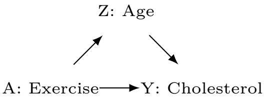
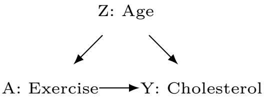


Not a causal DAG



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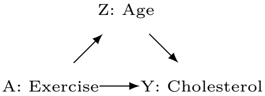

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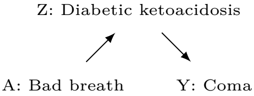

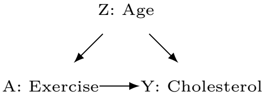

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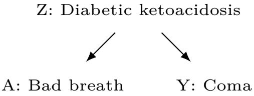

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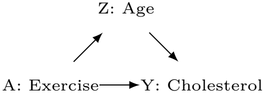

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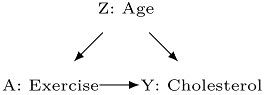

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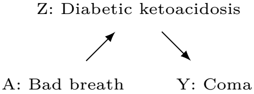

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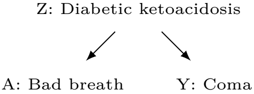
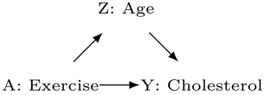


Causal DAG

Bayesian networks vs causal DAGs



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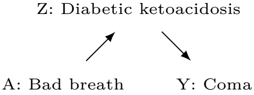
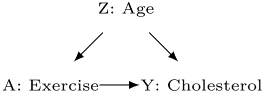


Not a causal DAG

Oracle for conditional independence



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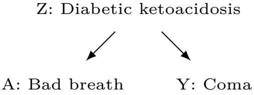
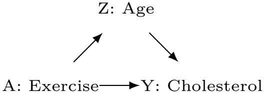


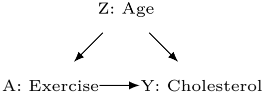
Causal DAG

Oracle for intervention

Bayesian networks vs causal DAGs



Not a causal DAG



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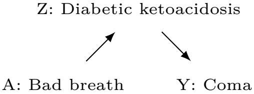

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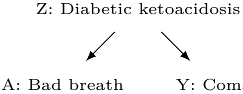

Causal DAG

Oracle for conditional independence

Oracle for intervention

It is impossible to determine, without additional assumptions, which of the two DAGs is causal based solely on the observed distribution!

Truncated factorization [2]: If we intervene on a subset $\mathbb{S} \subset \mathbb{V}$, then

$$P(v_1, \cdots, v_d \mid do(s)) = \prod_{V_i \notin \mathbb{S}} P(v_i \mid Pa(v_i))$$

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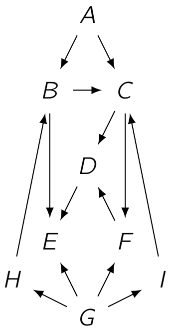
Let $P(\mathbb{V})$ be a probability distribution and let P_* denote the set of all interventional distributions $P(\mathbb{V} \mid do(s))$. A bayesian network $\mathcal{G}$ is said to be a causal DAG compatible with P_* iff $\mathcal{G}$ and P_* satisfy the truncated factorization.

Structural causal model



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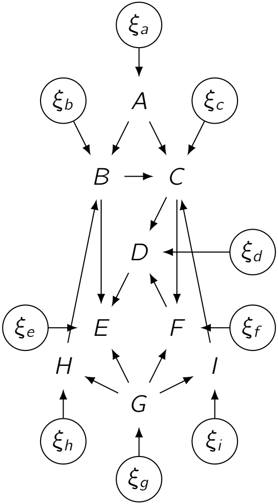

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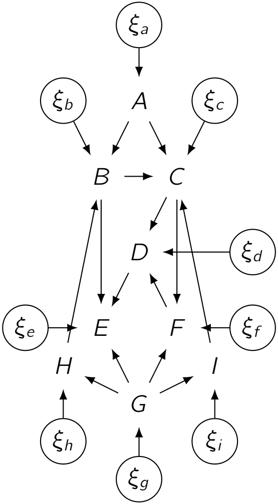

$$M : \begin{cases} A := f_a(\xi_a) \\ B := f_b(A, H, \xi_b) \\ C := f_c(A, B, I, \xi_c) \\ D := f_d(C, F, \xi_d) \\ E := f_e(B, G, \xi_e) \\ F := f_f(C, G, \xi_f) \\ G := f_g(\xi_g) \\ H := f_h(G, \xi_h) \\ I := f_i(G, \xi_i) \end{cases}$$

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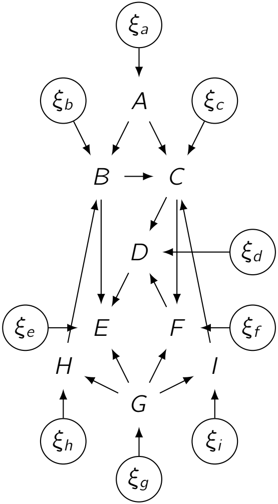

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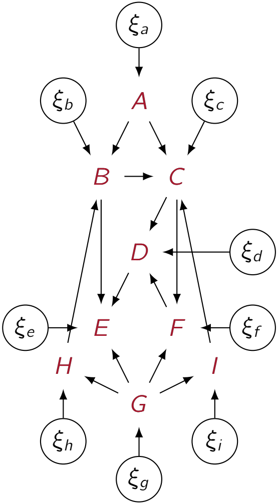
Structural causal model



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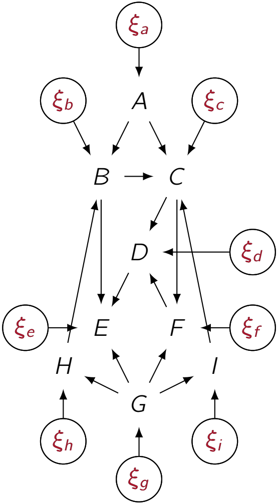
A structural causal model (SCM) is a tuple that contains:

- Endogenous variables

Structural causal model



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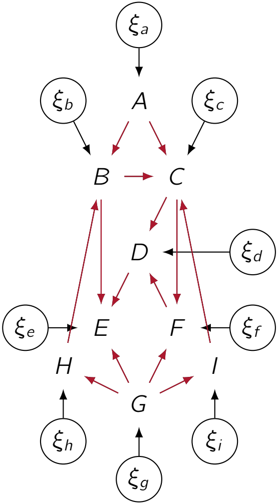

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A structural causal model (SCM) is a tuple that contains:

- Endogenous variables
- Exogenous variables
- Causal mechanisms for generating endogenous variables

Guidlines for constructing causal DAGs

In causal DAGs, the information lies in the absence of edges and the direction of arrows, rather than merely in the presence of edges.

