Identifiability of Direct Effects from Summary Causal Graphs

including appendix

Simon Ferreira[1,2], **Charles K. Assaad**[1]

[1]EasyVista
[2] ENS de Lyon
simon.ferreira@ens-lyon.fr, cassaad@easyvista.com

Abstract

Dynamic structural causal models (SCMs) are a powerful framework for reasoning in dynamic systems about direct effects which measure how a change in one variable affects another variable while holding all other variables constant. The causal relations in a dynamic structural causal model can be qualitatively represented with an acyclic full-time causal graph. Assuming linearity and no hidden confounding and given the full-time causal graph, the direct causal effect is always identifiable. However, in many application such a graph is not available for various reasons but nevertheless experts have access to the summary causal graph of the full-time causal graph which represents causal relations between time series while omitting temporal information and allowing cycles. This paper presents a complete identifiability result which characterizes all cases for which the direct effect is graphically identifiable from a summary causal graph and gives two sound finite adjustment sets that can be used to estimate the direct effect whenever it is identifiable.

Introduction

Structural causal models (SCMs) are a powerful framework for representing and reasoning about causal relations between variables with a long history in many fields such as genetics (Wright 1920, 1921), econometrics (Haavelmo 1943), social sciences (Duncan 1975; Goldberger 1972), epidemiology (Hernan and Robins 2023), and artificial intelligence (Pearl 2000). In particular, SCMs are useful for reasoning about direct effects which measure how a change in one variable affects another variable while holding all other variables constant (Pearl 2012). The identification and estimation of direct effects are important in many application, *e.g.*, epidemiologists are interested in measuring how smoking affects lung cancer risk without being mediated by genetic susceptibility (Zhou et al. 2021); ecologists are usually focus on understanding direct effects such as competition, herbivory, and predation (Connell 1961); IT experts can localize root causes of system failures by comparing the direct impact of different components on each other before and after the failure (Assaad, Ez-Zejjari, and Zan 2023).

In the framework of (non-dynamic) SCMs, assuming linearity and no hidden confounding and given a causal graph which qualitatively represents causal relations between different variables, the direct effect between two variables is always identifiable and there exists a complete graphical tool, called the single-door criterion (Pearl 1998; Spirtes et al. 1998; Pearl 2000) that finds all possible adjustment sets that allow to estimate the direct effect from data. These results are directly applicable in dynamic SCMs (Rubenstein et al. 2018) given an acyclic full-time causal graph—which qualitatively represents all causal relations between different temporal instants of the dynamic SCM—and assuming consistency throughout time. However, in many dynamic systems, experts have difficulties in building a full-time causal graph (Aït-Bachir et al. 2023), while they can usually build a summary causal graph (Assaad, Devijver, and Gaussier 2022) which is an abstraction of the full-time causal graph where temporal information is omitted and cycles are allowed. So far, the problem of identifying direct effects has solely been tackled for summary causal graphs with no cycles exceeding a size of 2 (Assaad, Ez-Zejjari, and Zan 2023). Specifically, it has been shown that in the absence of cycles greater than 2, direct effects are always identifiable, and in these cases, an adjustment set has been proposed for estimating the direct effect from data. However, in many applications, there exist cycles with a size greater than 2 (Veilleux 1979; Staplin et al. 2016).

In this work, we focus on the identifiability of direct effects from summary causal graphs without restricting the size of cycles. Our main contribution is twofold. First, we give a complete identifiability result which characterizes all cases for which a direct effect is graphically identifiable from a summary causal graph. Then, we present two finite adjustment sets that can be used to estimate the direct effect from data whenever it is identifiable.

The remainder of the paper is organized as follows: in the next section, we give the definitions of direct effect and summary causal graph and recall all necessary graphical preliminaries. In the section that follows it, we present the complete identifiability result in addition to a weaker but interesting result that is implied by the complete identifiability result. Then, in another section, we provide two finite adjustment sets that can be used to estimate the direct effect from data whenever it is identifiable. Finally, in the last section, we conclude the paper while elaborating on promising directions for further research.

Problem Setup

In this section, we first introduce some terminology, tools, and assumptions which are standard for the major part. Then, we formalize the problem we are going to solve. Without any opposite mention, in the following, a capital letter correspond to either to a variable or a vertex and a calligraphic letter correspond to a set with the exception of the set of natural numbers and the set of all integers which are respectivly represented by $\mathbb{N}$ and $\mathbb{Z}$.

In this work, we consider that a dynamic system can be represented by a linear dynamic SCM.

Definition 1 (Linear Dynamic SCM). *Considering a finite set of observed times series $\mathcal{V}$, a* linear dynamic SCM *is a set of equations in which each instant $t \in \mathbb{Z}$ of a time series (e.g., Y_t) is defined as a linear function of past instants of itself (e.g., $Y_{t-\gamma},\ \gamma > 0$), past or present instants of other times series (e.g., $X_{t-\gamma},\ X \neq Y,\ \gamma \geq 0$) and of some unobserved noise (e.g., ξ_{Y_t}):*

$$Y_t := \sum_{\gamma>0} \alpha_{Y_{t-\gamma},Y_t} * Y_{t-\gamma} + \sum_{X \neq Y,\ \gamma \geq 0} \alpha_{X_{t-\gamma},Y_t} * X_{t-\gamma} + \xi_{Y_t},$$

where any coefficient α can be zero and for each $Y \in \mathcal{V}$ the noises $\{\xi_{Y_t}, t \in \mathbb{Z}\}$ are identically distributed.

Note that in nonlinear SCMs, the direct effect is not uniquely defined, potentially varying depending on the values taken by other variables (Pearl 2000). In contrast, linear SCMs do not encounter such issues. In the following, we give the definition of direct effects in the case of linear dynamic SCMs.

Definition 2 (Direct Effect, (Pearl 2000)). *In a linear dynamic SCM, the* direct effect *of $X_{t-\gamma_{xy}}$ on Y_t is fully specified by the structural coefficient*[1] $\alpha_{X_{t-\gamma_{xy}},Y_t}$.

In the following, we explicitly state further assumptions we make in this paper.

Assumption 1 (No hidden Confounding, (Spirtes, Glymour, and Scheines 2000; Pearl 2000)). *The noise variables in the linear dynamic SCM are assumed to be jointly independent (i.e., $\forall X, Y \in \mathcal{V}\ \forall t', t \in \mathbb{Z},\ X_{t'} \neq Y_t \implies \xi_{X_{t'}} \perp\!\!\!\perp \xi_{Y_t}$).*

Assumption 2 (Stationarity). *The causal mechanisms of the system considered do not change over time and therefore, $\forall X \neq Y \in \mathcal{V}$, $\forall t-\gamma \leq t \in \mathbb{Z}$, $\alpha_{X_{t-\gamma},Y_t} = \alpha_{X_{t-\gamma+1},Y_{t+1}}$, and $\forall t-\gamma < t \in \mathbb{Z}$, $\alpha_{Y_{t-\gamma},Y_t} = \alpha_{Y_{t-\gamma+1},Y_{t+1}}$. There exists a maximum lag γ_{max} of a dynamic SCM defined as $\gamma_{max} := max\{\gamma \in \mathbb{N} | \exists X, Y \in \mathcal{V},\ \alpha_{X_{t-\gamma},Y_t} \neq 0\}$.*

Definition 3 (Full-Time Causal Graph). *Considering a finite set of times series $\mathcal{V}$ and a dynamic SCM, one can define the* full-time causal graph (FTCG) *$\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ associated to the dynamic SCM in the following way:*

$$\begin{aligned} \mathcal{V}_f &:= \{Y_t \quad |\forall Y \in \mathcal{V},\ \forall t \in \mathbb{Z}\}, \\ \mathcal{E}_f &:= \{X_{t-\gamma} \rightarrow Y_t \quad |\forall X_{t-\gamma}, Y_t \in \mathcal{V}_f \\ &\qquad \text{such that } \alpha_{X_{t-\gamma},Y_t} \neq 0\}. \end{aligned}$$

In addition, for every vertex $Y_t \in \mathcal{V}_f$ we note:

- $Par(Y_t, \mathcal{G}_f) = \{U_{t'} \in \mathcal{V}_f | U_{t'} \rightarrow Y_t \text{ in } \mathcal{E}_f\}$,
- $Anc(Y_t, \mathcal{G}_f) = \bigcup_{n \in \mathbb{N}} P_n$ *where* $P_0 = \{Y_t\}$ *and* $P_{k+1} = \bigcup_{U_{t'} \in P_k} Par(U_{t'}, \mathcal{G}_f)$, *and*
- $Desc(Y_t, \mathcal{G}_f) = \bigcup_{n \in \mathbb{N}} C_n$ *where* $C_0 = \{Y_t\}$ *and* $C_{k+1} = \bigcup_{U_{t'} \in C_k} \{W_{t''} \in \mathcal{V}_f | U_{t'} \rightarrow W_{t''} \text{ in } \mathcal{E}_f\}$.

Assumption 3 (Acyclicity of the FTCG). *Every FTCG is acyclic.*

The FTCG is the most natural way to represent a dynamic SCM but it is unpractical as it is infinite. Of course, given Assumption 2, it is possible to represent an FTCG in a finite graph, but even in this case sometimes, it is difficult to construct this type of graphs from prior knowledge due to the uncertainty regarding temporal lags. Furthermore, causal discovery methods are not always efficient (Aït-Bachir et al. 2023) due to the strong assumptions they require that are not always satisfied in real applications, so constructing this type of graphs from data is not always a valid option. Therefore, it is much more reliable to construct an abstraction of this type of graphs which does not contain temporal information. This abstraction is usually referred to as a summary causal graph (Assaad, Devijver, and Gaussier 2022).

Definition 4 (Summary Causal Graph). *Considering a finite set of times series $\mathcal{V}$ and an FTCG $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$, one can define the* summary causal graph (SCG) *$\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ compatible with the FTCG in the following way:*

$$\begin{aligned} \mathcal{V}_s &:= \mathcal{V}, \\ \mathcal{E}_s &:= \{X \rightarrow Y \quad |\forall X, Y \in \mathcal{V}, \\ &\qquad \exists t' \leq t \in \mathbb{Z} \text{ such that } X_{t'} \rightarrow Y_t \in \mathcal{E}_f\}. \end{aligned}$$

In addition, for every vertex $Y \in \mathcal{V}_s$ we note:

- $Par(Y, \mathcal{G}_s) = \{U \in \mathcal{V}_s | U \rightarrow Y \text{ or } U \rightleftarrows Y \text{ in } \mathcal{E}_s\}$,
- $Anc(Y, \mathcal{G}_s) = \bigcup_{n \in \mathbb{N}} P_n$ *where* $P_0 = \{Y\}$ *and* $P_{k+1} = \bigcup_{U \in P_k} Par(U, \mathcal{G}_s)$, *and*
- $Desc(Y, \mathcal{G}_s) = \bigcup_{n \in \mathbb{N}} C_n$ *where* $C_0 = \{Y\}$ *and* $C_{k+1} = \bigcup_{U \in C_k} \{U \in \mathcal{V}_s | U \rightarrow W \text{ or } U \rightleftarrows W \text{ in } \mathcal{E}_s\}$.

