Quick and dirty introduction to perverse sheaves

This is a very quick introduction to the definition of perverse sheaves, that also tries to give some motivation (coming from the theory of D-modules). These were the notes of a lecture in a semester-long seminar about mixed Hodge modules, so it makes reference to other lectures for motivation.

In this talk, X will be a variety (separated scheme of finite type) over $\mathbb{C}$, we will most often identify with its set of complex points and we will use the usual topology on $X(\mathbb{C})$. Note however that you can define étale torsion or ℓ-adic perverse sheaves on varieties over any field of characteristic $\neq \ell$ (see the book [1] of Beilinson-Bernstein-Deligne for this and most of the results of this note). More generally, once we have a theory of ℓ-adic sheaves over more general bases (see Ekedahl's article [3] and the article [2] of Bhatt-Scholze), it is possible to use Gabber's results (see [4] and the notes of the Gabber working seminar in [6]) to define a category of perverse sheaves in a very general setting.

1 Motivation

We have seen in one of Sam's talks that, if X is smooth, then there is an equivalence between the category of $\mathbb{C}$-local systems (i.e., locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces) of X and pairs $(\mathscr{E}, \nabla)$, where $\mathscr{E}$ is a vector bundle on X and ∇ is an integrable (flat) connection on $\mathscr{E}$. (This equivalence sends a pair $(\mathscr{E}, \nabla)$ to the sheaf $\mathscr{E}^\nabla$ and it sends a local system L to the vector bundle $L \otimes_{\mathbb{C}} \mathscr{O}_X$ with the connection given by the usual connection on $\mathscr{O}_X$.)

This is nice because it allows us to define the notion of a $\mathbb{Q}$-structure (or a $\mathbb{R}$-structure) on a pair $(\mathscr{E}, \nabla)$: it is simply the data of a $\mathbb{Q}$-local system L on X and of an isomorphism $L \otimes_{\mathbb{Q}} \mathscr{O}_X \simeq (\mathscr{E}, \nabla)$. Remember that we use this when we define variations of Hodge structures on X.

In Sam's other talks we have learned about things called D-modules which generalize pairs $(\mathscr{E}, \nabla)$. They form an abelian category $D-\mathrm{mod}$, with a subcategory $D-\mathrm{hmod}$ of holonomic

D-modules that has a lot of good properties. For example, $D-\mathrm{hmod}$ is Artinian and Noetherian, we know what its simple objects look like, it contains all the D-modules whose underlying $\mathscr{O}_X$-module is a vector bundle (in particular, all our $(\mathscr{E}, \nabla)$ from above), it is stable by D-module version of Verdier duality, and we have the 4 operations f_*, f^*, $f_!$, $f^!$ (by which I mean the triangulated forms) between the categories $D^b(D-\mathrm{hmod})$. In fact, one of the points of restricting ourselves to holonomic D-modules was that, for general D-modules, only $f^!$ and f_* are defined.

We have only talked about D-modules on smooth varieties, but that was just to simplify the exposition. The theory extends to general varieties. (If X admits a closed embedding into a smooth variety Y, consider D_Y-modules with support in X. In general, choose an covering of X by opens that embed into smooth varieties, glue and check that the result does not depend on the covering or the embeddings.)

Remember that the goal of this seminar is to define (and study) mixed Hodge modules, which generalize variations of Hodge structures. If holonomic D-modules generalize the $(\mathscr{E}, \nabla)$ side, we need something that will generalize the $\mathbb{Q}$-local system side, plus a generalization of the equivalence of categories above, so that we can talk about $\mathbb{Q}$-structures on holonomic D-modules. This is exactly what perverse sheaves and the Riemann-Hilbert correspondence will do : once we define the category $\mathrm{Perv}(X, \mathbb{C})$ of perverse sheaves on X with coefficients in $\mathbb{C}$, we'll have an equivalence of categories between $\mathrm{Perv}(X, \mathbb{C})$ and the category of regular holonomic D-modules on X, due to Mebkhout and Kashiwara (independently), often called the Riemann-Hilbert correspondence, and extending the correspondence between $\mathbb{C}$-local systems and vector bundles with flat connection if X is smooth.

2 Complexes of sheaves

The situation is a bit different from the D-module side, as we won't have to find our good abelian category in a huge abelian category of D-modules, but rather in a triangulated category.

As we want a theory that will work with coefficients $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, let's just fix a field F that will serve as coefficients.