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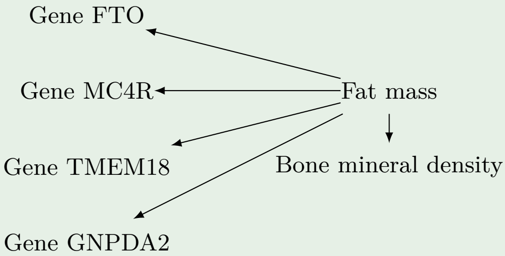
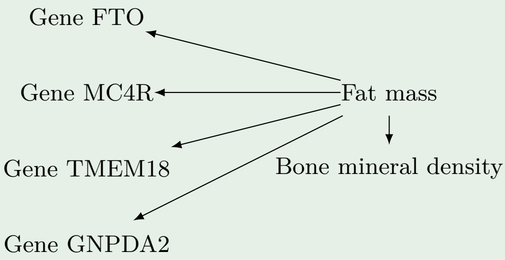
If you cannot keep the graph acyclic, do not worry, there exists new tools for cyclic graphs.
But in this lecture, we will focus only on acyclic graphs.

Example



Is this DAG causal?

Example

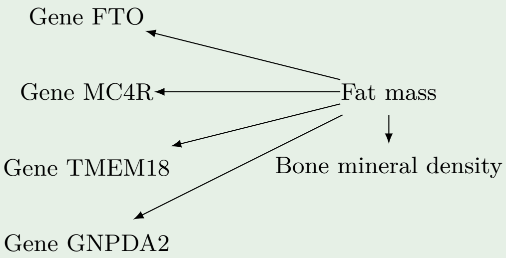


Is this DAG causal?

No

DAG Vs DAG causal: exemples

Example



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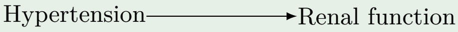

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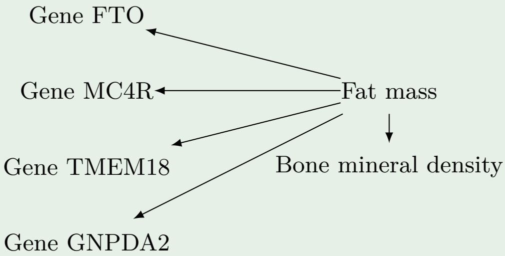
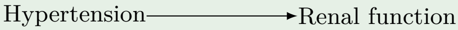


Is this DAG causal?
No



Is this DAG causal?
Yes

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4

Causal reasoning

Identifiability and positivity

A causal effect is said to be identifiable if it is uniquely computable from an observational distribution $P(\mathbb{V})$.

The causal effect of A on Y
$= \mathbb{E}(\mathbb{Y} \mid do(A = a)) - \mathbb{E}(Y \mid do(A = a'))$

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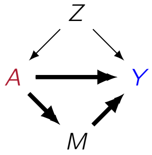

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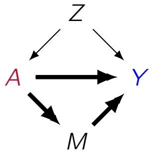

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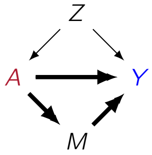

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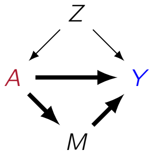
Causal reasoning involves utilizing a causal DAG to determine whether $P(y \mid do(a))$ is identifiable.

Key challenges of not using a causal DAG (1/3)

Confounding bias (e.g., Simpson paradox):



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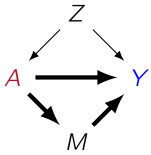

We must eliminate all confounding bias by adjusting for confounders.

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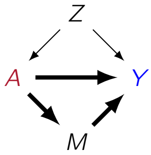

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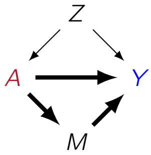
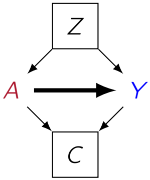


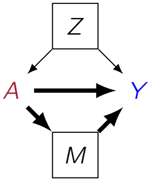
We must eliminate all confounding bias by adjusting for confounders.
Should we always adjust for all available variables? No!

Key challenges of not using a causal DAG (2/3)

Bias due to adjusting for colliders:



Bias due to incorrect adjustment for mediators:



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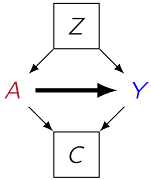
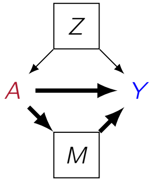


Bias due to incorrect adjustment for mediators:



Should we adjust on everything that is temporally prior to the exposure?

Key challenges of not using a causal DAG (3/3)

Let $\mathbb{Z}$ be the set of observed variables in a problem that are not affected by A. The set $\mathbb{Z}$ satisfies Association Criterion if each element $Z \in \mathbb{Z}$ meets the following conditions:

- Z is associated with A; and
- Z is associated with Y, conditionally on A.

.

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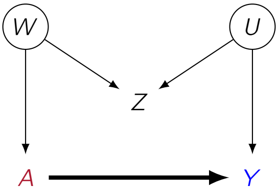
Counterexample



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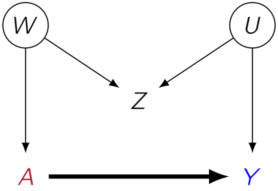

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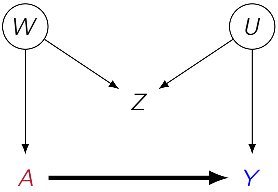

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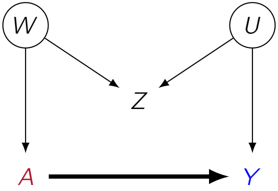

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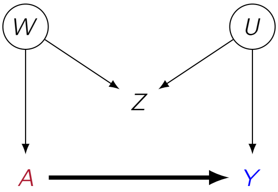
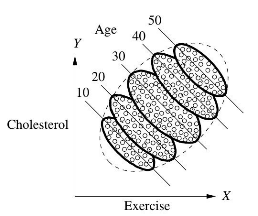


Z is associated with A and Z is associated with Y, conditionally on A. Should we adjust on Z? No!
This criterion is incorrect!

Simpson paradox and a simple solution

Example

In a study, we measure weekly exercise and cholesterol levels for various age groups.



What is the effect of exercise on cholesterol $P(c \mid do(e))$?