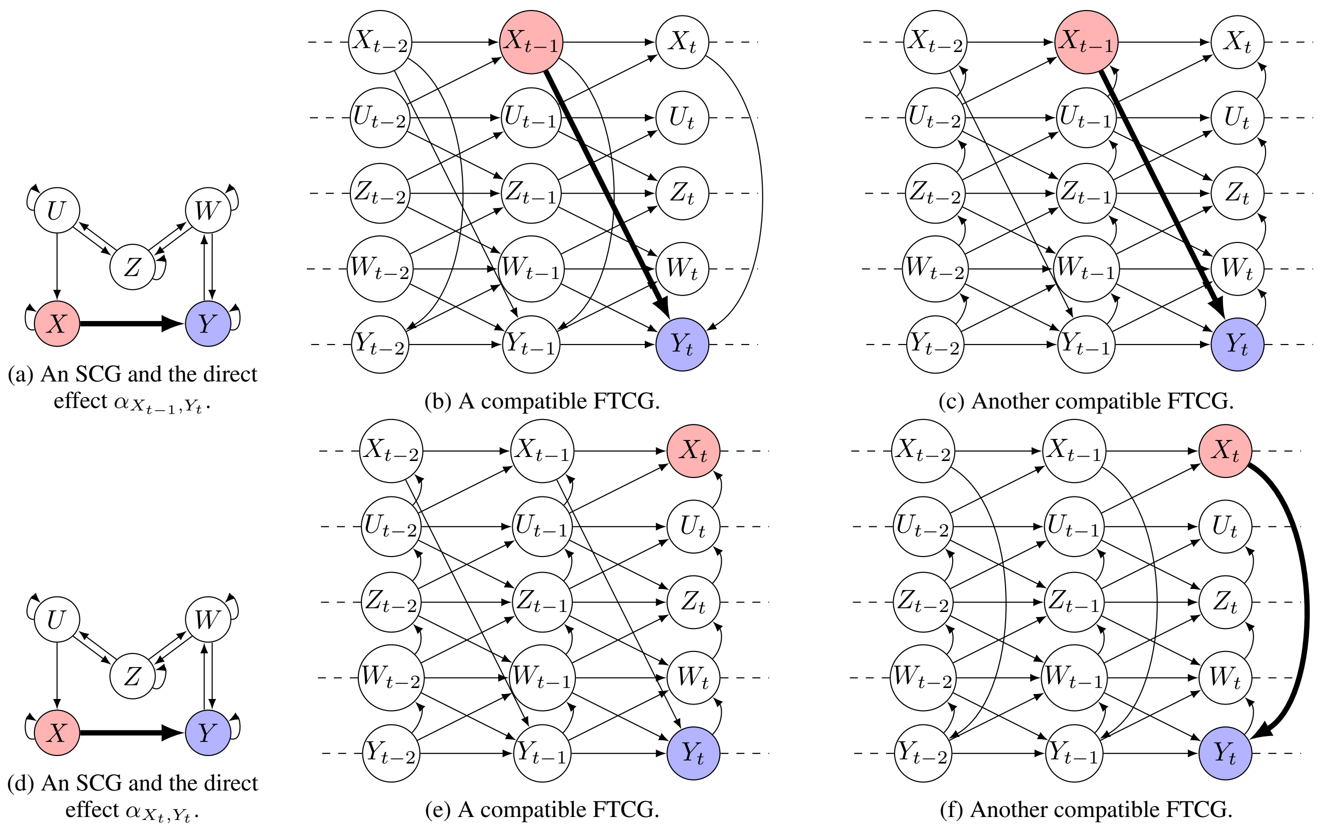
Notice that an SCG may have cycles and in particular two arrows in opposite directions, i.e., if in the FTCG we have $X_{t'} \rightarrow Y_t$ and $Y_{t''} \rightarrow X_t$ then in the SCG we have $X \rightleftarrows Y$.

The abstraction of SCGs entails that, even though there is exactly one SCG compatible with a given FTCG, there are in general several FTCGs compatible with a given SCG. For example, the FTCGs in Figure 1b and 1c correspond to the same SCG given in Figure 1a.

Now that we have defined direct effects and SCGs, we can formally define graphical identifiability.

Definition 5 (Graphical Identifiability of a Direct Effect from an SCG). *The direct effect of a time instant $X_{t-\gamma_{xy}}$ on another time instant Y_t, i.e., $\alpha_{X_{t-\gamma_{xy}},Y_t}$, in a linear dynamic SCM is said to be* identifiable *from an SCG if the quantity $\alpha_{X_{t-\gamma_{xy}},Y_t}$ can be computed uniquely from the observed distribution without any further assumption on the distribution and without knowing the FTCG.*

[1] In Sewall Wright's terminology the structural coefficient is called path coefficient (Wright 1920, 1921).



(a) An SCG and the direct effect α_{X_{t-1},Y_t}.

(b) A compatible FTCG.

(c) Another compatible FTCG.

(d) An SCG and the direct effect α_{X_t,Y_t}.

(e) A compatible FTCG.

(f) Another compatible FTCG.

Figure 1: An example of an SCG in (a and b) with four of its compatible FTCGs in (b), (c), (e) and (f). The pair of red and blue vertices in the SCG and in the FTCGs represents the direct effect of interest, *i.e.*, α_{X_t,Y_t} and α_{X_{t-1},Y_t}. In this example, α_{X_{t-1},Y_t} is non-identifiable given the SCG because for this direct effect, in the FTCG in (b), X_t should be in every valid adjustment set but X_t should not be in any valid adjustment set in the FTCG in (c). Similarly, α_{X_t,Y_t} is non-identifiable given the SCG because for this direct effect (equal to zero), in the FTCG in (e), at least one vertex in $\{U_t, Z_t, W_t\}$ should be in every valid adjustment set but none of the vertices in $\{U_t, Z_t, W_t\}$ should be in any valid adjustment set in the FTCG in (f).

Note that given the true FTCG, computing the direct effect uniquely from the observed distribution without any further assumption on the distribution usually consists of removing all confounding bias and non-direct effects by adjusting on a set that do not create any selection bias. Removing confounding bias usually necessitates adjusting on some suitable ancestors of the cause or the effect, removing non-direct effects usually consists of adjusting on some suitable ancestors of the effect that are not ancestors of the cause, and creating selection bias consists of adjusting on descendants of the effect. In the linear setting, given such an adjustment set $\mathcal{Z}_f$, the direct effect $\alpha_{X_{t-\gamma},Y_t}$ can be estimated using the partial linear regression coefficient $r_{X_{t-\gamma}Y_t.\mathcal{Z}_f}$ where $r_{X_{t-\gamma}Y_t.\mathcal{Z}_f}$ represents the correlation between $X_{t-\gamma}$ and Y_t after $\mathcal{Z}_f$ is "partialled out" (Pearl 2000). The difficulty of this paper lies in finding such an adjustment set $\mathcal{Z}_f$ using the SCG and without knowing the true FTCG. This consists in finding an adjustment set that is valid for every FTCG compatible with the SCG.

In the following, we recall several preliminaries related to FTCGs and SCGs.

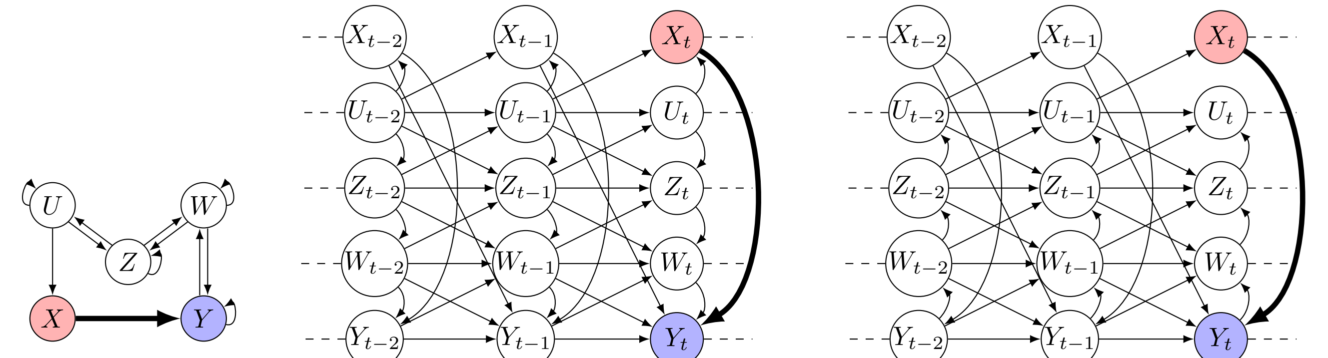
Definition 6 (Paths in FTCGs)**.** *A* path *between two vertices $X_{t'}$ to Y_t is an ordered sequence of vertices denoted as $\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n}\rangle$ such that $V^1_{t^1} = X_{t'}$, $V^n_{t^n} = Y_t$ and $\forall 1 \leq i < n$, $V^i_{t^i}$ and $V^{i+1}_{t^{i+1}}$ are adjacent (i.e., $V^i_{t^i} \rightarrow V^{i+1}_{t^{i+1}}$ or $V^i_{t^i} \leftarrow V^{i+1}_{t^{i+1}}$) and $\forall 1 \leq i < j \leq n$, $V^i_{t^i} \neq V^j_{t^j}$. In cases where orientations are crucial in the context under discussion, in the path π_f, we will replace commas with orientations. In this paper, a path π_f from $X_{t'}$ to Y_t is said to be* non-direct *if $\pi_f \neq \langle X_{t'} \rightarrow Y_t\rangle$.*

Definition 7 (Paths in SCGs)**.** *A* path *between two vertices X to Y is an ordered sequence of vertices denoted as $\pi_s = \langle V^1, \dots, V^n\rangle$ such that $V^1 = X$, $V^n = Y$ and $\forall 1 \leq i < n$, V^i and V^{i+1} are adjacent (i.e., $V^i \rightarrow V^{i+1}$ or $V^i \leftarrow V^{i+1}$ or $V^i \rightleftarrows V^{i+1}$) and $\forall 1 \leq i < j \leq n$, $V^i \neq V^j$. In cases where orientations are crucial in the context under discussion, in the path π_s, we will replace commas with orientations. In this paper, a path π_s from X to Y is said to be* non-direct *if $\pi_s \neq \langle X \rightarrow Y\rangle$.*

A path is said to be *directed* if it only contains $\rightarrow$ and $\leftrightarrows$.

Definition 8 (Cycles in SCGs)**.** *In an SCG, a* cycle *is an ordered sequence of vertices $\pi_s = \langle V^1, \dots, V^n\rangle$ such that:*

- $V^1 = V^n$,



(a) An SCG and the direct effect α_{X_t,Y_t}.

(b) A compatible FTCG.

(c) Another compatible FTCG.

Figure 2: An example of an SCG in (a) with two of its compatible FTCGs in (b) and (c). The pair of red and blue vertices in the SCG and in the FTCGs represents the direct effect of interest, *i.e.*, α_{X_t,Y_t}. In this example, α_{X_t,Y_t} is non-identifiable given the SCG because for this direct effect at least one vertex in $\{U_t, Z_t, W_t\}$ should be in every valid adjustment set for the first FTCG but none of the vertices in $\{U_t, Z_t, W_t\}$ should be in any valid adjustment set for the second FTCG.

- $\forall 1 \leq i < n,\ V^i \rightarrow V^{i+1}$ *or* $V^i \rightleftarrows V^{i+1}$*, and*
- $\forall 1 \leq i < j \leq n,\ V^i = V^j \implies i = 1$ *and* $j = n$*.*

The set of cycles with endpoints $Y \in \mathcal{V}_s$ *in an SCG* $\mathcal{G}_s$ *is written* $Cycles(Y, \mathcal{G}_s)$*.*

An important graphical notion used in causal reasoning is the notion of blocked path (Pearl 1998) for which the classical definition was introduced for directed acyclic graphs and thus can be directly used for FTCGs under Assumption 3. Note that an adjustment set that removes all confounding bias and non-direct effects and that do not create any selection bias between $X_{t-\gamma_{xy}}$ and Y_t consists of non descendants of Y_t different from $X_{t-\gamma_{xy}}$ that block all non-direct paths between $X_{t-\gamma_{xy}}$ and Y_t. In the following, we firstly give the definition of blocked path in FTCGs and then we give a similar notion for SCGs.