The first observation is that F-local systems are special sheaves of F-vector spaces. For any variety X, denote by $\mathrm{Sh}(X, F)$ the category of sheaves of F-vector spaces on X. This is an abelian category and it has enough injectives, but it is too big for our purposes : as we'll play with left and right derived functors (and with things that are not even derived functors), we'll want to be in the bounded derived category $D^b(\mathrm{Sh}(X, F))$, but it is not clear that, say, the (derived) direct image functor always preserves $D^b(\mathrm{Sh}(X, F))$. So let's impose finiteness conditions.

Definition 2.1 We say that a sheaf $\mathscr{F} \in \mathrm{Sh}(X, F)$ is *constructible* if there exists a stratification of X such that the restriction of $\mathscr{F}$ to each stratum is a local system. (For our purposes, a *stratification* of X will just be a finite partition of X by locally closed (algebraic) subvarieties, called *strata*, such that the closure of each stratum is a union of strata. Without loss of generality, we can (and we will) assume that all the strata are smooth and connected.)

We say that a complex $K \in D(\mathrm{Sh}(X, F))$ is constructible if $\mathrm{H}^i K$ is constructible for every $i \in \mathbb{Z}$, and 0 for almost every i.

We denote by $D^b_c(X, F)$ the category of constructible complexes of sheaves of F-vector spaces on X. It is a triangulated subcategory of $D(\mathrm{Sh}(X, F))$ (because the category of constructible sheaves is an abelian subcategory of $\mathrm{Sh}(X, F)$ that is stable by extensions).

If $\mathscr{S}$ is a stratification of X, we denote by $D^b_{\mathscr{S}}(X, F)$ the full subcategory of $D^b_c(X, F)$ of complexes whose cohomology sheaves are local systems on all the strata of $\mathscr{S}$.

The miracle (far from trivial or obvious a priori) is that all the usual sheaf operations preserve $D^b_c(X, F)$. Let's review them. Let $f : X \to Y$ be a morphism.

- We derive the Hom and tensor product functors in the usual way, to get bifunctors $R\,\underline{\mathrm{Hom}}$ and $\otimes^L$ on $D^b_c(X, F)$. If X_1 and X_2 are two varieties, we also have an exterior tensor product functor $\boxtimes : D^b_c(X_1, F) \boxtimes D^b_x(X_2, F) \to D^b_c(X_1 \times X_2, F)$, that is obtained by trivially deriving the exact exterior tensor product functor on sheaves.
- We have a direct image functor $f_* : \mathrm{Sh}(X, F) \to \mathrm{Sh}(Y, F)$ that I hope you all know how to define. It is left exact and can be derived to give a triangulated functor $D^+(\mathrm{Sh}(X, F)) \to D^+(\mathrm{Sh}(Y, F))$, that preserves constructible complexes. From now on, we'll use the notation f_* for the functor from $D^b_c(X, F)$ to $D^b_c(Y, F)$. If I want to talk about the functor on the level of sheaves, I'll call it ${}^0 f_*$.
- The functor ${}^0 f_*$ has a left adjoint f^*, the inverse image functor. This one is exact, so it's easy to derive it, and we'll still write f^* for the derived functor.
- Then there is the direct image with proper support functor $f_! : \mathrm{Sh}(X, F) \to \mathrm{Sh}(Y, F)$. (As we are in the topological case, it is easier to define : for every $\mathscr{F} \in \mathrm{Sh}(X, F)$ and every open $U \subset Y$, $f_!\mathscr{F}(U) = \{s \in \mathscr{F}(f^{-1}(U)) |\ \text{the restriction of } f \text{ to } \operatorname{supp} s \text{ is proper}\}$.) It is left exact, and its right derived functor preserves constructible complexes. As before, we'll reserve the notation $f_!$ for the functor from $D^b_c(X, F)$ to $D^b_c(Y, F)$, and use ${}^0 f_!$ for the functor between categories of sheaves.
- The functor $f_! : D^b_c(X, F) \to D^b_c(Y, F)$ (derived version !) admits a right adjoint, usually denoted $f^!$ and called exceptional inverse image functor. This functor $f^!$ is not in general a derived functor.
- We define the *dualizing complex* on X by $a_X^! F$, where a_X is the obvious map from X to $\operatorname{Spec} \mathbb{C}$. Then the Verdier duality functor sends $K \in D^b_c(X, F)$ to $D(K) := R\,\underline{\mathrm{Hom}}(K, a_X^! F)$. It is an anti-involution, and we have canonical isomorphisms $D \circ f_* \simeq f_! \circ D$ and $D \circ f^* \simeq f^! \circ D$.