Simpson paradox and a simple solution

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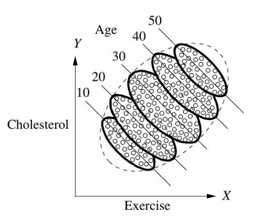

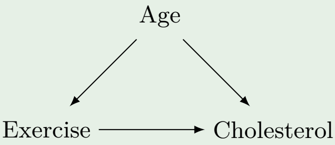

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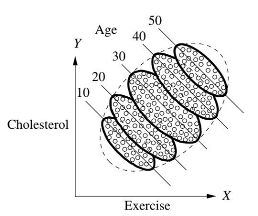

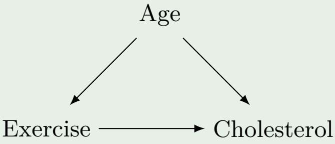

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$$P(a, e, c) = P(a)P(e \mid a)P(c \mid a, e) \qquad \text{(Compatibility)}$$

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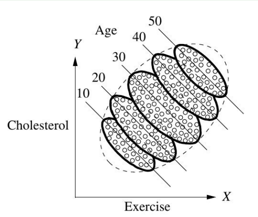

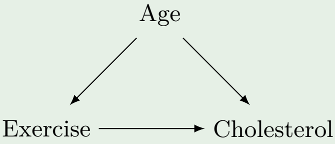

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$$P(c \mid do(e)) = \sum_a P(a)P(c \mid a, e) \quad \text{(Marginalizing)}$$

A simple solution: direct causes adjustment [2]

Theorem

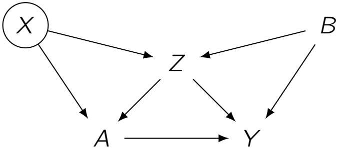
Given a causal DAG $\mathcal{G}$ in which a subset $\mathbb{V}$ of variables are measured, the causal effect $P(y \mid do(a))$ is identifiable whenever $\{A \cup Y \cup Parents(A)\} \subseteq \mathbb{V}$, and is given by:

$$P(y \mid do(a)) = \sum_{Z \in Pa(A)} P(y \mid a, z) P(z)$$

Limitations of the direct causes adjustment

- Sometimes the set of parents is too large. Is it possible to find a smaller set?

- Sometimes the set of observed parents is not sufficient for adjustment. Is it possible to find another set?



The back-door criterion

A set of variables $\mathbb{Z}$ satisfies the back-door criterion relative to an ordered pair of variables (A, Y) in causal DAG $\mathcal{G}$ if:

- No node in $\mathbb{Z}$ is a descendant of A; and
- $\mathbb{Z}$ blocks all paths between A and Y that contain an arrow pointing toward A.

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Theorem

If $\mathbb{Z}$ satisfies the back-door criterion with respect to (A, Y) and if $P(a, z) > 0$, then $P(y \mid do(a))$ is identifiable and is given by:

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$$P(y \mid do(a)) = \sum_z P(y \mid a, z) P(z).$$

This criterion can be extended to a set $\mathbb{A}$ and a set $\mathbb{Y}$.

The back-door criterion: intuition

- Why ”no node in $\mathbb{Z}$ is a descendant of A”?

- Why ”$\mathbb{Z}$ blocks all paths between A and Y that contain an arrow pointing toward A”?

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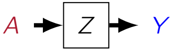

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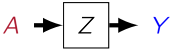

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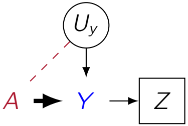

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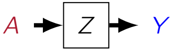

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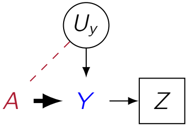

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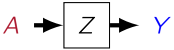

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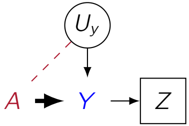

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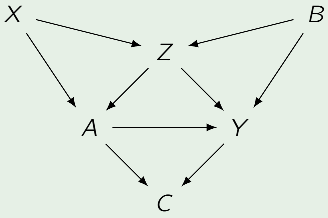
- Why "$\mathbb{Z}$ blocks all paths between A and Y that contain an arrow pointing toward A"?

To eliminate confounding bias.

Back-door criterion in action (1/2)

Example

$P(y \mid do(a))$?



Back-door criterion in action (1/2)

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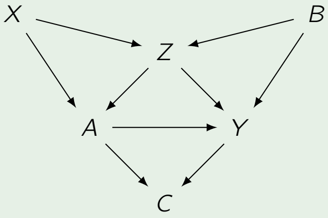


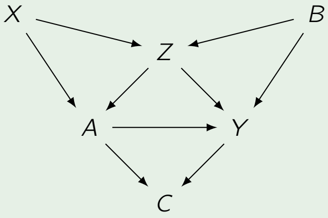
Z ?

Back-door criterion in action (1/2)

Example

$P(y \mid do(a))$?

Z ? No



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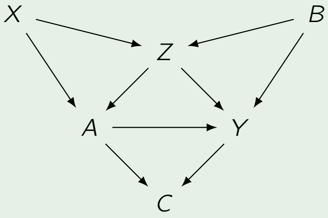

Z ? No

X ?

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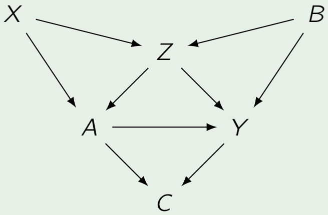

Z ? No

X ? No

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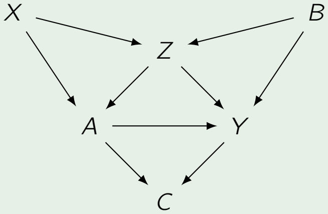

Z ? No

X ? No

B ?

Back-door criterion in action (1/2)

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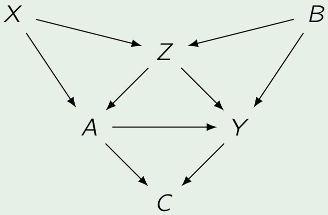

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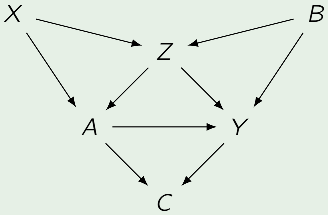

Z ? No

X ? No

B ? No

C ?

Back-door criterion in action (1/2)

Example

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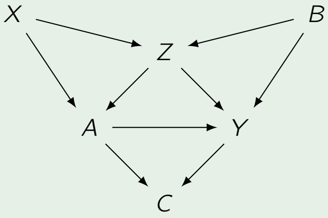

Z ? No
X ? No
B ? No
C ? No

Back-door criterion in action (1/2)

Example

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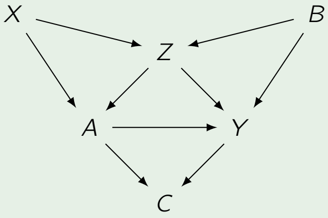

Z ? No

X ? No

B ? No

C ? No

X, B?

Back-door criterion in action (1/2)

Example

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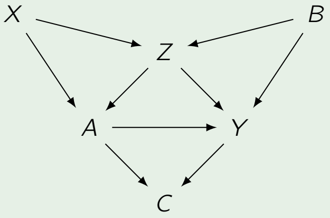


Z ? No

X ? No

B ? No

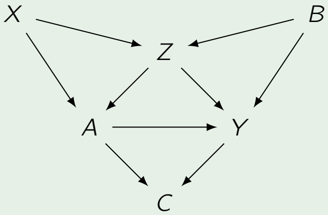
C ? No

X, B? No

Back-door criterion in action (1/2)

Example

$P(y \mid do(a))$?