Definition 9 (Blocked Path in FTCGs)**.** *In an FTCG* $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$*, a path* $\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n}\rangle$ *is said to be* blocked *by a set of vertices* $\mathcal{Z}_f \subseteq \mathcal{V}_f$ *if:*

1. $V^1_{t^1} \in \mathcal{Z}_f$ *or* $V^n_{t^n} \in \mathcal{Z}_f$*, or*
2. $\exists 1 < i < n$ *such that* $V^{i-1}_{t^{i-1}} \leftarrow V^i_{t^i}$ *or* $V^i_{t^i} \rightarrow V^{i+1}_{t^{i+1}}$ *and* $V^i_{t^i} \in \mathcal{Z}_f$*, or*
3. $\exists 1 < i < n$ *such that* $\rightarrow V^i_{t^i} \leftarrow$ *and* $Desc(V^i_{t^i}, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$*.*

A path which is not blocked is said to be active*. When the set* $\mathcal{Z}_f$ *is not specified, it is implicit that we consider* $\mathcal{Z}_f = \emptyset$*.*

The classical definition of blocked path is usually used in directed acyclic graph and since the SCG compatible with an FTCG can be cyclic, one needs to adapt it. (Spirtes 1993) explains that under the linearity assumption the notion of blocked path is readily extended. Moreover, (Forré and Mooij 2017) introduced a more recent and general (non-parametric and allow for hidden confounding) adaptation called σ-blocked path. Here we will focus on the definition used in (Spirtes 1993) since we assume linearity but we adapt it such that in SCGs extremity vertices of a path do not necessarily block the path.

Definition 10 (Blocked Path in SCGs)**.** *In an SCG* $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$*, a path* $\pi_s = \langle V^1, \dots, V^n\rangle$ *is said to be* blocked *by a set of vertices* $\mathcal{Z}_s \subseteq \mathcal{V}_s$ *if:*

1. $\exists 1 < i < n$ *such that* $V^{i-1} \leftarrow V^i$ *or* $V^i \rightarrow V^{i+1}$ *and* $V^i \in \mathcal{Z}_s$*, or*
2. $\exists 1 < i \leq j < n$ *such that* $\langle V^{i-1} \rightarrow V^i \rightleftarrows \cdots \rightleftarrows V^j \leftarrow V^{j+1}\rangle$ *and* $Desc(V^i, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$*.*

A path which is not blocked is said to be active*. When the set* $\mathcal{Z}_s$ *is not specified, it is implicit that we consider* $\mathcal{Z}_s = \emptyset$*.*

Condition 1 in Definition 10 is a direct adaptation of condition 2 in Definition 9. Condition 2 of Definition 10 is explained by the fact that for a path $\pi_s = \langle V^1, \dots, V^n\rangle$ in an SCG $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ and a set of vertices $\mathcal{Z}_s \subseteq \mathcal{V}_s$, if $\exists 1 < i \leq j < n$ such that

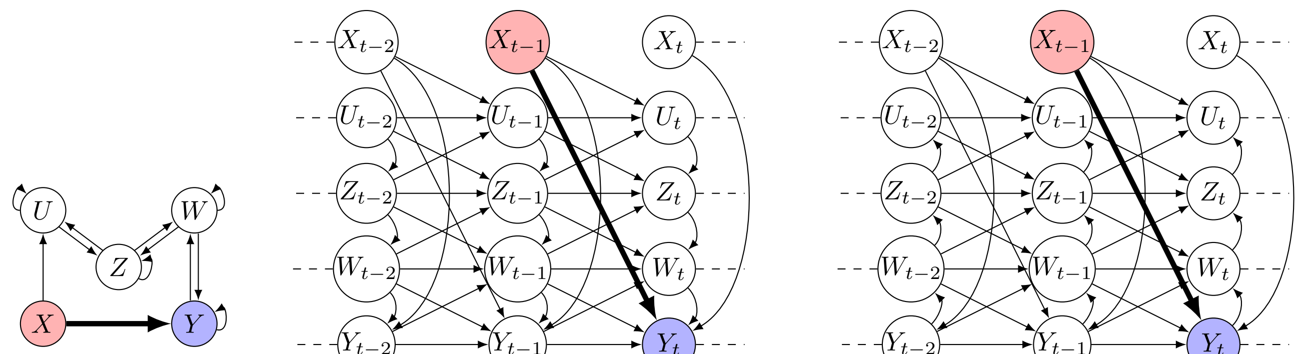
$$\langle V^{i-1} \rightarrow V^i \rightleftarrows \cdots \rightleftarrows V^j \leftarrow V^{j+1}\rangle \text{ and}$$
$$Desc(V^i, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$$

then $\forall \pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n}\rangle, \exists 1 < i \leq k \leq j < n$ such that

$$\rightarrow V^k_{t^k} \leftarrow \text{ and } Desc(V^k, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$$

therefore $Desc(V^k_{t^k}, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$ where $\mathcal{Z}_f \subseteq \{V_{t'} | V \in \mathcal{Z}_s,\ t' \in \mathbb{Z}\}$. Notice that there is no adaptation of condition 1 as having $V^1 \in \mathcal{Z}_s$ or $V^n \in \mathcal{Z}_s$ does not mean that instants of interests $V^1_{t^1}$ or $V^n_{t^n}$ which are endpoints of compatible paths of interests are in $\mathcal{Z}_f \subseteq \{V_{t'} | V \in \mathcal{Z}_s,\ t' \in \mathbb{Z}\}$. Moreover, in this paper we are interested in the direct effect of $V^1_{t^1} = X_{t-\gamma_{xy}}$ on $V^n_{t^n} = Y_t$ and therefore we cannot adjust on them and thus we will always have $V^1_{t^1}, V^n_{t^n} \notin \mathcal{Z}_f$.

Note that a set $\mathcal{Z}_s$ that blocks all paths between two vertices X and Y in an SCG does not necessary have a compatible *finite* set $\mathcal{Z}_f \subseteq \{V_{t'} | V \in \mathcal{Z}_s,\ t' \in \mathbb{Z}\}$ that block all paths between two vertices $X_{t-\gamma_{xy}}$ and Y_t in every FTCG compatible with the given SCG. For example, in Figure 1d, U blocks all paths between X and Z but in Figure 1e, $\nexists i \in \mathbb{N}$ such that $\{U_{t-i}, \cdots, U_t\}$ blocks all paths between X_t and Z_t because there will always be an active path between them passing by U_{t-i-1}.



(a) An SCG and the direct effect α_{X_{t-1},Y_t}.

(b) A compatible FTCG.

(c) Another compatible FTCG.

Figure 3: An example of an SCG in (a) with two of its compatible FTCGs in (b) and (c). The pair of red and blue vertices in the SCG and in the FTCGs represents the direct effect of interest, *i.e.*, α_{X_{t-1},Y_t}. In this example, α_{X_{t-1},Y_t} is non-identifiable given the SCG because for this direct effect at least one vertex in $\{U_t, Z_t, W_t\}$ should be in every valid adjustment set for the first FTCG but none of the vertices in $\{U_t, Z_t, W_t\}$ should be in any valid adjustment set for the second FTCG.

Complete Graphical Identifiability from SCGs

In this section, we start by presenting the complete identifiability result followed by its proof as well as several toy examples to demonstrate it. Then we give a weaker but interesting result that is implied by the identifiability result.

Theorem 1. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be an SCG that represents a linear dynamic SCM verifying Assumptions 1,2,3, $\gamma_{max} \geq 0$ a maximum lag and $\alpha_{X_{t-\gamma_{xy}},Y_t}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X, Y \in \mathcal{V}_s$ and $X \neq Y$. The direct effect $\alpha_{X_{t-\gamma_{xy}},Y_t}$ is non-identifiable from $\mathcal{G}_s$ if and only if $X \in Par(Y, \mathcal{G}_s)$ and one of the following conditions holds:*

1. *$X \in Desc(Y, \mathcal{G}_s)$ and $\exists C \in Cycles(X, \mathcal{G}_s)$ with $Y \notin C$, or*
2. *There exists an active non-direct path $\pi_s = \langle V^1, \ldots, V^n \rangle$ from X to Y in $\mathcal{G}_s$ such that $\langle V^2, \ldots, V^{n-1} \rangle \subseteq Desc(Y, \mathcal{G}_s)$, and one of the following conditions holds:*
   (a) $\gamma_{xy} = 0$, or
   (b) $\gamma_{xy} > 0$, $n \geq 3$ and $\nexists 1 \leq i < n$, $V^i \leftarrow V^{i+1}$ (i.e., $\forall i$, $V^i \rightarrow V^{i+1}$ or $V^i \rightleftarrows V^{i+1}$).

In the following, we provide an additional definition that will be utilized in proving Theorem 1, followed by the proof itself. A more detailed proof of Theorem 1 is also available in the supplementary materials.

Definition 11 (A First Finite Adjustment Set)**.** *Consider an SCG $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$, a maximal lag γ_{max}, two vertices X and Y with $X \in Par(Y, \mathcal{G}_s)$ and a lag γ_{xy}. $\mathcal{Z}_f^{Def\,11} = \mathcal{A}_{\leq t} \cup \mathcal{D}_{<t}$ is an adjustment set relative to $(X_{t-\gamma_{xy}}, Y_t)$ such that:*

$$\mathcal{D}_{<t} = \{V_{t'} \quad | V \in Desc(Y, \mathcal{G}_s), \; t - \gamma_{max} \leq t' < t\} \backslash \{X_{t-\gamma_{xy}}\} \quad and$$
$$\mathcal{A}_{\leq t} = \{V_{t'} \quad | V \in \mathcal{V}_s \backslash Desc(Y, \mathcal{G}_s), \; t - \gamma_{max} \leq t' \leq t\} \backslash \{X_{t-\gamma_{xy}}\}.$$

Proof. Firstly, let us prove the backward implication.

Let $\gamma_{xy} \geq 0$ and $X \in Par(Y, \mathcal{G}_s)$.

- Suppose Condition 1 holds. Let $C = \langle V^1, \ldots, V^n \rangle$ be a cycle on X with $Y \notin C$. If $\gamma_{xy} > 0$ and since $X \in Par(Y, \mathcal{G}_s)$, there exists a compatible FTCG $\mathcal{G}_f^1$ where there exists a directed path $\pi_f^1 = \langle V^1_{t-\gamma_{xy}} = X_{t-\gamma_{xy}}, V_t^2, \ldots, V_t^n = X_t, Y_t \rangle$. Obviously, each set of vertices $\mathcal{Z}_f^1 \subset \mathcal{V}_f^1$ that blocks all non-direct paths between $X_{t-\gamma_{xy}}$ and Y_t have to contain at least one vertex from $\langle V_t^2, \ldots, V_t^{n-1} \rangle$. Similarly, since $X \in Desc(Y, \mathcal{G}_s)$, there exists a compatible FTCG $\mathcal{G}_f^2$ where $\{V_t^2, \ldots, V_t^n = X_t\} \subseteq Desc(Y_t, \mathcal{G}_f^2)$. Thus each set of vertices $\mathcal{Z}_f^2 \subset \mathcal{V}_f^2$ that blocks all non-direct paths between $X_{t-\gamma_{xy}}$ and Y_t should not contain any vertex from $\langle V_t^2, \ldots, V_t^n \rangle$ in order to be a valid adjustment set. If $\gamma_{xy} = 0$, then since $X \in Desc(Y, \mathcal{G}_s)$ there exists $\langle V^n, \ldots, V^1 \rangle$ a directed path from Y to X and there exists $\pi_f = \langle V_t^1, \ldots, V_t^n \rangle$ in a compatible FTCG. If $n = 2$, then $\pi_f = \langle V_t^1 \leftarrow V_t^n \rangle$ cannot be blocked and if $n \geq 3$, then every set $\mathcal{Z}_f$ that blocks this path contains a descendant of Y_t in another compatible FTCG.
- Suppose Condition 2 holds. Let $\pi_s = \langle V^1, \ldots, V^n \rangle$ as described. If $\pi_s = \langle X \rightleftarrows Y \rangle$ and $\gamma_{xy} = 0$, then there exists a compatible FTCG $\mathcal{G}_f$ in which the path $\pi_f = \langle X_t \leftarrow Y_t \rangle$ exists and cannot be blocked. Else, $n \geq 3$ and π_s is active so there exists a compatible FTCG $\mathcal{G}_f^1$ in which the path $\pi_f = \langle V^1_{t-\gamma_{xy}}, V_t^2, \ldots, V_t^{n-1}, V_t^n \rangle$ exists and is active. Notice that there exists another compatible FTCG $\mathcal{G}_f^2$ in which $\{V_t^2, \ldots, V_t^{n-1}\} \subseteq Desc(Y_t, \mathcal{G}_f^2)$ and every set $\mathcal{Z}_f$ that blocks π_f in $\mathcal{G}_f^1$ contains a vertex in $Desc(Y_t, \mathcal{G}_f^2)$.