A few particular cases to get a feel for these functors (especially $f_!$ and $f^!$) :

- If f is proper, then $f_* = f_!$. If moreover f is finite (eg a closed immersion), then f_* is exact.

- If f is smooth of relative dimension d (say X and Y are connected to simplify), then $f^! = f^*[2d](d)$. In particular, if f is étale (eg an open embedding), then $f^! = f^*$. Also, if X is smooth of dimension d, then the dualizing complex on X is just $F_X[2d](d)$ (where F_X, or just F, is the constant sheaf with fiber F on X).

- If f is an open embedding, then $f_!$ is the functor "extension by 0".

- If f is a closed embedding, then $f^!$ is the right derived functor of a functor ${}^0f^! : \mathrm{Sh}(Y, F) \to \mathrm{Sh}(X, F)$, and ${}^0f^!$ is the functor "sections with support in X".

In conclusion, if f is an open embedding or a closed embedding, then we have a handle on the functors $f_!$ and $f^!$. In general, if we want to calculate $f_!$, then we write $f = g \circ j$ where j is an open embedding and g is proper (this is always possible), and if we want to calculate $f^!$, then we try to write $f = h \circ i$ where i is a closed embedding and h is smooth (this always possible locally on the source of f).

3 The perverse t-structure

Let's come back to the original problem. If X is smooth, then, inside of $D_c^b(X, F)$, we have the abelian category of F-local systems on X. We want to extend this category to some Artinian and Noetherian abelian category whose simple objects we understand, stable by duality, and on which we have some version of the 4 operations.

The most obvious idea is to just take the category of constructible sheaves on X. Unfortunately, it is neither Artinian nor stable by duality :

(1) Take $X = \mathbb{A}^1$, let $(z_n)_{n\in\mathbb{N}}$ be a sequence of distinct points of X and, for every $n \in \mathbb{N}$, let j_n be the open immersion $X - \{z_0, \dots, z_n\} \to X$. Then $(j_{n!}F)_{n\in\mathbb{N}}$ is an infinite decreasing sequence of constructible subsheaves of F_X, and it does not stabilize.

(2) Let j be the open immersion of $\mathbb{G}_m$ into $\mathbb{A}^1$, and let $\mathscr{F} = j_! F_{\mathbb{G}_m}$. Then $D(\mathscr{F}) = j_* D(F_{\mathbb{G}_m}) = j_* F_{\mathbb{G}_m}[2](1)$ is not concentrated in one degree, so, even ignoring the problems with shifts, it cannot be a sheaf.

Actually, I wrote about ignoring the problems with shifts above, but this is exactly what we should not do. Take the first example above. For each n, there is an exact triangle

$$j_{n!}F \to F_X \to \mathscr{F}_n \xrightarrow{+1},$$

where $\mathscr{F}_n$ is a direct sum of skyscraper sheaves, and what makes the map $j_{n!}F \to F_X$ injective is the fact that F_X and the skyscraper sheaves are allowed to live in the same degree. In the second example, there is an exact triangle

$$j_! F_{\mathbb{G}_m} \to F_{\mathbb{A}^1} \to F_{\{0\}} \xrightarrow{+1},$$

hence an exact triangle

$$D(F_{\{0\}}) = F_{\{0\}} \to D(F_{\mathbb{A}^1}) = F_{\mathbb{A}^1}[2](1) \to D(j_! F_{\mathbb{G}_m}) \xrightarrow{+1},$$

and now the problem is that, though we allow both $F_{\mathbb{A}^1}$ and $F_{\{0\}}$ to be sheaves, their duals are not in the same degree.

More generally, if X is smooth of dimension d and $\mathscr{F}$ is a local system on X, then $D(\mathscr{F}) = \mathscr{F}^\vee[2d](d)$, where $\mathscr{F}^\vee$ is the dual local system. This is not a sheaf and we cannot ignore this problem if we want to work with things more general than local systems. Here is the solution we will adopt : in our abelian category, we will accept local systems placed in degree $-d$ instead of local systems (because the dual of a local system placed in degree $-d$ is again a local system placed in degree $-d$). More generally, if Z is a smooth subvariety of X of dimension e, we will accept local systems on Z placed in degree $-e$. For example, in the first example above, the sum of skyscraper sheaves $\mathscr{F}_n$ would still be an object of our category, but the constant sheaf F_X would now be concentrated in degree 1, so $j_{n!}F$ would also have to be concentrated in degree 1, and the map $j_{n!}F[1] \to F_X[1]$ would be surjective.