Z ? No

X ? No

B ? No

C ? No

X, B? No

Z, X ?

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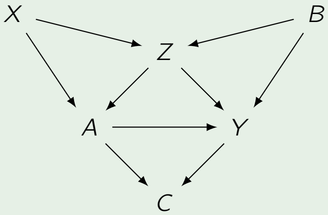

Z ? No

X ? No

B ? No

C ? No

X, B? No

Z, X ? Yes

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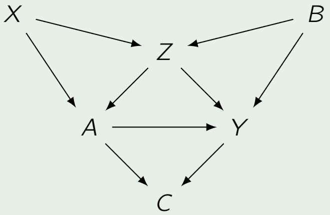

Z ? No

X ? No

B ? No

C ? No

X, B? No

Z, X ? Yes

Z, B ?

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Back-door criterion in action (1/2)

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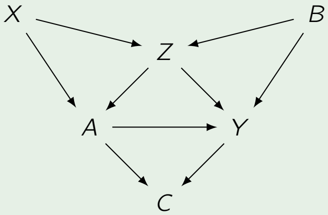

Z ? No

X ? No

B ? No

C ? No

X, B? No

Z, X ? Yes

Z, B ? Yes

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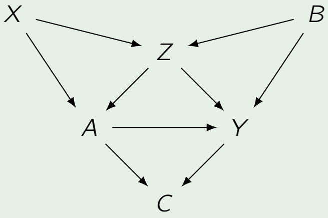


Z ? No

X ? No

B ? No

C ? No

X, B? No

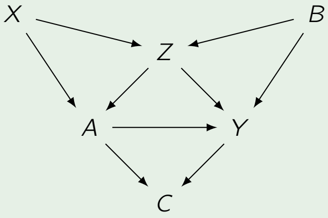
Z, X ? Yes

Z, B ? Yes

Z, X, B ?

Example

$P(y \mid do(a))$?



Z ? No

X ? No

B ? No

C ? No

X, B? No

Z, X ? Yes

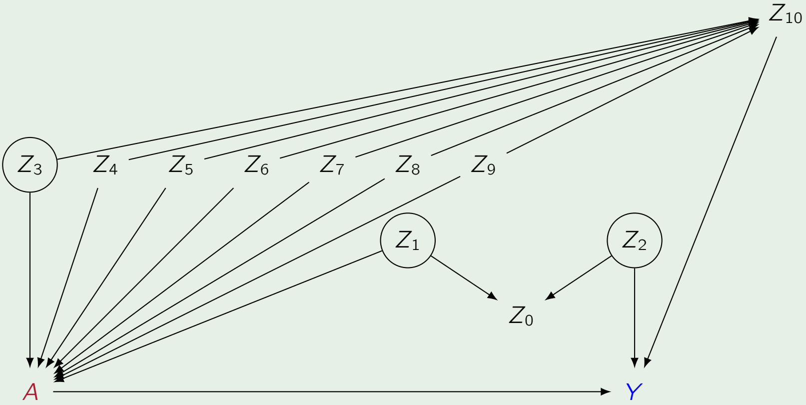
Z, B ? Yes

Z, X, B ? Yes

Back-door criterion in action (2/2)

Example

Suppose that all variable are temporally prior to A and Y.
$P(y \mid do(a))$?



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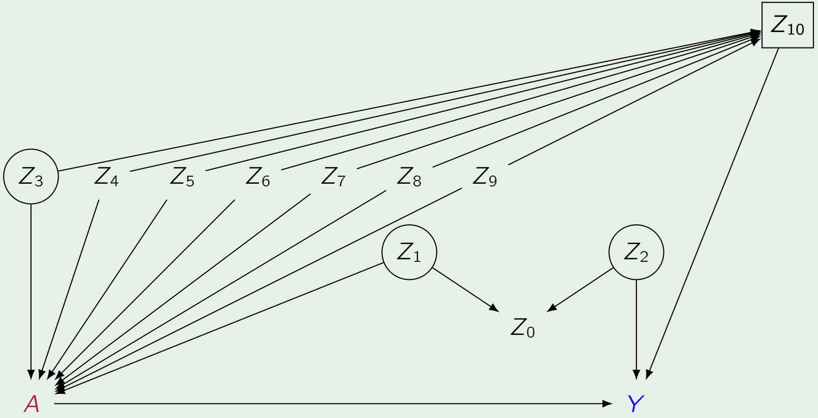

Back-door criterion in a real application

Example

In this study, we aim to estimate the effect of the neighborhood's socioeconomic status (N) on the reduction of walking within the neighborhood (R), $P(r \mid do(n))$?

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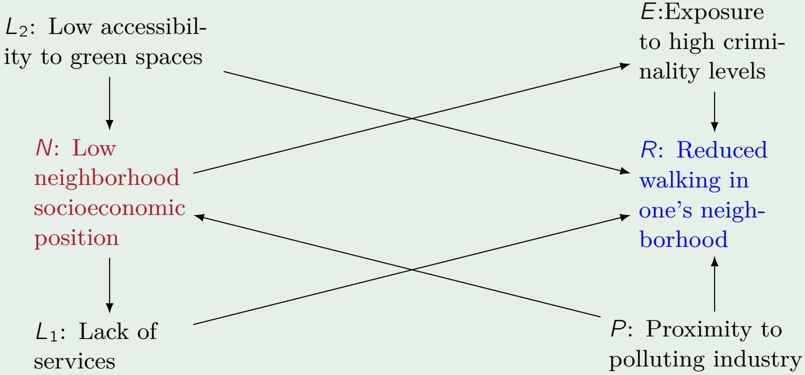

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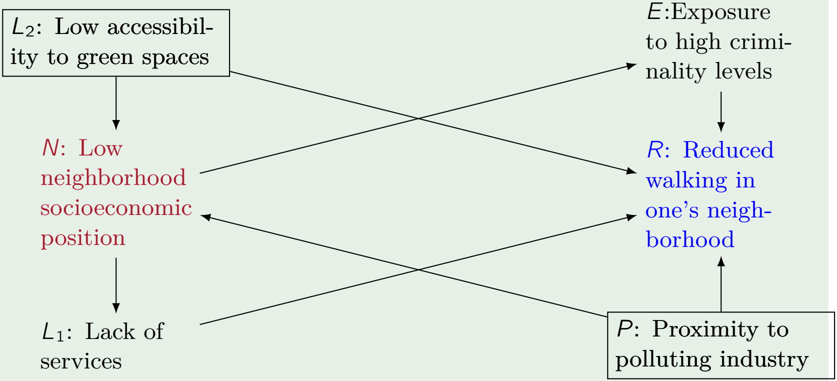