Secondly, let us prove the forward implication. Let $\mathcal{G}_f$ be an FTCG, $\gamma_{xy} \geq 0$ a lag, $X_{t-\gamma_{xy}}$ and Y_t two vertices and $\mathcal{G}_s$ the SCG compatible with $\mathcal{G}_f$. If $X \notin Par(Y, \mathcal{G}_s)$ then there is no direct effect from $X_{t-\gamma_{xy}}$ to Y_t, *i.e.*, $\alpha_{X_{t-\gamma_{xy}},Y_t} = 0$. Suppose that $X \in Par(Y, \mathcal{G}_s)$ and that the conditions of Theorem 1 are not verified and let us show that $\mathcal{Z}_f^{\text{Def 11}}$ is a valid adjustment set to estimate the direct effect $\alpha_{X_{t-\gamma_{xy}},Y_t}$ by showing that $\mathcal{Z}_f^{\text{Def 11}}$ verifies:

- $\mathcal{Z}_f^{\text{Def 11}} \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$, and
- $\mathcal{Z}_f^{\text{Def 11}}$ blocks every non-direct path from $X_{t-\gamma_{xy}}$ to Y_t in every compatible FTCG $\mathcal{G}_f$ of maximal lag at most γ_{max}.

The first point, $\mathcal{Z}_f^{\text{Def 11}} \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$ is immediate. In order to prove the second point, let $\pi_f = \langle V_{t^1}^1, \dots, V_{t^n}^n \rangle$ be a non-direct path from $X_{t-\gamma_{xy}}$ to Y_t. Notice the following properties:

- If $\exists i_{max} = max\{1 < i < n | t^i < t\}$ then $t - \gamma_{xy} \leq t^{i_{max}} < t$ and $V_{t^{i_{max}}}^{i_{max}} \rightarrow V_{t^{i_{max}+1}}^{i_{max}+1}$ thus π_f is blocked by $\mathcal{Z}_f^{\text{Def 11}}$ as $V_{t^{i_{max}}}^{i_{max}} \in \mathcal{Z}_f^{\text{Def 11}}$.
- Else, if $I = \{1 < i < n | t^i > t\} \neq \emptyset$ then $\exists i \in I$ such that $\rightarrow V_{t^i}^i \leftarrow$ in π_f and π_f is $\mathcal{Z}_f^{\text{Def 11}}$-blocked as $Desc(V_{t^i}^i, \mathcal{G}_f) \cap \mathcal{Z}_f^{\text{Def 11}} = \emptyset$.
- Else, if $\exists i_{max} = max\{1 < i < n | V^i \notin Desc(Y, \mathcal{G}_s)\}$ then $t^{i_{max}} = t$ and $V_{t^{i_{max}}}^{i_{max}} \rightarrow V_{t^{i_{max}+1}}^{i_{max}+1}$ thus π_f is blocked by $\mathcal{Z}_f^{\text{Def 11}}$ as $V_{t^{i_{max}}}^{i_{max}} \in \mathcal{Z}_f^{\text{Def 11}}$.

Therefore, if π_f is $\mathcal{Z}_f^{\text{Def 11}}$-active then $\langle V^2, \dots, V^n \rangle \subseteq Desc(Y, \mathcal{G}_s)$ and either $\pi_f = \langle X_{t-\gamma_{xy}} \leftarrow Y_t \rangle$ which forces $\gamma_{xy} = 0$ or $n \geq 3$ and $t^2 = \dots = t^{n-1} = t$. Notice that π_f and $\pi_s = \langle V^1, \dots, V^n \rangle$ are active and non-direct.

- Suppose $\pi_f = \langle X_t \leftarrow Y_t \rangle$ ($\gamma_{xy} = 0$). Given that $X \in Par(Y, \mathcal{G}_s)$, this means that $\langle X \rightleftarrows Y \rangle$ is a path in $\mathcal{G}_s$ and thus Condition 2a is verified.
- Suppose $n \geq 3$ and $t^2 = \dots = t^{n-1} = t$. We make the following *observation*: *if* $\gamma_{xy} > 0$, *then* $t - \gamma_{xy} = t^1 < t^2 = t$ *so* $V_{t^1}^1 \rightarrow V_{t^2}^2$ *and given that* π_f *is active,* $\forall 1 \leq i < n$, $V_{t^i}^i \rightarrow V_{t^{i+1}}^{i+1}$ *and* $\nexists 1 \leq i < n$, $V^i \leftarrow V^{i+1}$. In addition, since π_f is a path and $t^2 = \dots = t^{n-1} = t$, either π_s is a path or $\gamma_{xy} > 0$ and $\exists 1 < i < n$ such that $V^i = X$ and $\forall 1 \leq j_1 < j_2 \leq n$, $V^{j_1} = V^{j_2} \implies j_1 = 1$ and $j_2 = i$.
  - Suppose π_s is a path. In the case $\gamma_{xy} = 0$, Condition 2a would be verified. In the case $\gamma_{xy} > 0$, by the previous *observation*, Condition 2b would be verified.
  - Suppose $\gamma_{xy} > 0$ and $\exists 1 < i < n$ such that $V^i = X$ and $\forall 1 \leq j_1 < j_2 \leq n$, $V^{j_1} = V^{j_2} \implies j_1 = 1$ and $j_2 = i$. Notice that $X = V^i \in Desc(Y, \mathcal{G}_s)$ and that, using the previous *observation*, $Y \notin \langle V^1, \dots, V^i \rangle \in Cycles(X, \mathcal{G}_s)$. Thus Condition 1 is verified.

In conclusion, when the conditions of Theorem 1 are not verified, there is no non-direct $\mathcal{Z}_f^{\text{Def 11}}$-active path π_f between $X_{t-\gamma_{xy}}$ and Y_t. □

To graphically illustrate the backward implication of Theorem 1, we consider three different examples which respectively correspond to Figures 1, 2, and 3 and to Conditions 1, 2a, and 2b of the theorem.

Example 1. *Given the SCG in Figure 1a,* α_{X_{t-1},Y_t} *is non-identifiable since Condition 1 of Theorem 1 is satisfied. This can be illustrated by looking at the two FTCGs in Figures 1b and 1c which are compatible with the given SCG. In the first FTCG, it is obvious that it is important to adjust on* X_t *in order to identify* α_{X_{t-1},Y_t} *(to block the path* $\langle X_{t-1}, X_t, Y_t \rangle$*). However, in the other FTCG,* X_t *is a descendant of* Y_t *therefore it is important not to adjust on it. Which means that without knowing which is the true FTCG and given only the SCG,* α_{X_{t-1},Y_t} *is non-identifiable. Notice that* α_{X_{t-1},Y_t} *remains non-identifiable if we remove the cycle of size* 2 *on* X *and we replace* $X \leftarrow U$ *by* $X \leftrightarrows U$ *since* $X \leftrightarrows U$ *induces a cycle of size* 3*. This cycle means that there might exists an active path* $\langle X_{t-1} \rightarrow U_t \rightarrow X_t \rightarrow Y_t \rangle$ *in one FTCG and that* $\{U_t, X_t\}$ *can be descendants of* Y_t *in another FTCG.*

Similarly, given the SCG in Figure 1d, α_{X_t,Y_t} *is non-identifiable because in the first FTCG, at least one vertex in* $\{U_t, Z_t, W_t\}$ *should be in every valid adjustment set but in the second FTCG, these vertices are descendants of* Y_t*.*

Example 2. *Given the SCG in Figure 2a (which is similar to the SCGs considered in Example 1 but where the cycle of size* 2 *on* X *is removed so Condition 1 of Theorem 1 is no longer satisfied),* α_{X_t,Y_t} *is non-identifiable since Condition 2a of Theorem 1 is satisfied. This can be illustrated as in Example 1. Notice that in this case* α_{X_{t-1},Y_t} *becomes identifiable as the path* $\langle X_{t-1}, X_t, Y_t \rangle$ *can no longer exist.*

Example 3. *Given the SCG in Figure 3a (which is similar to the SCGs considered in Example 2 but where the orientation between* X *and* U *is reversed),* α_{X_{t-1},Y_t} *is non-identifiable since Condition 2b of Theorem 1 is satisfied. This can be illustrated by looking at the two FTCGs in Figures 3b and 3c. In the first FTCG, at least one vertex in* $\{U_t, Z_t, W_t\}$ *should be in every valid adjustment set but in the second FTCG, these vertices are descendants of* Y_t*. Notice that we have the same result for* α_{X_t,Y_t}*.*

We give additional examples of SCGs in Figure 4 where the direct effect $\alpha_{X_{t-\gamma_{xy}},Y_t}$ is identifiable for all γ_{xy}. Note that $\alpha_{X_{t-\gamma_{xy}},Y_t}$ remains identifiable in Figures 4a,4b,4c,4d even if we replace $X \rightarrow U$ or $X \leftarrow U$ by $X \leftrightarrows U$.

It is important to highlight that Theorem 1 encompasses the identifiability result presented in (Assaad, Ez-Zejjari, and Zan 2023). Specifically, when there are no cycles of size greater than 2 in the SCG, conditions 1 and 2 of the theorem are not satisfied, indicating the identifiability of the direct effect. Interestingly, Theorem 1 also shows that under some conditions we can directly know that there exist at least one non-identifiable instantaneous direct effect. This is given by the following Corollary.

Corollary 1. *Let* $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ *be an SCG. If there exists a cycle in* $\mathcal{G}_s$ *of length strictly greater than* 2*, then* $\exists \alpha_{X_t,Y_t}$ *which is non-identifiable.*

Proof. Let $\langle V^1, \dots, V^n \rangle$ be a cycle in $\mathcal{G}_s$ with $n \geq 3$. $\alpha_{V_t^1,V_t^2}$ from V_t^1 to V_t^2 is non-identifiable because the path $\langle V^n, V^{n-1}, \dots, V^2 \rangle$ from $V^1 = V^n$ to V^2 in the SCG, verifies Condition 2a of Theorem 1. □

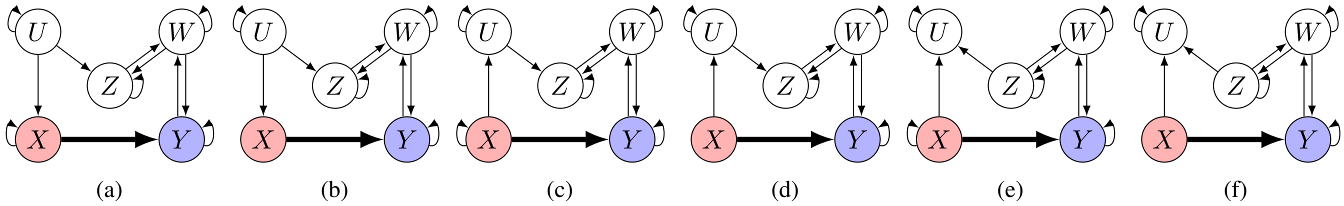


Figure 4: Examples of SCGs with 5 vertices where the direct effect $\alpha_{X_{t-\gamma_{xy}},Y_t}$ is identifiable for all γ_{xy}. Red and blue vertices respectively represent the cause and the effect we are interested in and the thick edge corresponds to the the edge between them. All SCGs share the same skeleton, the edges $X \rightarrow Y$, $Y \leftrightarrows W$, and $Z \leftrightarrows W$ and the cycles of size 2 on Y, W, Z and U.