This is nice and vague, but there is a technical problem : We are trying to define an abelian subcategory of the triangulated category $D^b_c(X, F)$. How do we do that concretely ?

The answer is called a *t-structure*, it's a machine designed specifically for that usage (constructing abelian subcategories of triangulated categories).

In general, let $\mathscr{T}$ be a triangulated category (think $D^b_c(X, F)$). A t-structure on $\mathscr{T}$ is a couple $(D^{\leq 0}, D^{\geq 0})$ of full subcategories of $\mathscr{T}$ that is trying to imitate $(D^{\leq 0}_c(X, F), D^{\geq 0}_c(X, F))$, where

$$D^{\leq 0}_c(X, F) = \{K \in D^b_c(X, F) | \mathrm{H}^i K = 0 \text{ for } i > 0\}$$
$$D^{\geq 0}_c(X, F) = \{K \in D^b_c(X, F) | \mathrm{H}^i K = 0 \text{ for } i < 0\}.$$

So we ask that $(D^{\leq 0}, D^{\geq 0})$ satisfy the following properties :

(a) For every $A \in D^{\leq 0}$ and $B \in D^{\geq 0}$, $\mathrm{Ext}^{-1}(A, B)(:= \mathrm{Hom}(A, B[-1])) = 0$.

(b) $D^{\leq 0} \subset D^{\leq 0}[-1]$ and $D^{\geq 0} \supset D^{\geq 0}[-1]$.

(c) For every $X \in \mathscr{T}$, there exists an exact triangle $A \to X \to B \xrightarrow{+1}$ with $A \in \mathrm{D}^{\leq 0}$ and $B \in D^{\geq 0}[-1]$.

We write (in analogy with our example) $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$.

It turns out that these properties are enough to ensure the following (the proof is an intricate but elementary series of diagram chases; see [1] 1.3 for the solution or try it yourself) :

(i) The inclusion functor $D^{\leq n} \to \mathscr{T}$ (resp. $D^{\geq n} \to \mathscr{T}$) admits a right (resp. left) adjoint, that we will denote $\tau_{\leq n}$ (resp. $\tau_{\geq n}$). For a given $X \in \mathscr{T}$, the exact triangle in (c) is unique up to unique isomorphism, and isomorphic to $\tau_{\leq 0} X \to X \to \tau_{\geq 1} X \xrightarrow{+1}$, where the first two maps are adjunction maps.

(ii) The full subcategory $\mathscr{C} = D^{\leq 0} \cap D^{\geq 0}$ is an abelian category, and it is stable by extensions in $\mathscr{T}$ (i.e., if $A \to B \to C \xrightarrow{+1}$ is an exact triangle with A and C in $\mathscr{C}$, then B is also in $\mathscr{C}$). We call it the heart of the t-structure. In our example, it is simply the category of constructible sheaves on X.

(iii) The functor $\mathrm{H}^n = \tau_{\leq n}\tau_{\geq n}[n] = \tau_{\geq n}\tau_{\leq n}[n]$ sends $\mathscr{T}$ to $\mathscr{C}$, and it is a cohomological functor.

(iv) We say that the t-structure is non-degenerate if $\bigcap D^{\leq n} = \bigcap D^{\geq n} = \{0\}$. In that case, then the family $(\mathrm{H}^n)_{n\in\mathbb{Z}}$ is a system of conservative functors on $\mathscr{T}$ (i.e. a morphism in $\mathscr{T}$ is 0 if and only if its images by all the H^n are 0), and an object X of $\mathscr{T}$ is in $D^{\leq n}$ (resp. $D^{\geq n}$) if and only if $\mathrm{H}^i X = 0$ for every $i > n$ (resp. $i < n$).

(v) If $\mathscr{T}$ admits a filtered version (this is made precise in section 3.1 of [1][1]), then the inclusion $\mathscr{C} \subset \mathscr{T}$ extends to a triangulated functor $D^b(\mathscr{C}) \to \mathscr{T}$. (This is almost always the case in practice, and will be the case for perverse sheaves.)

We'll apply this to $\mathscr{T} = D^b_c(X, F)$. To try to apply the remarks above, let's define $D^{\leq 0}$ in the following way : Let $K \in D^b_c(X, F)$, and let $\mathscr{S}$ be a stratification of X such that $K \in D^b_{\mathscr{S}}(X, F)$. Then K is in $D^{\leq 0}$ if and only, for each stratum $i : S \to X$ of $\mathscr{S}$, i^*K is concentrated in degree $\leq -\dim S$. (Remember that we are assuming that each S is smooth and connected.)