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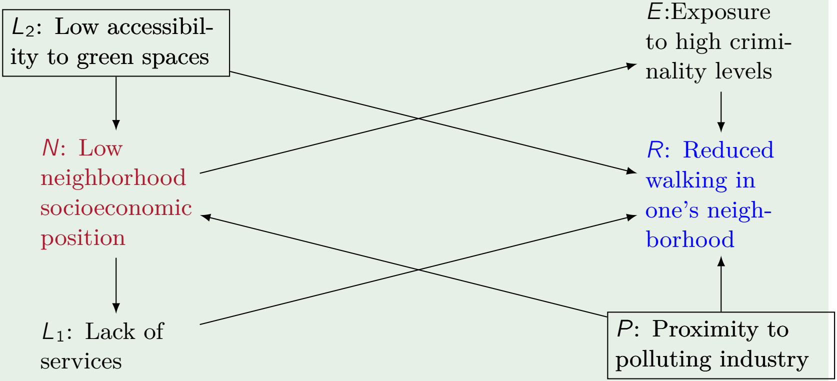


$$P(r \mid do(n)) = \sum P(r|n, l_2, p)P(l_2, p)$$

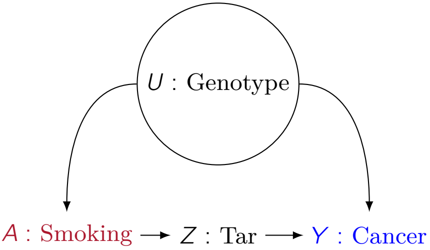
Incompleteness of the back-door criterion

- The back-door criterion is sound but not complete:

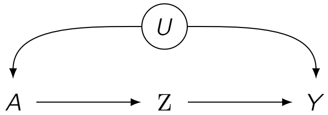
Incompleteness of the back-door criterion

- The back-door criterion is sound but not complete:
  - If there exists a set that satisfy the back-door criterion for $P(y \mid do(a))$, then $P(y \mid do(a))$ is identifiable;
  - If there exists a no set that satisfy the back-door criterion for $P(y \mid do(a))$, then $P(y \mid do(a))$ is not necesarly not identifiable.

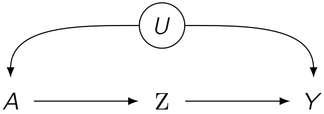
Going beyond the back-door



Going beyond the back-door



Going beyond the back-door



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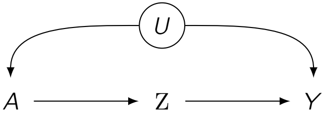

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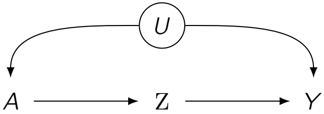

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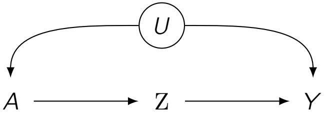

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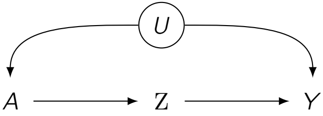
Front-door criterion



A set of variables $\mathbb{Z}$ satisfies the front-door criterion relative to an ordered pair of variables (A, Y) in causal DAG $\mathcal{G}$ if:

- $\mathbb{Z}$ intercepts all directed paths from A to Y;
- There is no back-door path from A to $\mathbb{Z}$;
- All back-door paths from $\mathbb{Z}$ to Y are blocked by A.

Theorem ([1])

If $\mathbb{Z}$ satisfies the front-door criterion relative to (A, Y) and if $P(a, z) > 0$, then the causal effect of A on Y is identifiable and is given by

$$P(y \mid do(a)) = \sum_{z} P(z \mid a) \sum_{a'} P(y \mid a', z) P(a').$$

Back-door and Front-door criteria in a real application

In this study, Piccininni et al. were interested in estimating the effect of Mobile Stroke Unit dispatch (M) on functional outcomes (F). In other words, $P(f \mid do(m))$?

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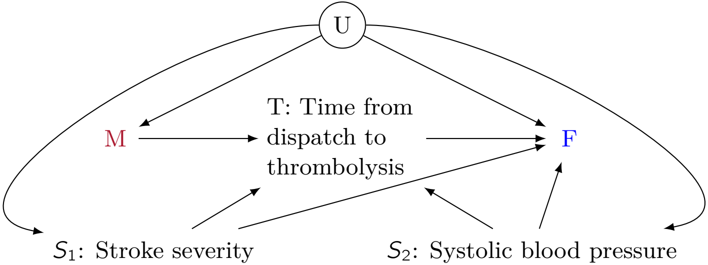

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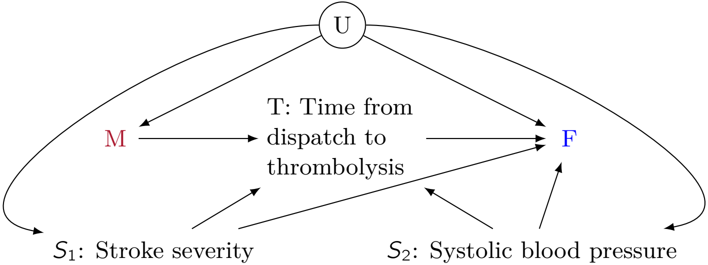

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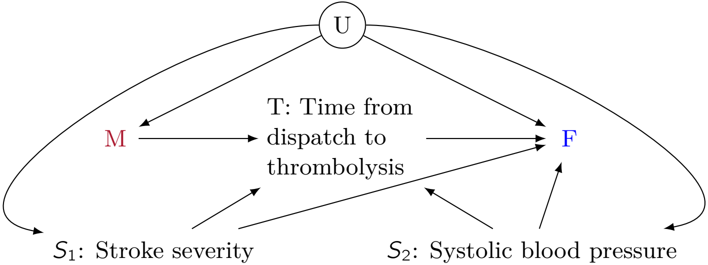

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Best paper in Epidemiology in 2024!

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- The front-door criterion is sound but not complete:
  - If there exists a set that satisfy the front-door criterion for $P(y \mid do(x))$, then $P(y \mid do(x))$ is identifiable;
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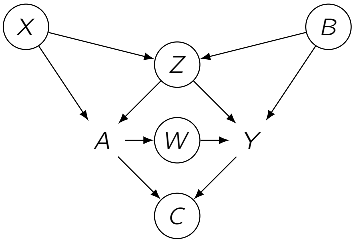
The combination of the back-door and front-door criterions is also not complete.

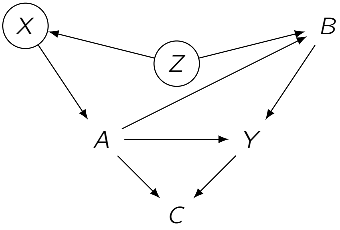
Exercise 1

Consider that in the following causal DAG, only A and Y, and one additional variable can be measured. Which variable would allow the identification of $P(y \mid do(a))$?