Two Sound Adjustment Sets

In this section, we provide two finite adjustment sets that can be used to estimate the direct effect whenever it is identifiable. In an effort to be succinct we only give a sketch for the proof of the soundness of the second adjustment. For the full proof, please refer to the supplementary material.

For a given FTCG $\mathcal{G}_f$, to estimate $\alpha_{X_{t-\gamma_{xy}},Y_t}$ from data, it is necessary and sufficient to adjust on a finite set $\mathcal{Z}_f$ such that $\mathcal{Z}_f \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}, Y_t\}) = \emptyset$ which blocks every non-direct path from $X_{t-\gamma_{xy}}$ to Y_t. Thus, given an SCG $\mathcal{G}_s$, one needs to find a set $\mathcal{Z}_f$ such that $\mathcal{Z}_f \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}, Y_t\}) = \emptyset$ which blocks every non-direct path from $X_{t-\gamma_{xy}}$ to Y_t in every FTCG $\mathcal{G}_f$ compatible with $\mathcal{G}_s$ of maximal lag at most γ_{max}.

The following corollary formally indicates the soundness of the finite adjustment set defined in Definition 11.

Corollary 2. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be an SCG, $\gamma_{max} \geq 0$ a maximum lag. Consider two vertices X and Y such that $X \in Par(Y, \mathcal{G}_s)$ and $\alpha_{X_{t-\gamma_{xy}},Y_t}$ is identifiable following Theorem 1. Then $\mathcal{Z}_f^{Def\ 11} = \mathcal{A}_{\leq t} \cup \mathcal{D}_{<t}$ as defined in Definition 11 is a valid adjustment set for $\alpha_{X_{t-\gamma_{xy}},Y_t}$.*

Proof. The proof of the forward implication of Theorem 1 proves this corollary. □

Note that many valid adjustment sets may exist. An estimator of direct effects based on any of these sets is unbiased, but the estimation variance may vary for different sets. Thus, it is interesting to search for many, and ideally all, such sets in order to optimize the estimation of direct effects. In the following, we give a smaller valid adjustment set.

Definition 12 (A Second Finite Adjustment Set)**.** *Consider an SCG $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$, a maximal lag γ_{max}, two vertices X and Y with $X \in Par(Y, \mathcal{G}_s)$ and a lag γ_{xy}. $\mathcal{Z}_f^{Def\ 12} = \mathcal{D}_{<t}^{Anc(Y)} \cup \mathcal{A}_{\leq t}^{Anc(Y)}$ is an adjustment set relative to $(X_{t-\gamma_{xy}}, Y_t)$ such that:*

$$\mathcal{D}_{<t}^{Anc(Y)} = \{V_{t'} \quad | V \in Anc(Y, \mathcal{G}_s) \cap Desc(Y, \mathcal{G}_s), \quad t - \gamma_{max} \leq t' < t\} \backslash \{X_{t-\gamma_{xy}}\} \qquad \textit{and}$$

$$\mathcal{A}_{\leq t}^{Anc(Y)} = \{V_{t'} \quad | V \in Anc(Y, \mathcal{G}_s) \backslash Desc(Y, \mathcal{G}_s), \quad t - \gamma_{max} \leq t' \leq t\} \backslash \{X_{t-\gamma_{xy}}\}.$$

The following proposition formally indicates the soundness of the finite adjustment set defined in Definition 12.

Proposition 1. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be an SCG, $\gamma_{max} \geq 0$ a maximum lag. Consider two vertices X and Y such that $X \in Par(Y, \mathcal{G}_s)$ and $\alpha_{X_{t-\gamma_{xy}},Y_t}$ is identifiable following Theorem 1. Then $\mathcal{Z}_f^{Def\ 12}$ as defined in Definition 12 is a valid adjustment set for $\alpha_{X_{t-\gamma_{xy}},Y_t}$.*

Proof Sketch. Corollary 2 shows that the adjustment set $\mathcal{Z}_f^{\text{Def 11}}$ allows to estimate the direct effect $\alpha_{X_{t-\gamma_{xy}},Y_t}$. Firstly, since $\mathcal{Z}_f^{\text{Def 12}} \subseteq \mathcal{Z}_f^{\text{Def 11}}$ and $\mathcal{Z}_f^{\text{Def 11}} \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$, it is immediate that $\mathcal{Z}_f^{\text{Def 12}} \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$. Moreover, it is known that adjusting on ancestors of the cause Y_t is sufficient to estimate the direct effect in directed acyclic graphs. Therefore, it is intuitive that the restriction of $\mathcal{Z}_f^{\text{Def 11}}$ to ancestors of Y_t, $\mathcal{Z}_f^{\text{Def 11}} \cap \{V_{t'} | V \in Anc(Y, \mathcal{G}_s),\ t' \leq t\} = \mathcal{Z}_f^{\text{Def 12}}$, allows to estimate the direct effect $\alpha_{X_{t-\gamma_{xy}},Y_t}$. The full proof is given in the supplementary materials. □

Conclusion

In this paper, we developed a new graphical criteria for the identifiability of direct effects in linear dynamic structural causal models from summary causal graphs. Theorem 1 has important ramifications to the theory and practice of observational studies in dynamic systems. It implies that the key to graphical identifiability of the direct effect of $X_{t-\gamma_{xy}}$ on Y_t from summary causal graphs lies not only in finding a set of non-descendants of Y in the summary causal graph that are able of blocking paths between X and Y but also in some descendants of Y in the case $\gamma_{xy} > 0$. Furthermore, in case of identifiability, we presented two adjustments sets that can be used to estimate the direct effects from data.

The finding of this paper should be useful for many applications such as root cause identification in dynamic systems and it should open new research questions. Namely, for future works, it would be interesting to have a criterion along with a completeness result describing every possible adjustment set. In addition, since in many real world applications causal relations can be nonlinear, it would be interesting to extend this work to nonlinear SCMs and consider non-parametric direct effects (Robins and Greenland 1992; Pearl 2001). Finally, as many other works, we assumed that the FTCG is acyclic but we think that this assumption can be relaxed, so it would be interesting to formally check the validity of our results for cyclic FTCGs.

Technical Appendix

In Section "*Further Necessary Definitions*" we give several definitions that are needed for a more detailed proof of Theorem 1. In Section "*Proofs*" we start by given several lemmas and properties along with their proofs and then we use them to prove Theorem 1. Finally, at the end of the Section, we give the proof of Proposition 1.

Further Necessary Definitions

In order to prove Theorem 1, one needs to further define walks in SCGs, adapt the concept of blocked paths for walks and introduce some new notions related to paths and walks.

Definition 13 (Walks and Paths in SCGs, adaptation of Definition 7)**.** *A* walk *between two vertices X to Y is an ordered sequence of vertices denoted as $\pi_s = \langle V^1, \dots, V^n \rangle$ such that $V^1 = X$, $V^n = Y$ and $\forall 1 \leq i < n$, V^i and V^{i+1} are adjacent (i.e., $V^i \rightarrow V^{i+1}$ or $V^i \leftarrow V^{i+1}$ or $V^i \rightleftarrows V^{i+1}$). In this paper, a walk π_s between X to Y is said to be* non-direct *if $\pi_s \neq \langle X \rightarrow Y \rangle$. A path is a walk with no two identical vertices.*

Definition 14 (Blocked Walk in SCGs, adaptation of Definition 10)**.** *In a SCG $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$, a walk $\pi_s = \langle V^1, \dots, V^n \rangle$ is said to be* blocked *by a set of vertices $\mathcal{Z}_s \subseteq \mathcal{V}_s$ if:*

1. *$\exists 1 < i < n$ such that $V^{i-1} \leftarrow V^i$ or $V^i \rightarrow V^{i+1}$ and $V^i \in \mathcal{Z}_s$, or*
2. *$\exists 1 < i \leq j < n$ such that $V^{i-1} \rightarrow V^i \rightleftarrows \dots \rightleftarrows V^j \leftarrow V^{j+1}$ and $Desc(V^i, \mathcal{G}_s) \cap \mathcal{Z}_s = \emptyset$.*

A walk which is not blocked is said to be active*. When the set $\mathcal{Z}_s$ is not specified, it is implicit that we consider $\mathcal{Z}_s = \emptyset$. In the case of condition 1, we say that π_s is manually $\mathcal{Z}_s$-blocked by V^i and in the case of condition 2 we say that π_s is passively $\mathcal{Z}_s$-blocked by $\{V^k | i \leq k \leq j\}$.*

As SCGs represent FTCGs, the walks in a SCG can represent the paths of compatible FTCGs. This is gives rise to the notion of compatible walk.

Definition 15 (Compatible Walk)**.** *Let $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ be a FTCG and $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ the compatible SCG. A path $\pi_f = \langle V^1_{t_1}, \dots, V^n_{t_n} \rangle$ in $\mathcal{G}_f$ can be uniquely represented as a walk $\pi_s = \langle V^1, \dots, V^n \rangle$ in $\mathcal{G}_s$ in which the temporal information has been removed. We refer to π_s as π_f's* compatible walk *and we write $\pi_s = \phi(\pi_f)$. e.g., $\phi(\langle X_{t-1}, X_t, Y_t, Z_t, Z_{t+1}, X_{t+1} \rangle) = \langle X, X, Y, Z, Z, X \rangle$.*

Furthermore, since there exists an infinite number of walks in a SCG, it is hard in practice to say verify anything about walks. Therefore, it is necessary to have a notion which creates a link between walks and paths of a SCG. This is the purpose of the following notion of primary path.

Definition 16 (Primary path)**.** *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be a SCG and $\pi_s = \langle V^1, \dots, V^n \rangle$ a walk from X to Y. $\pi'_s = \langle U^1, \dots, U^m \rangle$ such that $U^1 = V^1$ and $U^{k+1} = V^{max\{i | V^i = U^k\}+1}$ is called the primary path of π_s.*

Lastly, in the following, we write

$$t_{min}(\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n} \rangle) = min\{t^i | 1 \leq i \leq n\}$$

and

$$t_{max}(\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n} \rangle) = max\{t^i | 1 \leq i \leq n\}.$$

Proofs

In this section, we prove the identifiability result. We start by stating a trivial lemma that will be needed for the general identifiability result.

Lemma 1. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be a SCG, $\gamma_{max} \geq 0$ a maximum lag and $\alpha_{Y_t, X_{t-\gamma_{xy}}}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X, Y \in \mathcal{V}_s$, $X \neq Y$ and $0 \leq \gamma_{xy} \leq \gamma_{max}$. If $X \notin Par(Y, \mathcal{G}_s)$ then $\alpha_{Y_t, X_{t-\gamma_{xy}}}$ is identifiable.*

Proof. Suppose $X \notin Par(Y, \mathcal{G}_s)$. Then $X_{t-\gamma_{xy}} \notin Par(Y_t, \mathcal{G}_f)$ and the direct effect is equal to zero (*i.e.*, $\alpha_{Y_t, X_{t-\gamma_{xy}}} = 0$). □

The aim of this work is to identify direct effects that can be estimated from data. This implies that we are interested in figuring out if for a given $X_{t-\gamma_{xy}}$ and a given Y_t, it is possible to find at least some *finite* adjustment set that removes all confounding bias and non-direct effects and that does not create any selection bias between $X_{t-\gamma_{xy}}$ and Y_t. Therefore, in the following lemma, we show that what we are trying to achieve is possible by pointing out that infinite sets are not necessary to block paths between $X_{t-\gamma_{xy}}$ and Y_t in a given FTCG.