Note that then the following conditions are equivalent :

(1) K is in $D^{\leq 0}$.

(2) For every $k \in \mathbb{Z}$, $\dim \operatorname{supp} \mathrm{H}^i K \leq -i$.

(3) For every (schematic) point $s \xrightarrow{i_s} X$ of X, $i_s^* K$ is concentrated in degree $\leq -\dim \overline{\{s\}}$.

In particular, the condition defining $D^{\leq 0}$ is independent of the choice of $\mathscr{S}$.

Now for $D^{\geq 0}$. We want the resulting abelian category to be stable by Verdier duality, so the only thing we can do is to set $D^{\geq 0} = D(D^{\leq 0})$. In other words, if $K \in D^b_c(X, F)$, then the following conditions are equivalent :

(1) K is in $D^{\geq 0}$.

(2) $D(K)$ is in $D^{\leq 0}$.

(3) For every stratification $\mathscr{S}$ of X such that $K \in D^b_{\mathscr{S}}(X, F)$, for every stratum $i : S \to X$ of $\mathscr{S}$, $i^! K$ is concentrated in degree $\geq -\dim(S)$.

(4) Condition (3) for just one $\mathscr{S}$.

(5) For any (schematic) point $s \xrightarrow{i_s} X$ of X, $i_s^! K$ is concentrated in degree $\geq -\dim \overline{\{s\}}$.

Theorem 3.1 *$(D^{\leq 0}, D^{\geq 0})$ is a t-structure on $D^b_c(X, F)$.*

[1]It would probably be better to use the concept of dg enhancement, that was not available at the time of [1].

Let's sketch the proof. Condition (b) is obvious. Condition (a) is also easy to check : Let $K \in D^{\leq 0}$ and $L \in D^{\geq 1}$. For every point $s \xrightarrow{i_s} X$ of X, the complex $i_s^* K$ is concentrated in degree $\leq -\dim \overline{\{s\}}$ and the complex $i_s^! L$ is concentrated in degree $\geq 1 - \dim \overline{\{s\}}$, so that

$$i_s^* R \underline{\mathrm{Hom}}(K, L) = R \underline{\mathrm{Hom}}(i_s^* K, i_s^! L)$$

is concentrated in degree ≥ 1. In particular, $\mathrm{Hom}(K, L) = 0$.

Condition (c) is the hardest. First notice that $D_c^b(X, F) = \bigcup_{\mathscr{S}} D_{\mathscr{S}}^b(X, F)$. This stays true if we only consider stratifications $\mathscr{S}$ that satisfy the following condition (eg Whitney stratifications) : For every strata $i_S : S \to X$ and $i_T : T \to X$ such that $T \subset \overline{S}$, for every local system $\mathscr{F}$ on S, the cohomology sheaves of $i_T^* i_{S*} \mathscr{F}$ are local systems on T.

Suppose that $\mathscr{S}$ is a stratification satisfying the above condition. We show that a complex $K \in D_{\mathscr{S}}^b(X, F)$ satisfies condition (c) by induction on the number of strata. It's obvious for one stratum, and the induction is an intricate diagram chase (that uses the octaedron axiom).

The category of perverse sheaves on X is $\mathrm{Perv}(X, F) := D^{\leq 0} \cap D^{\geq 0}$. We write ${}^p\mathrm{H}^k : D_c^b(X, F) \to \mathrm{Perv}(X, F)$ for the cohomology functors given by the t-structure.

Why do we call them sheaves when they are obviously complexes of sheaves ? It is because of the following property (proved in proposition 3.2.2 and théorème 3.2.4 of [1]) :

Proposition 3.2 *The categories of perverse sheaves on open subsets of X form a stack.*

In other words, we can define perverse sheaves locally, just like ordinary sheaves.

Examples 3.3

- Skyscraper sheaves are perverse.
- If $i : Z \to X$ is a closed immersion with Z smooth of dimension d and $\mathscr{F}$ is a local system on Z, then $i_* \mathscr{F}[d] \in \mathrm{Perv}(X, F)$.
- Let $j : \mathbb{G}_m \to \mathbb{A}^1$ be the embedding, and let $\mathscr{F}$ be a local system on $\mathbb{G}_m$. Then the complexes $j_! \mathscr{F}[1]$, $({}^0 j_* \mathscr{F})[1]$ and $j_* \mathscr{F}[1]$ are perverse.