Exercise 2

- Consider the following causal DAG. List all sets of variables that satisfy the back-door criterion for $P(y \mid do(a))$;



Exercise 3

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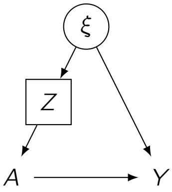

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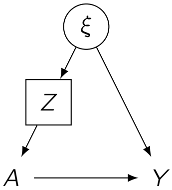

- Z blocks all back-door paths and it is not a descendant of A $\implies$ $\{Z\}$ is a good adjustment set.

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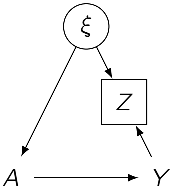

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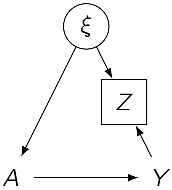

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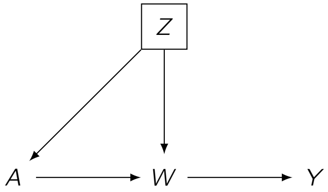

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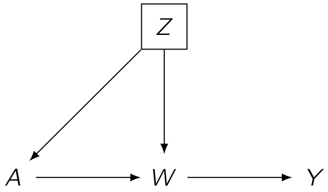

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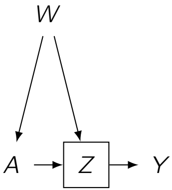

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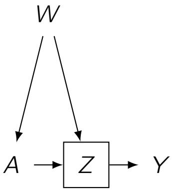

- Z d-separates A from Y
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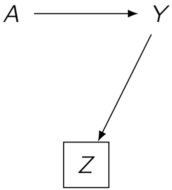

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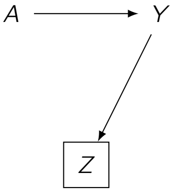

- Selection bias
  $\implies$ $\{Z\}$ is a bad control.

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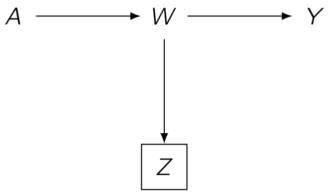

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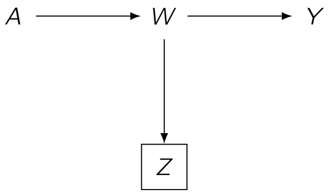

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5

Estimation

Linear regression and back-door criterion

Goal: estimate the causal effect

$$\mathrm{Q} := \mathbb{E}[Y \mid do(A=1)] - \mathbb{E}[Y \mid do(A=0)] \quad \text{or} \quad \mathrm{Q} := \frac{d}{da}\mathbb{E}[Y \mid do(A=a)]$$

Assume the back-door criterion holds with covariates $\mathbb{Z}$.

$$\begin{aligned}
\mathbb{E}[Y \mid do(A=a)] &= \int_{\mathbb{z}} \mathbb{E}[Y \mid A=a, \mathbb{Z}=\mathbb{z}] P(\mathbb{z}) d\mathbb{z} \\
&= \int_{\mathbb{z}} (\alpha_1 A + \alpha_2 Z + \alpha_0) P(\mathbb{z}) d\mathbb{z} \qquad \text{(linear model)} \\
&= \alpha_1 A + \alpha_2 \mathbb{E}[Z] + \alpha_0
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Causal interpretation (under correct specification):

$$\widehat{\mathrm{Q}} = \hat{\alpha}_1$$

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Assume the front-door criterion holds with variable Z.

Linear model:

$$\mathbb{E}[Z \mid do(A=a)] = \alpha_0 + \alpha_1 a \qquad \text{(can be estimated by OLS)}$$

$$\mathbb{E}[Y \mid do(Z=z)] = \beta_0 + \beta_1 z + \beta_2 \mathbb{E}[A] \qquad \text{(can be estimated by OLS)}$$

$$\begin{aligned} \mathbb{E}[Y \mid do(A=a)] &= \beta_0 + \beta_1\, \mathbb{E}[Z \mid do(A=a)] + \beta_2 \mathbb{E}[A] \\ &= \beta_0 + \beta_1(\alpha_0 + \alpha_1 a) + \beta_2 \mathbb{E}[A] \\ &= \beta_0 + \beta_1 \alpha_0 + \beta_1 \alpha_1 a + \beta_2 \mathbb{E}[A] \end{aligned}$$

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$$\widehat{\mathrm{Q}} = \hat{\beta}_1 \times \hat{\alpha}_1$$

Limitations of linear regression

- Model misspecification: incorrect functional form leads to biased causal effects.
- Poor extrapolation: limited overlap in covariates can make linear predictions unreliable.

Machine learning (ML)?

- ML methods can automatically capture nonlinear effects and interactions without explicit specification.
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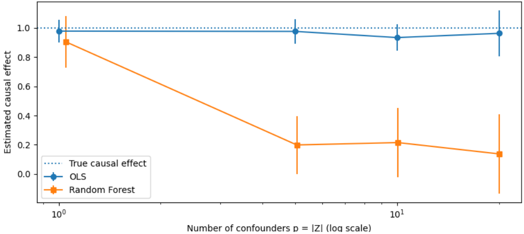

(simulated using a linear model)

Targeted Maximum Likelihood Estimation

TMLE relies on two nuisance models:

$$
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O(A, \mathbb{Z}) &= \mathbb{E}[Y \mid A, \mathbb{Z}] \quad \text{(outcome model)}, \\
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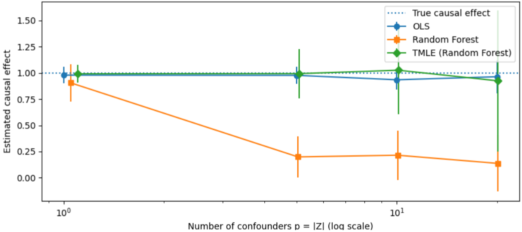

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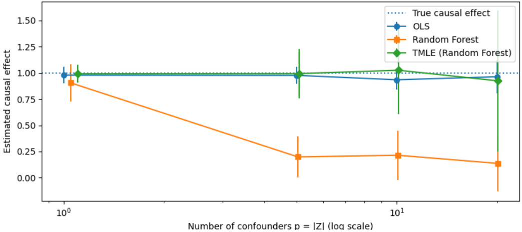

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TMLE is doubly robust: Consistent $\hat{Q}$ if either $\hat{O}$ or $\hat{E}$ is correctly specified.

Cross-fitting

- Flexible ML models can overfit.
- Overfitting leads to biased estimates and invalid confidence intervals.

Cross-fitting idea:

- Split data into folds.
- Estimate nuisance models on one fold.
- Predict on another fold.
- Combine results across folds.

Take-home: TMLE + ML + cross-fitting = flexible estimation with valid confidence intervals.