Lemma 2. *Let $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ be a FTCG of maximal lag at most $\gamma_{max} \geq 0$ and $\alpha_{Y_t, X_{t-\gamma_{xy}}}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X_{t-\gamma_{xy}}, Y_t \in \mathcal{V}_f$, $X \neq Y$ and $0 \leq \gamma_{xy} \leq \gamma_{max}$. Let $\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n} \rangle$ a path from $X_{t-\gamma_{xy}}$ to Y_t in $\mathcal{G}_f$. If $t_{max}(\pi_f) > t$ then π_f is passively blocked by any $\mathcal{Z}_f \subseteq \mathcal{V}_f$ such that $\mathcal{Z}_f \cap \{V_{t'} \in \mathcal{V}_f | t' > t\} = \emptyset$. If $t_{min}(\pi_f) < t - \gamma_{xy}$ then π_f is manually blocked by any $\mathcal{Z}_f \subseteq \mathcal{V}_f$ such that $\{V_{t'} \in \mathcal{V}_f | t - \gamma_{max} \leq t' < t\} \backslash \{X_{t-\gamma_{xy}}\} \subseteq \mathcal{Z}_f$.*

Proof. Let $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ be a FTCG, $X_{t-\gamma_{xy}} \neq Y_t \in \mathcal{V}_f$ and $\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n} \rangle$ a path from $X_{t-\gamma_{xy}}$ to Y_t in $\mathcal{G}_f$.

- Suppose $t_{max}(\pi_f) > t$. Since $t - \gamma_{xy} \leq t < t_{max}(\pi_f)$ there exists $1 < i \leq j < n$ such that $t^{i-1} < t^i$, $t^j > t^{j+1}$ and $\forall i \leq k \leq j, t^k = t_{max}(\pi_f)$. Therefore, $V^{i-1}_{t^{i-1}} \rightarrow V^i_{t^i}$ and $V^j_{t^j} \leftarrow V^{j+1}_{t^{j+1}}$ in π_f. Thus, $\exists i \leq k \leq j$ such that $V^{k-1}_{t^{k-1}} \rightarrow V^k_{t^k} \leftarrow V^{k+1}_{t^{k+1}}$ in π_f and $t^k = t_{max}(\pi_f) > t$. In conclusion, π_f is passively blocked by any $\mathcal{Z}_f \subseteq \mathcal{V}_f$ such that $Desc(V^k_{t^k}, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$ so by any $\mathcal{Z}_f$ such that $\mathcal{Z}_f \cap \{V_{t'} \in \mathcal{V}_f | t' > t\} = \emptyset$
- Suppose $t_{min}(\pi_f) < t - \gamma_{xy}$. Since $t_{min}(\pi_f) < t - \gamma_{xy} \leq t$ and $t^n = t$ there exists $1 < i < n$ such that $t^i < t \leq t^{i+1}$. Therefore, $V^i_{t^i} \rightarrow V^{i+1}_{t^{i+1}}$ in π_f and $t - \gamma_{max} \leq t^i < t \leq t^{i+1}$ so $V^i_{t^i} \in \{V_{t'} \in \mathcal{V}_f | t - \gamma_{max} \leq t' < t\} \backslash \{X_{t-\gamma_{xy}}\}$. In conclusion, π_f is manually blocked by any $\mathcal{Z}_f \subseteq \mathcal{V}_f$ such that $V^i_{t^i} \in \mathcal{Z}_f$ so by any $\mathcal{Z}_f$ such that $\{V_{t'} \in \mathcal{V}_f | t - \gamma_{max} \leq t' < t\} \backslash \{X_{t-\gamma_{xy}}\} \subseteq \mathcal{Z}_f$.

□

Lemma 1 and Lemma 2 respectively show that the case where $X \notin Par(Y, \mathcal{G}_s)$ is trivially identifiable and that any path π_f where $t_{min}(\pi_f) < t - \gamma_{xy}$ and $t_{max}(\pi_f) > t$ are easily blocked by $\mathcal{Z}_f$ as defined in Definition 11. Thus we will consider $X \in Par(Y, \mathcal{G}_s)$ and $t - \gamma_{xy} \leq t_{min}(\pi_f) \leq t_{max}(\pi_f) \leq t$ in the following Lemmas.

In these cases, one might think that to block all active non-direct paths between $X_{t-\gamma_{xy}}$ and Y_t, it is simply sufficient to adjust on all vertices in the FTCG which do not temporally succeed the effect Y_t and have compatible vertices on an active path between X and Y in the SCGs. However, in general this is not and this is given by the following lemma.

Lemma 3. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be a SCG, $\gamma_{max} \geq 0$ a maximum lag and $\alpha_{Y_t, X_{t-\gamma_{xy}}}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X, Y \in \mathcal{V}_s$, $X \neq Y$ and $0 \leq \gamma_{xy} \leq \gamma_{max}$. For every non-direct walk $\pi_s = \langle V^1, \ldots, V^n \rangle$ between X and Y such that $\langle V^2, \ldots, V^{n-1} \rangle \not\subseteq Desc(Y, \mathcal{G}_s)$[2], every compatible path $\pi_f \in \phi^{-1}(\pi_s)$ from $X_{t-\gamma_{xy}}$ to Y_t can be blocked by the adjustment set $\mathcal{Z}_f = \mathcal{A}_{\leq t} \cup \mathcal{D}_{<t}$ defined in Definition 11.*

Proof. Suppose there exists a non-direct walk $\pi_s = \langle V^1, \ldots, V^n \rangle$ between X and Y such that $\langle V^2, \ldots, V^{n-1} \rangle \not\subseteq Desc(Y, \mathcal{G}_s)$. Then, take $\pi_f = \langle V^1_{t^1}, \ldots, V^n_{t^n} \rangle$ from $X_{t-\gamma_{xy}}$ to Y_t compatible with π_s (*i.e.*, $\pi_f \in \phi^{-1}(\pi_s)$). Take $j = max\{1 < i < n | V^i \notin Desc(Y, \mathcal{G}_s)\}$. Notice that $V^j \notin Desc(Y, \mathcal{G}_s)$ and $V^{j+1} \in Desc(Y, \mathcal{G}_s)$ so $V^j_{t^j} \rightarrow V^{j+1}_{t^{j+1}} \in \pi_f$. Therefore, since $t - \gamma_{xy} \leq t_{min}(\pi_f) \leq t_{max}(\pi_f) \leq t$ by Lemma 2, $t - \gamma_{xy} \leq t^j \leq t$ and thus π_f is manually blocked by $V^j_{t^j} \in \mathcal{A}_{\leq t} \subseteq \mathcal{Z}_f$. □

One important factor that we did not take into account in these first lemmas is the value of γ_{xy}. Indeed, when $\gamma_{xy} > 0$ it is safe to say that the problem should become easier as in this case we know that the parents of $X_{t-\gamma_{xy}}$ cannot be descendants of Y_t. Therefore, distinguishing the case where $\gamma_{xy} = 0$ and the case where $\gamma_{xy} = 0$ is important to reach a general identifiability result.

Lemma 4. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be a SCG, $\gamma_{max} \geq 0$ a maximum lag and $\alpha_{Y_t, X_{t-\gamma_{xy}}}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X, Y \in \mathcal{V}_s$, $X \neq Y$ and $0 < \gamma_{xy} \leq \gamma_{max}$. For every non-direct walk $\pi_s = \langle V^1, \ldots, V^n \rangle$ from X to Y such that $\exists 1 \leq i < n$, $V^i \leftarrow V^{i+1}$ (i.e., not $V^i \rightarrow V^{i+1}$ and not $V^i \rightleftarrows V^{i+1}$), every compatible path $\pi_f \in \phi^{-1}(\pi_s)$ from $X_{t-\gamma_{xy}}$ to Y_t can be blocked by $\mathcal{Z}_f = \mathcal{A}_{\leq t} \cup \mathcal{D}_{<t}$ because $\mathcal{Z}_f \cap \mathcal{D}_{\geq t} = \emptyset$ where:*

- *$\mathcal{A}_{\leq t}$, $\mathcal{D}_{<t}$ are defined in Definition 11 and*
- *$\mathcal{D}_{\geq t}$ is the set of instants of descendants of Y greater or equal to t, i.e., $\mathcal{D}_{\geq t} = \{V_{t'} | V \in Desc(Y, \mathcal{G}_s),\ t' \geq t\}$.*

Note that $\pi_s = \langle V^1, \ldots, V^n \rangle$ from X to Y is non-direct, $X \in Par(Y, \mathcal{G}_s)$ by Lemma 1 and $\exists 1 \leq i < n$, $V^i \leftarrow V^{i+1}$ implies that $n \geq 3$.

Proof. Let $\gamma_{xy} > 0$ and $\pi_s = \langle V^1, \ldots, V^n \rangle$ be a non-direct walk between X and Y such that $\exists 1 \leq i < n$, $V^i \leftarrow V^{i+1}$. Then, take a path $\pi_f = \langle V^1_{t^1}, \ldots, V^n_{t^n} \rangle$ from $X_{t-\gamma_{xy}}$ to Y_t compatible with π_s (*i.e.*, $\pi_f \in \phi^{-1}(\pi_s)$) such that $t_{min}(\pi_f) \geq t - \gamma_{xy}$ and $t_{max}(\pi_f) \leq t$ by Lemma 2 and take $1 \leq i < n$ such that $V^i_{t^i} \leftarrow V^{i+1}_{t^{i+1}}$. If $\langle V^2, \ldots, V^n \rangle \not\subseteq Desc(Y, \mathcal{G}_s)$ then Lemma 3 states that π_f is $\mathcal{Z}_f$-blocked. Therefore, on can suppose $\langle V^2, \ldots, V^n \rangle \subseteq Desc(Y, \mathcal{G}_s)$ and in particular $V^{i+1} \in Desc(Y, \mathcal{G}_s)$. If $t^{i+1} < t$ then $V^{i+1}_{t^{i+1}} \in \mathcal{D}_{<t} \subseteq \mathcal{Z}_f$ and π_f is manually blocked by $V^{i+1}_{t^{i+1}} \in \mathcal{Z}_f$. Else, $V^{i+1}_{t^{i+1}} \in \mathcal{D}_{\geq t}$. Since $\gamma_{xy} > 0$, one can take $j = max\{1 < j \leq i | V^{j-1}_{t^{j-1}} \rightarrow V^j_{t^j}\}$. Notice that $V^j_{t^j}$ is a collider and that $V^j_{t^j} \in Desc(V^i_{t^i}, \mathcal{G}_f)$ so $V^j_{t^j} \in \mathcal{D}_{\geq t}$ and $Desc(V^j_{t^j}, \mathcal{G}_s) \subseteq \mathcal{D}_{\geq t}$. Therefore, π_f is passively blocked by $V^j_{t^j} \in \mathcal{Z}_f$. □

Now we give the complementary of Lemma 4 in the case where $X \rightleftarrows Y$.