4 Exactness properties of the 4 operations

Definition 4.1 If $T : \mathscr{T}_1 \to \mathscr{T}_2$ is a triangulated functor between triangulated categories equipped with t-structures, then we say that T is left (resp. right) t-exact if $T(D^{\geq 0}) \subset D^{\geq 0}$ (resp. $T(D^{\leq 0}) \subset D^{\leq 0}$). We say that T is t-exact if it is both left and right t-exact.

From now on, we will always use the perverse t-structure on the categories $D_c^b(X, F)$. Here is a summary of the t-exactness properties of the usual sheaf operations (they are all proved in section 4 of [1]) :

1. The Verdier duality functor $D : D^b_c(X,F)^{op} \to D^b_c(X,F)$ is t-exact. ([1] 4.0.)
2. If $f : X \to Y$ is an affine morphisme, then $f_* : D^b_c(X,F) \to D^b_c(Y,F)$ is right t-exact and $f_! : D^b_c(X,F) \to D^b_c(Y,F)$ is left t-exact. ([1] théorème 4.1.1 and corollaire 4.1.2.) This is a reformulation of the theorem on the cohomological dimension of affine schemes of SGA 4 XIV.
3. If the dimension of the fibers of $f : X \to Y$ is $\leq d$, then $f_!$ and f^* are of (perverse) cohomological amplitude $\leq d$, and $f^!$ and f_* are of (perverse) cohomological amplitude $\geq -d$. ([1] 4.2.4)
4. The exterior product functor $\boxtimes$ is t-exact. ([1] proposition 4.2.8.)

This implies easily the following properties :

1. If f is a quasi-finite morphism (for example, a locally closed embedding), then $f_!$ is right t-exact and f_* is left t-exact.
2. If j is an affine open embedding, then $j_!$, j_* and $j^* = j^!$ are t-exact.
3. If i is a finite morphism, then $i_! = i_*$ is t-exact, i^* is right t-exact and $i^!$ is left t-exact.
4. If f is smooth of relative dimension d, then $f^*[d] = f^!(-d)[-d]$ is t-exact. In particular, if f is étale, then $f^! = f^*$ is t-exact; if f is finite étale, then $f_* = f_!$ is also t-exact.
5. If f is proper and the dimension of its fibers is $\leq d$, then $f_! = f_*$ is of cohomological amplitude $[-d, d]$.
6. The derived tensor product $\otimes^L$ is left t-exact.
7. The derived internal Hom functor $R\underline{\mathrm{Hom}}$ is left t-exact, by which we mean that it sends $D^{\leq 0} \times D^{\geq 0}$ to $D^{\geq 0}$.

Remark 4.2 We can also show that the nearby and vanishing cycles functors are t-exact when suitably normalized (see corollaires 4.5 and 4.6 of Illusie's [5]).

5 Intermediate extension and simple objects

Definition 5.1 Let $j : Z \to X$ be a locally closed immersion. For every $K \in \mathrm{Perv}(Z,F)$, we set :

$$j_{!*}K = \mathrm{Im}({}^p\mathrm{H}^0 j_! K \to {}^p\mathrm{H}^0 j_* K).$$

This defines a functor $j_{!*} : \mathrm{Perv}(Z,F) \to \mathrm{Perv}(X,F)$, called the *intermediate extension functor*.

Note that, by the results of previous section, $j_!$ is right t-exact and j_* is left t-exact. But the functor $j_{!*}$ is neither left nor right t-exact in general. (It preserves injective and surjective maps, but is not exact "in the middle".)

We can now give the description of the simple objects of $\mathrm{Perv}(X, F)$. (See section 4.3 of [1] for a proof.)

Theorem 5.2 *(i) If $K \in \mathrm{Perv}(X, F)$ is simple and $j : Z \to X$ is an open embedding with everywhere dense image, then we have a canonical isomorphism $K = j_{!*}j^*K$.*

(ii) If $j : Z \to X$ is a locally closed embedding and $K \in \mathrm{Perv}(Z)$, then K is simple if and only if $j_{!*}K$ is simple.

(iii) The category $\mathrm{Perv}(X, F)$ is Artinian and Noetherian, and its simple objects are all of the form $j_{!*}(\mathscr{F}[d])$, where $j : Z \to X$ is a locally closed embedding, Z is smooth of dimension d and $\mathscr{F}$ is an irreducible local system on Z.

Note that by (i), we can replace the couple $(Z, \mathscr{F})$ in (iii) by any $(U, \mathscr{F}_{|U})$, where $U \subset Z$ is open dense.

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