Python code: causal estimation using zepid package

```
#Load data
import pandas as pd
data = pd.read_csv("./data.csv")
#Load zepid and sklearn
from zepid.causal.doublyrobust import TMLE
from sklearn.ensemble import RandomForestRegressor as rf
#Estimate causal effect
tmle = TMLE( df=data, exposure='A', outcome='Y' )
tmle.exposure_model('Z')
tmle.outcome_model('A+Z', custom_model=rf())
tmle.fit()
#Show results
tmle.summary()
```

Remark: cross-fitting is not implemented!

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6

Causal discovery

Causal discovery using conditional independencies

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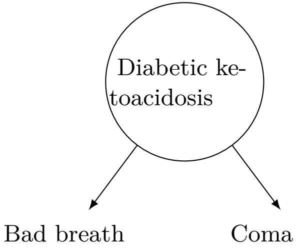

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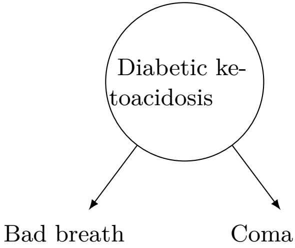

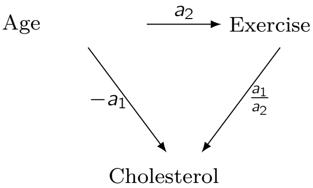

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Markov equivalence class [7]

Equivalence in terms of conditional independence

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Theorem

Two causal DAGs are Markov equivalent iff they have the same skeleton and the same unshielded colliders.

- All equivalent graphs can be represented by a completed partially DAG (CPDAG)
- This CPDAG is called the representative of the Markov equivalence class

Given observational data, is it possible to infer a CPDAG using conditional independencies under the assumptions of faithfulness and causal sufficiency? Yes!

Finding skeleton and unshielded colliders

Theorem

If $P(\mathbb{V})$ is faithful to some causal DAG $\mathcal{G}$ with vertex $\mathbb{V}$ then:

- For $X, Y \in \mathbb{V}$, X and Y are adjacent iff $\forall \mathbb{S} \subseteq \mathbb{V} \backslash \{X, Y\}$, $X \not\perp\!\!\!\perp_P Y \mid \mathbb{S}$;

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Finding skeleton and unshielded colliders

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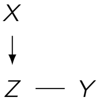
- Point 1 can be used to discover the skeleton of $\mathcal{G}$ from $P(\mathbb{V})$;
- Given the skeleton of $\mathcal{G}$, point 2 can be used to find all unshielded colliders.

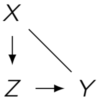
Orientation rules

Suppose we already found the skeleton and all unshielded colliders:

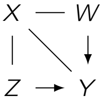
Meek-Rule 1:



Meek-Rule 2:



Meek-Rule 3:



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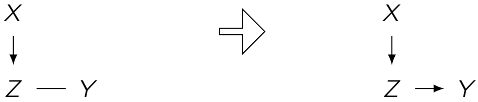

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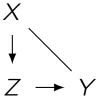

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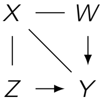

Orientation rules

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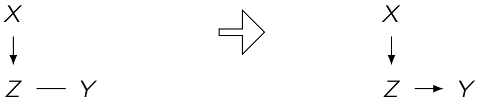
Meek-Rule 1:



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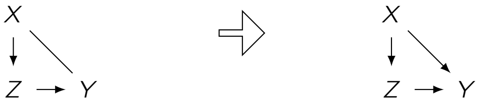

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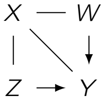

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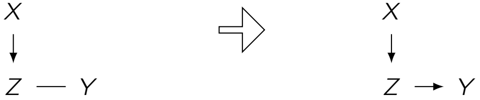

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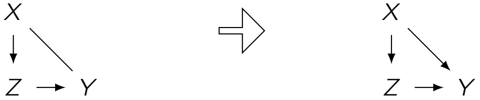

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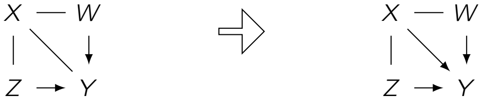

The PC algorithm [6]

- Step 1: skeleton construction:
  - Construct a complete non oriented graph
  - Prune unnecessary edges (optimal) from the skeleton using $\perp\!\!\!\perp_P$

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- Step 2: orientation
  - Find unshielded colliders (optimal)
  - Meek-Rules 1, 2, 3

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  - Meek-Rules 1, 2, 3

Theorem ([6])

Assume the distribution P is compatible and faithful to some causal DAG $\mathcal{G}$ and assume that we are given perfect conditional independence information about all pairs of variables. The PC algorithm returns the CPDAG of $\mathcal{G}$.

Pseudocode of the PC algorithm

Algorithm 1 PC

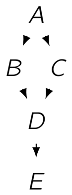
Input: $P(\mathbb{V})$

Output: CPDAG $\mathcal{G}^*$

1: Initialize a complete undirected graph $\mathcal{G}^*$ on vertex set $\mathbb{V}$
2: Let $n = 0$
3: **repeat**
4:     **for all** $X - Y$ in $\mathcal{G}^*$ such that $|Ne(X, \mathcal{G}^*) \backslash \{Y\}| \geq n$
    and subsets $\mathbb{S} \subseteq Ne(X, \mathcal{G}^*) \backslash \{Y\}$ such that $|\mathbb{S}| = n$ **do**
5:         **if** $X \perp\!\!\!\perp_P Y \mid \mathcal{S}$ **then**
6:             Delete edge $X - Y$ from $\mathcal{G}^*$
7:             Let $sepset(X, Y) = sepset(Y, X) = \mathbb{S}$
8:         **end if**
9:     **end for**
10:     Let $n = n + 1$
11: **until** for each pair of adjacent vertices (X, Y), $|Ne(X, \mathcal{G}^*) \backslash \{Y\}| < n$
12: For each triple $X - Z - Y$ in $\mathcal{G}^*$, if $X \notin Ne(Y, \mathcal{G}^*)$ and $Z \notin sepsep(X, Y)$
then orient the triple as $X \rightarrow Z \leftarrow Y$
13: Recursively apply Meek-Rules until no more edges can be oriented
14: Return $\mathcal{G}^*$

PC in action

- Suppose the causal DAG on the right
- Input: Observational data
- Output: CPDAG
- Assumptions: causal sufficiency, faithfulness



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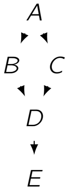
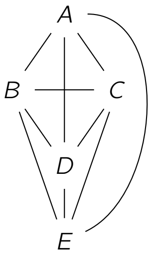


Skeleton construction:



$card = 0$

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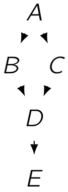

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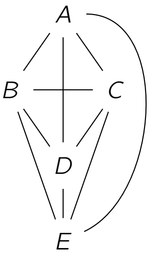
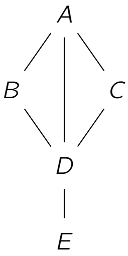