Lemma 5. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be a SCG, $\gamma_{max} \geq 0$ a maximum lag and $\alpha_{Y_t, X_{t-\gamma_{xy}}}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X, Y \in \mathcal{V}_s$, $X \neq Y$ and $0 < \gamma_{xy} \leq \gamma_{max}$. For every walk $\pi_s = \langle V^1, \ldots, V^n \rangle$ from X to Y where $\exists 1 < i < n$, $V^i = Y$, every compatible path $\pi_f \in \phi^{-1}(\pi_s)$ from $X_{t-\gamma_{xy}}$ to Y_t can be blocked by $\mathcal{Z}_f = \mathcal{A}_{\leq t} \cup \mathcal{D}_{<t}$ (notice $\mathcal{Z}_f \cap \mathcal{D}_{\geq t} = \emptyset$) where $\mathcal{A}_{\leq t}$, $\mathcal{D}_{<t}$ are defined in Definition 11 and $\mathcal{D}_{\geq t}$ is defined in Lemma 4.*

Proof. Let $\pi_s = \langle V^1, \ldots, V^n \rangle$ be a walk between X and Y as described. Then, take $\pi_f = \langle V^1_{t^1}, \ldots, V^n_{t^n} \rangle$ from $X_{t-\gamma_{xy}}$ to Y_t compatible with π_s (*i.e.*, $\pi_f \in \phi^{-1}(\pi_s)$) and $1 < i < n$ such that $V^i = Y$. Since π_f is a path, $t^i \neq t$ so by Lemma 2, $t - \gamma_{xy} \leq t^i < t$. If $\leftarrow V^i_{t^i}$ or $V^i_{t^i} \rightarrow$ then π_f is manually blocked by $V^i_{t^i} \in \mathcal{D}_{<t} \subseteq \mathcal{Z}_f$. If $\rightarrow V^i_{t^i} \leftarrow V^{i+1}_{t^{i+1}}$ then $t^{i+1} \leq t^i < t$ and π_f is a path so $V^{i+1}_{t^{i+1}} \neq X_{t-\gamma_{xy}}$. Therefore, π_f is manually blocked by $V^{i+1}_{t^{i+1}} \in \mathcal{Z}_f$. □

Since Lemmas 3,4 and 5 consider walks in SCGs but it is computationally easier to consider paths in SCGs, in the following we provide a list of properties to reconcile these two notions.

Let π_s be a walk and π'_s its primary path. They verify the following properties.

Property 1. *If π'_s is passively blocked by U^i then π_s is passively blocked by at least a descendant of U^i.*

Proof. Suppose π'_s is passively blocked by U^i then there exists $i_1 \leq i \leq i_2$ such that $\pi'_s = \langle \cdots \rightarrow U^{i_1} \rightleftarrows \cdots \rightleftarrows U^i \rightleftarrows \cdots \rightleftarrows U^{i_2} \leftarrow \ldots \rangle$. Thus there exists $j_1 \leq j \leq j_2$ such that $V^{j_1} = U^{i_1}$, $V^j = U^i$, $V^{j_2} = U^{i_2}$ and $\pi_s = \langle \cdots \rightarrow V^{j_1} \ldots V^{j_2} \leftarrow \ldots \rangle$. Let $k_1 = max\{j_1 \leq k \leq j | V^{k-1} \rightarrow V^k\}$ and $k_2 = min\{j \leq k \leq j_2 | V^k \leftarrow V^{k+1}\}$. Notice that $V^{k_1}, V^{k_2} \in Desc(V^j, \mathcal{G}_s)$ and since $V^j = U^i$, this corresponds to $V^{k_1}, V^{k_2} \in Desc(U^i, \mathcal{G}_s)$. Moreover, at least one of V^{k_1} and V^{k_2} is a collider in π_s. Indeed, if all edges between V^{k_1} and V^{k_2} are $\rightleftarrows$ then both V^{k_1} and V^{k_2} are colliders. If $\exists k_1 \leq k < k_2$ such that $V^k \rightarrow V^{k+1}$ then by definition of k_1, one can deduct that $k \geq j$ and V^{k_2} is a collider. Similarly, if $\exists k_1 < k \leq k_2$ such that $V^{k-1} \leftarrow V^k$ then by definition of k_2, one can deduct that $k \leq j$ and V^{k_1} is a collider. Thus, π_s is passively blocked by at least a descendant of U^i. □

[2]For simplification, we sometimes abuse the notation of walks. Here $\langle V^2, \ldots, V^{n-1} \rangle = \{V^i | 2 \leq i \leq n-1\}$ can be empty if $n \leq 2$.

Property 2. *If $m = 2$ then either $n = 2$ or $\exists 1 < i < n$ such that $V^i = X$ or $V^i = Y$.*

Proof. If $m = 2$ then $V^{max\{i|V^i=V^1\}+1} = V^n$. Firstly, if $max\{i|V^i = V^1\} + 1 = n$ then either $n = 2$ or $i = max\{i|V^i = V^1\}$ verifies $1 < i < n$ and $V^i = V^1 = X$. Secondly, if $max\{i|V^i = V^1\} + 1 < n$ then $i = max\{i|V^i = V^1\} + 1$ verifies $1 < i < n$ and $V^i = V^n = Y$. $\square$

Property 3. *If $\langle U^2, \dots, U^{m-1}\rangle \not\subseteq Desc(Y, \mathcal{G}_s)$ then $\langle V^2, \dots, V^{n-1}\rangle \not\subseteq Desc(Y, \mathcal{G}_s)$.*

Proof. Since $\{U^1, \dots, U^m\} \subseteq \{V^1, \dots, V^n\}$, $\langle U^2, \dots, U^{m-1}\rangle \not\subseteq Desc(Y, \mathcal{G}_s) \implies \langle V^2, \dots, V^{n-1}\rangle \not\subseteq Desc(Y, \mathcal{G}_s)$. $\square$

Property 4. *If $\exists 1 \leq i < m,\ U^i \leftarrow U^{i+1}$ then $\exists 1 \leq i < n,\ V^i \leftarrow V^{i+1}$.*

Proof. $\forall 1 \leq i < m,\ \exists 1 \leq j < n$ such that $V^j = U^i$ and $V^{j+1} = U^{i+1}$, therefore $\exists 1 \leq i < m,\ U^i \leftarrow U^{i+1} \implies \exists 1 \leq j < n,\ V^j \leftarrow V^{j+1}$. $\square$

Lemma 1 deals with the trivial case of identifiability. Lemma 2 shows that every path outside of the time slices of $X_{t-\gamma_{xy}}$ and Y_t are easily blocked. Property 1 shows that passively blocked paths are not problematic for identification. Lemma 3 together with Property 3 states that we will always be able to block paths in which some vertices are not descendants of Y. Lemmas 4 and 5 together with Properties 2 and 4 show that in the case of positive lag (*i.e.*, $\gamma_{xy} > 0$) we can use this temporal information to block other specific paths. Together, these lemmas give a set of sufficient conditions for the direct effect to be identifiable. These conditions are in fact necessary and sufficient. This is summarized in Theorem 1.

Theorem 1. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be an SCG that represents a linear dynamic SCM verifying Assumptions 1,2,3, $\gamma_{max} \geq 0$ a maximum lag and $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ the direct effect of $X_{t-\gamma_{xy}}$ on Y_t such that $X, Y \in \mathcal{V}_s$ and $X \neq Y$. The direct effect $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ is non-identifiable from $\mathcal{G}_s$ if and only if $X \in Par(Y, \mathcal{G}_s)$ and one of the following conditions holds:*

1. *$X \in Desc(Y, \mathcal{G}_s)$ and $\exists C \in Cycles(X, \mathcal{G}_s)$ with $Y \notin C$, or*
2. *There exists an active non-direct path $\pi_s = \langle V^1, \dots, V^n\rangle$ from X to Y in $\mathcal{G}_s$ such that $\langle V^2, \dots, V^{n-1}\rangle \subseteq Desc(Y, \mathcal{G}_s)$, and one of the following conditions holds:*
   - *(a) $\gamma_{xy} = 0$, or*
   - *(b) $\gamma_{xy} > 0$, $n \geq 3$ and $\nexists 1 \leq i < n,\ V^i \leftarrow V^{i+1}$ (i.e., $\forall i,\ V^i \rightarrow V^{i+1}$ or $V^i \rightleftarrows V^{i+1}$).*

Proof. Firstly, let us prove the backward implication. Let $\gamma_{xy} \geq 0$ and $X \in Par(Y, \mathcal{G}_s)$.

- Suppose Condition 1 holds. Let C be a cycle on X with $Y \notin C$ and since $X \in Par(Y, \mathcal{G}_s)$, $\pi_s = C + Y = \langle V^1, \dots, V^n\rangle$ is a walk in $\mathcal{G}_s$. If $\gamma_{xy} > 0$ then $\pi_f = \langle V^1_{t-\gamma_{xy}}, V^2_t, \dots, V^n_t\rangle$ is a path that exists in a compatible FTCG and every set $\mathcal{Z}_f$ that blocks this path contains a descendant of Y_t in another compatible FTCG. If $\gamma_{xy} = 0$, then because $X \in Desc(Y, \mathcal{G}_s)$ one can take $\langle V^n, \dots, V^1\rangle$ a directed path from Y to X and $\pi_f = \langle V^1_t, \dots, V^n_t\rangle$ is a path that exists in a compatible FTCG and every set $\mathcal{Z}_f$ that blocks this path contains a descendant of Y_t in another compatible FTCG.
- Suppose Condition 2 holds. Let $\pi_s = \langle V^1, \dots, V^n\rangle$ as described. If $\pi_s = X \rightleftarrows Y$ and $\gamma_{xy} = 0$, then there exists a compatible FTCG $\mathcal{G}_f$ in which the path $\pi_f = X_t \leftarrow Y_t$ exists and cannot be blocked. Else, $n \geq 3$ and π_s is active so there exists a compatible FTCG $\mathcal{G}^1_f$ in which the path $\pi_f = \langle V^1_{t-\gamma_{xy}}, V^2_t, \dots, V^{n-1}_t, V^n_t\rangle$ exists and is active. Notice that there exists another compatible FTCG $\mathcal{G}^2_f$ in which $\{V^2_t, \dots, V^{n-1}_t\} \subseteq Desc(Y_t, \mathcal{G}^2_f)$ and every set $\mathcal{Z}_f$ that blocks π_f in $\mathcal{G}^1_f$ contains a vertex in $Desc(Y_t, \mathcal{G}^2_f)$.

Lemma 1 gives the first trivial condition $X \notin Par(Y, \mathcal{G}_s) \implies \alpha_{Y_t, X_{t-\gamma_{xy}}}$ identifiable, so we assume in the remaining of the proof $X \in Par(Y, \mathcal{G}_s)$. To prove the forward implication it suffices to prove that if we suppose that $\mathcal{G}_s$ does not verify any condition of Theorem 1, then there exists an adjustment set that:

1. does not contain any descendant of Y_t in any FTCG that is compatible with $\mathcal{G}_s$ nor $X_{t-\gamma_{xy}}$, and
2. blocks every non-direct path from $X_{t-\gamma_{xy}}$ to Y_t in every FTCG that is compatible with $\mathcal{G}_s$.

Consider $\mathcal{Z}_f$ as defined in Definition 11. By construction $\mathcal{Z}_f$ does not contain any descendant of Y_t nor $X_{t-\gamma_{xy}}$. To show the second point, let us consider $\pi_f = \langle V^1_{t^1}, \dots, V^n_{t^n}\rangle$ to be a path from $X_{t-\gamma_{xy}}$ to Y_t in a FTCG $\mathcal{G}_f$ compatible with $\mathcal{G}_s$. Let $\phi(\pi_f) = \pi_s = \langle V^1, \dots, V^n\rangle$ be its compatible walk in $\mathcal{G}_s$ and $\pi'_s = \langle U^1, \dots, U^m\rangle$ the primary path of π_s. In the following, we exhaustively explore the possible characteristics of π'_s and show how in every case either it is either direct, or $\mathcal{Z}_f$-blocked or one of the conditions of Theorem 1 is verified.