$card = 0$



$card = 1$

PC in action

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- Input: Observational data
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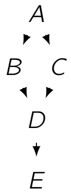
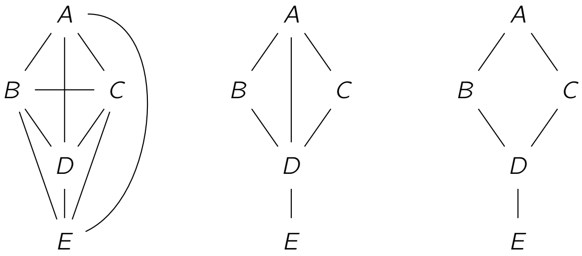
- Assumptions: causal sufficiency, faithfulness



Skeleton construction:



$card = 0$ $card = 1$ $card = 2$

PC in action

- Suppose the causal DAG on the right
- Input: Observational data
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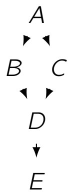
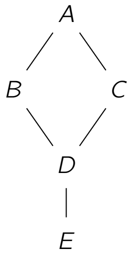


Orientation:



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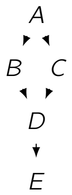

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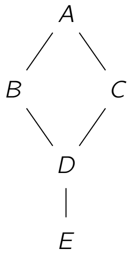

- Find unshielded colliders

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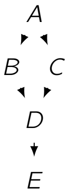

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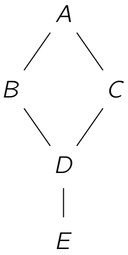

- Find unshielded colliders
  - $B \perp\!\!\!\perp_P C \mid A \implies B \rightarrow D \leftarrow C$

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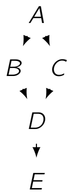

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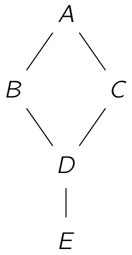

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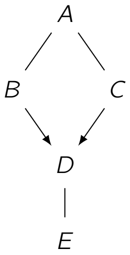

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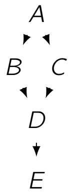
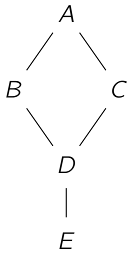


Orientation:



- Find unshielded colliders
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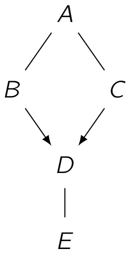
- Meek-Rule 1



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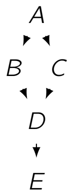

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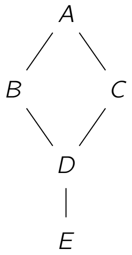

- Find unshielded colliders
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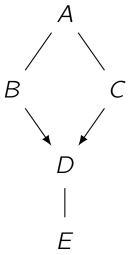
- Meek-Rule 1
  - $B \to D$ & $D - E \implies D \to E$



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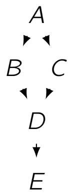

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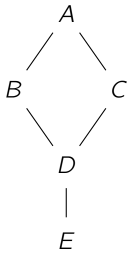

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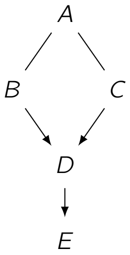


Incorporating background knowledge, extensions, and more

The PC algorithm can effectively incorporate background knowledge in the form of:

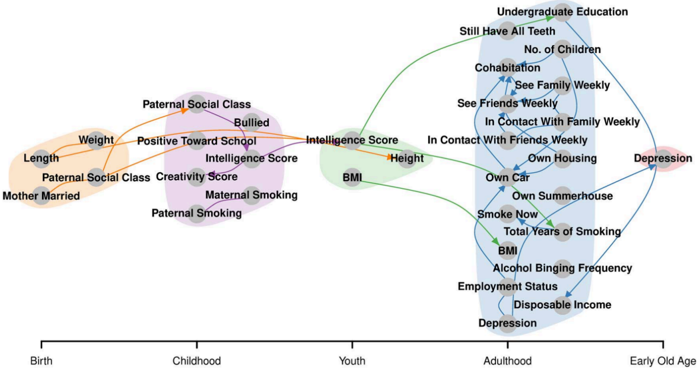
- Forbidden edges
- Required edges
- Forbidden orientations
- Required orientations

The PC algorithm has been extended to settings with unmeasured confounding: FCI algorithm [6]

Many other causal discovery algorithms have been proposed; some are able to recover the underlying DAG under additional assumptions, such as semiparametric modeling assumptions: LiNGAM, ANM, ... [4]

Real application of PC [5]

- Danish men born in 1953, followed from birth to age 65
- Data sources: surveys at ages 12 and 51 + administrative registers
- 33 variables measured across 5 life-course periods

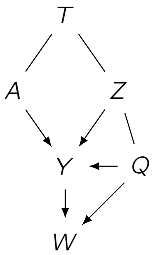


Python code: causal discovery using causallearn package

```
#Load data
import pandas as pd
data = pd.read_csv("./data.csv")
#Load the causallearn package
from causallearn.search.ConstraintBased.PC import pc
#Learn CPDAG
cg = pc(data, alpha=0.05, indep_test='fisherz')
#Plot CPDAG
cg.draw_pydot_graph()
```

Applicability of the back-door criterion to CPDAGs

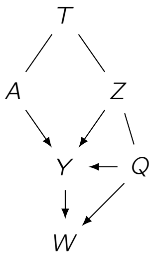
CPDAG



- Is it possible to identify the causal effect of A on Y using the CPDAG?
- What about causal effect of Y on W?

Applicability of the back-door criterion to CPDAGs

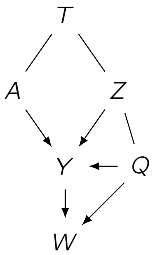
CPDAG



- Is it possible to identify the causal effect of A on Y using the CPDAG? No
- What about causal effect of Y on W? Yes

Applicability of the back-door criterion to CPDAGs

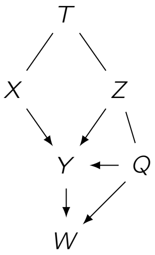
CPDAG



- Is it possible to identify the causal effect of A on Y using the CPDAG? No
- What about causal effect of Y on W? Yes

A set of variables $\mathbb{Z}$ satisfies the back-door criterion relative to an ordered pair of variables (A, Y) in a CPDAG if $\mathbb{Z}$ does not contain a possible descendant of A; and $\mathbb{Z}$ blocks all possible paths between A and Y that contain an arrow pointing toward A.

Exercise 4



Which orientation rules did PC use to orient this CPDAG?

7

References

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[1] Judea Pearl. "Causal diagrams for empirical research". In: Biometrika 82.4 (Dec. 1995), pp. 669–688. issn: 0006-3444. doi: 10.1093/biomet/82.4.669.

[2] Judea Pearl. Causality: Models, Reasoning, and Inference. Cambridge University Press, 2000.

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