Firstly, suppose $m = 2$. According to Property 2 this implies either $n = 2$ or $\exists 1 < i < n$ such that $V^i = Y$ or $\exists 1 < i < n$ such that $V^i = X$.

- If $n = 2$, then either π_s is direct and so is π_f or $\pi_s = X \rightleftarrows Y$. In this second case, if $\gamma_{xy} = 0$ then Condition 2a of Theorem 1 is verified and if $\gamma_{xy} > 0$ then π_f is direct as a path $X_{t-\gamma_{xy}} \leftarrow Y_t$ is not possible.
- If $\exists 1 < i < n$ such that $V^i = Y$, then, if $\gamma_{xy} > 0$, Lemma 5 shows that π_f is $\mathcal{Z}_f$-blocked. Moreover, since π_f is a path, one knows that $t^i \neq t$, and if $\gamma_{xy} = 0$, this forces $t_{max}(\pi_f) > t$ or $t_{min}(\pi_f) < t - \gamma_{xy}$ in which case Lemma 2 shows that π_f is $\mathcal{Z}_f$-blocked.

- If $\forall 1 < i < n, V^i \neq Y$ and $\exists 1 < i < n$ such that $V^i = X$, since π_f is a path, one knows that $t^i \neq t - \gamma_{xy}$, and if $\gamma_{xy} = 0$, this forces $t_{max}(\pi_f) > t$ or $t_{min}(\pi_f) < t - \gamma_{xy}$ in which case Lemma 2 shows that π_f is $\mathcal{Z}_f$-blocked. Additionally, if $\gamma_{xy} > 0$, Lemma 3 shows that if $V^i = X \notin Desc(Y, \mathcal{G}_s)$ then π_f is $\mathcal{Z}_f$-blocked and Lemma 4 shows that if $\exists 1 \leq j < n$ such that $V^j \leftarrow V^{j+1}$ then π_f is $\mathcal{Z}_f$-blocked. However, if $\forall 1 \leq j < n$, $V^j \rightarrow V^{j+1}$ or $V^j \rightleftarrows V^{j+1}$, then $C = \langle V^1, \ldots, V^i \rangle \in Cycles(X, \mathcal{G}_s)$ and $Y \notin C$ and together with $X \in Desc(Y, \mathcal{G}_s)$ this verifies Condition 1 of Theorem 1.

Now, suppose $m \geq 3$.

- If $\exists 1 < i < m$ such that $U^i \notin Desc(Y, \mathcal{G}_s)$. Then, Property 3 shows that $\exists 1 < j < n$ such that $V^j \notin Desc(Y, \mathcal{G}_s)$. In this case, Lemma 3 shows that π_f is $\mathcal{Z}_f$-blocked.
- If π'_s is not active, then it is passively blocked by U^i for $1 < i < m$. One can assume $U^i \in Desc(Y, \mathcal{G}_s)$ as the case $U^i \notin Desc(Y, \mathcal{G}_s)$ was previously treated. Then, Property 3 shows that π_s is passively blocked by $V^j \in Desc(U^i, \mathcal{G}_s) \subseteq Desc(Y, \mathcal{G}_s)$ for $1 < j < n$. If $\gamma_{xy} = 0$, then, either $t^j \neq t$ and Lemma 2 shows that π_f is $\mathcal{Z}_f$-blocked, either $t^j = t$ and thus $V^j_{t^j} \in \mathcal{D}_{\geq t}$ and π_f contains a collider $V^k_{t^k} \in Desc(V^j_{t^j}, \mathcal{G}_f) \subseteq \mathcal{D}_{\geq t}$ so π_f is $\mathcal{Z}_f$-blocked. If $\gamma_{xy} > 0$, then since π_s is passively blocked by $V^j \in Desc(U^i, \mathcal{G}_s) \subseteq Desc(Y, \mathcal{G}_s)$, there $\exists j \leq k$ such that $V^k \leftarrow V^{k+1}$ in π_s in which case Lemma 4 shows that π_f is $\mathcal{Z}_f$-blocked.
- If no previous condition is verified and $\gamma_{xy} = 0$ then Condition 2a of Theorem 1 is verified and if $\gamma_{xy} > 0$ and $\nexists 1 \leq i < m$ such that $U^i \leftarrow U^{i+1}$ in π'_s then Condition 2b of Theorem 1 is verified. Therefore, the last case remaining is when $\gamma_{xy} > 0$ and $\exists 1 \leq i < m$ such that $U^i \leftarrow U^{i+1}$ in π'_s. Property 4 shows that, in this case, $\exists 1 \leq j < n$ such that $V^j \leftarrow V^{j+1}$ in π_s. According to Lemma 4, π_f is $\mathcal{Z}_f$-blocked.

□

Proposition 1. *Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be an SCG, $\gamma_{max} \geq 0$ a maximum lag. Consider two vertices X and Y such that $X \in Par(Y, \mathcal{G}_s)$ and $\alpha_{X_{t-\gamma_{xy}}, Y_t}$ is identifiable following Theorem 1. Then $\mathcal{Z}_f^{Def\ 12}$ as defined in Definition 12 is a valid adjustment set for $\alpha_{X_{t-\gamma_{xy}}, Y_t}$.*

Proof. Let $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ be a SCG, $X, Y \in \mathcal{V}_s$ with $X \in Par(Y, \mathcal{G}_s)$ and γ_{xy} be a lag. Suppose the direct effect of $X_{t-\gamma_{xy}}$ on Y_t is identifiable following Theorem 1. Let $\mathcal{Z}_f$ be the adjustment set relative to $(X_{t-\gamma_{xy}}, Y_t)$ defined in Definition 12. Let $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ be a compatible FTCG of maximal lag at most γ_{max} compatible with $\mathcal{G}_s$.

- Firstly, using the decomposition $\mathcal{Z}_f = \mathcal{D}^{Anc(Y)}_{<t} \cup \mathcal{A}^{Anc(Y)}_{\leq t}$ as in Definition 12, and because $Desc(Y_t, \mathcal{G}_f) \subseteq \{V_{t'} | V \in Desc(Y, \mathcal{G}_s),\ t' \geq t\}$ it is clear that $\mathcal{Z}_f \cap (Desc(Y_t, \mathcal{G}_f) \cup \{X_{t-\gamma_{xy}}\}) = \emptyset$.
- Secondly, let $\pi_f = \langle V^1_{t^1}, \ldots, V^n_{t^n} \rangle$ be a non-direct path from $X_{t-\gamma_{xy}}$ to Y_t in FTCG $\mathcal{G}_f$ and $\pi_s = \langle V^1, \ldots, V^n \rangle = \phi(\pi_f)$ its compatible walk. If $t_{max}(\pi_f) > t$ then $\exists 1 < k < n$ such that $\rightarrow V^k_{t_{max}(\pi_f)} \leftarrow$ in π_f with $Desc(V^k_{t_{max}(\pi_f)}, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$ and thus π_f is passively blocked by $\mathcal{Z}_f$. Therefore, for the following we can suppose $t_{max}(\pi_f) \leq t$. Because the direct effect of $X_{t-\gamma_{xy}}$ on Y_t is identifiable following Theorem 1 we know that $\pi_f \neq \langle V^1_{t^1} \leftarrow \cdots \leftarrow V^n_{t^n} \rangle$. Therefore, $\exists k_{max} = max\{1 < k \leq n | V^{k-1}_{t^{k-1}} \rightarrow V^k_{t^k}\}$ with $t^{k_{max}} \geq t$ and $V^{k_{max}} \in Desc(Y, \mathcal{G}_s)$ and since π_f is non-direct this forces $n \geq 3$.
  - If $k_{max} < n$ then $\rightarrow V^{k_{max}}_{t^{k_{max}}} \leftarrow$ and $Desc(V^{k_{max}}_{t^{k_{max}}}, \mathcal{G}_f) \cap \mathcal{Z}_f = \emptyset$ so π_f is passively blocked by $\mathcal{Z}_f$.
  - If $k_{max} = n$ (*i.e.*, $V^{n-1}_{t^{n-1}} \rightarrow V^n_{t^n}$) and $\pi_f = \langle V^1_{t^1} \rightarrow \cdots \rightarrow V^n_{t^n} \rangle$ then because the direct effect of $X_{t-\gamma_{xy}}$ on Y_t is identifiable following Theorem 1 we know that $\exists d_{max} = \max\{1 < d < n | V^d \notin Desc(Y, \mathcal{G}_s)\}$ and $t - \gamma_{max} \leq t^{d_{max}} \leq t$, so since $V^{d_{max}} \in Anc(Y, \mathcal{G}_s) \backslash Desc(Y, \mathcal{G}_s)$, we have $V^{d_{max}}_{t^{d_{max}}} \in \mathcal{A}^{Anc(Y)}_{\leq t}$ and thus π_f is manually blocked by $\mathcal{Z}_f$.
  - If $k_{max} = n$ (*i.e.*, $V^{n-1}_{t^{n-1}} \rightarrow V^n_{t^n}$) and $\exists l_{max} = max\{1 < l < n | V^{l-1}_{t^{l-1}} \leftarrow V^l_{t^l}\}$ then $\langle V^{l_{max}}, \ldots, V^n \rangle \subseteq Anc(Y, \mathcal{G}_s)$
    * If $t^{l_{max}} < t$ then $\exists l_{max} \leq i$ such that $V^i_{t^i} \rightarrow V^{i+1}_{t^{i+1}}$ and $t - \gamma_{max} \leq t^i < t^{i+1} = t$ and since $V^i \in Anc(Y, \mathcal{G}_s)$, $V^i_{t^i} \in \mathcal{Z}_f$ and thus π_f is manually blocked by $\mathcal{Z}_f$.
    * If $V^{l_{max}} \notin Desc(Y, \mathcal{G}_s)$ and $t^{l_{max}} = t$ then, since $V^{l_{max}} \in Anc(Y, \mathcal{G}_s)$, $V^{l_{max}}_{t^{l_{max}}} \in \mathcal{A}^{Anc(Y)}_{\leq t}$ and thus π_f is manually blocked by $\mathcal{Z}_f$.
    * If $V^{l_{max}} \in Desc(Y, \mathcal{G}_s)$ and $t^{l_{max}} = t$ and $\exists r_{max} = max\{1 < r < l_{max} | V^{r-1}_{t^{r-1}} \rightarrow V^r_{t^r}\}$, then notice that $\langle \rightarrow V^{r_{max}}_{t^{r_{max}}} \leftarrow \cdots \leftarrow V^{l_{max}}_{t^{l_{max}}} \rangle$ forces $V^{r_{max}} \in Desc(Y, \mathcal{G}_s)$ and $t^{r_{max}} = t$ so π_f is passively blocked by $\mathcal{Z}_f$.
    * If $V^{l_{max}} \in Desc(Y, \mathcal{G}_s)$ and $t^{l_{max}} = t$ and $\nexists 1 < r < l_{max}$ such that $V^{r-1}_{t^{r-1}} \rightarrow V^r_{t^r}$, then $\pi_f = \langle X_{t-\gamma_{xy}} \leftarrow \cdots \leftarrow V^{l_{max}}_{t^{l_{max}}} \rightarrow \cdots \rightarrow Y_t \rangle$. Since $t^{l_{max}} = t$ and $t_{max}(\pi_f) \leq t$, $\forall 1 \leq i \leq n$, $t^i = t$ and in particular $\gamma_{xy} = 0$ and thus because π_f is a path, $\pi_s = \langle V^1, \ldots, V^n \rangle$ is also a path. Moreover, since $V^{l_{max}} \in Desc(Y, \mathcal{G}_s)$, $\forall 1 \leq i \leq n$, $V^i \in Desc(Y, \mathcal{G}_s)$. Therefore, π_s verifies Condition 2a which is impossible.

In conclusion, π_f is blocked by $\mathcal{Z}_f$.

□

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