Mixed ℓ-adic complexes for schemes over number fields

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If X is a variety over a number field, Annette Huber has defined in [14] a category of "horizontal" (or "almost everywhere unramified") ℓ-adic complexes and ℓ-adic perverse sheaves on X. For such objects, the notion of weights makes sense (in the sense of Deligne, see [9]), just as in the case of varieties over finite fields. However, contrary to what happens in that last case, mixed perverse sheaves (or mixed locally constant sheaves) on X do not have a weight filtration in general, even when X is a point. The goal of this paper is to show how to avoid this problem by working directly in the derived category of the abelian category of perverse sheaves that do admit a weight filtration. As an application, the methods of [21] to calculate the intermediate extension of a pure perverse sheaf apply over any finitely generated field, and not just over a finite field.

[1]

1 Introduction

Let k be a field of finite type over its prime subfield, let X be a separated scheme of finite type over k, and let ℓ be a prime number invertible in k. In her article [14] Annette Huber introduced a category $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X) = D^b_m(X, E)$ of mixed horizontal ℓ-adic sheaves on X, where E is an algebraic extension of $\mathbb{Q}_\ell$. The idea of [14] is to consider the category of ℓ-adic complexes on X that extend to a constructible ℓ-adic complex on a model $\mathscr{X}$ of X over a normal scheme $\mathscr{U}$ of finite type over $\mathbb{Z}$ and with field of fractions k; we also want the morphisms between complexes to extend to $\mathscr{X}$. There is a natural definition of weights (in the sense of Deligne's [9]) on such complexes, by considering their restriction to the fibers of $\mathscr{X}$ over closed points of $\mathscr{U}$. So we have a notion of pure sheaves, and mixed complexes are defined (as in [9]) as those complexes whose cohomology sheaves have a filtration with pure quotients.

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By sections 2 and 3 of [14], the 6 operations (usual and exceptional direct and inverse images, tensor products and internal Homs) exist on these categories of complexes. Moreover, it is shown in 2.5 and 3.2 of [14] that the category $D^b_m(X)$ has a (self-dual) perverse t-structure, whose heart $\mathrm{Perv}_m(X)$ is called the category of horizontal mixed perverse sheaves on X.

Also, the results of chapters 4 and 5 of Beilinson-Bernstein-Deligne's book [6] about the t-exactness (or perverse cohomological amplitude) of the 6 functors, and the way these 6 functors affect weights, can be extended to our situation thanks to Deligne's generic base change theorem (SGA 4 1/2 [Th. finitude] section 2), see for example 3.4 and 3.5 of [14].

Finally, there is a notion of weight filtration on an object of $\mathrm{Perv}_m(X)$ (see [14] 3.7); it is an increasing filtration whose quotients are pure perverse sheaves of increasing weights. This filtration is unique if it exists ([14] 3.8), but unfortunately it doesn't always exist, unless k is a finite field. As noted in the remark below [14] 3.8, the category of horizontal mixed perverse sheaves on X admitting a weight filtration is a full abelian subcategory $\mathrm{Perv}_{mf}(X)$ of $\mathrm{Perv}_m(X)$ which is stable by subquotients, but it is not stable by extensions.

As a consequence, if we start from a horizontal mixed perverse sheaf that does have a weight filtration and apply some sheaf operations, then it is not clear that the perverse cohomology sheaves of the resulting mixed complex will still have weight filtrations. (Although we would certainly expect that to be the case.) For example, this is a problem if we want to generalize the arguments of [21], that gives among other things a formula for the intersection complex of X.

The goal of this paper is to give a solution to this problem, inspired by Beilinson's theorem that, if k is a finite field, then the derived category of $\mathrm{Perv}_m(X)$ is canonically equivalent to $D^b_m(X)$ (see [4], [3]; note that Beilinson's result is more general). Beilinson also gives a way to reconstruct the derived direct image functors from their perverse versions, and formulas adapted to perverse sheaves for the unipotent nearby and vanishing cycles functors. Building on this, Morihiko Saito has shown in [23] and [24] how to recover the other operations (inverse images, tensor products and internal Homs) using only perverse sheaves.

In this paper, we will follow the ideas of Beilinson and M. Saito to construct all the sheaf operations on the bounded derived categories of the categories $\mathrm{Perv}_{mf}(X)$. The main point, which is taken as an axiom in [24], is the fact that these categories are stable by perverse direct images; in section 6.3, we show how to deduce it from Deligne's weight-monodromy theorem. Another difficulty is to state all the compatibilities that the sheaf operations should satisfy. We have chosen to use the formalism of crossed functors ("foncteurs croisés"), originally due to Deligne and developed by Voevodsky and Ayoub. In order to check that the constructions of Beilinson and M. Saito do fit into this formalism, we have had to rewrite some of them. (Another reason is that the categories $\mathrm{Perv}_{mf}(X)$ satisfy assumptions that are slightly different from the axioms of [24], and so certain proofs become simpler, and at least one proof has to be totally changed. However, most of the constructions are very similar to the ones in [24].)

Here is a quick description of the different parts of the paper. Section 1 is the introduction, and section 2 contains reminders about ℓ-adic perverse sheaves, the realization functor and a quick summary of the beginning of Huber's article [14], in particular the definition of the main object of study $\mathrm{Perv}_{mf}(X)$. In section 3, we state the main results of the paper, first informally and then using the language of crossed functors. Section 4 gives a list of functors that obviously preserve the categories $\mathrm{Perv}_{mf}(X)$. Section 5 contains reminders about Beilinson's construction of unipotent nearby and vanishing cycles. In section 6, we state the form of Deligne's weight-monodromy theorem that we will use, and deduce the crucial fact that perverse direct images also preserve the categories $\mathrm{Perv}_{mf}(X)$; we also give an application to complexes with support in a closed subscheme, that was already noted in 2.2.1 of [4] and Theorem 5.6 of [24]. Section 7 gives the proof of the first main theorem (Theorem 3.2.4, concerning the existence of the four operations $f^*, f_*, f_!, f^!$), and section 8 gives the proof of the second main theorem (Theorem 3.2.12, about the existence of tensor products and internal Homs). Finally, section 9 shows how the results of this article imply that we can extend the formalism of weight truncation functors defined in [21].

Here are some conventions that will be used throughout the paper :

- As we are considering sheaves for the étale topology or proétale topology, we only care about schemes up to universal homeomorphism. So we will allow ourselves to specify a closed subscheme of a scheme X by giving only the underlying closed subset.
- We are mostly interested in the triangulated versions of the sheaf operations, so we will denote them without the usual "R"'s or "L"'s. For example, the derived direct image functors will simply be denoted by f_*, and we will similarly write f^*, $f_!$ and $f^!$ for the other direct and image inverse functors, seen as functors between the triangulated categories of complexes of sheaves. The only exception we will make is for the functor $R\,\mathrm{Hom}$ (in an abelian category), in order to distinguish it from Hom.
- All the schemes will be assumed to be excellent and separated, and all the morphisms will be assumed to be of finite type. (We are only interested in schemes that are of finite type over $\mathbb{Z}$ or over a field, and these schemes are automatically excellent.) If we write "scheme over k", where k is a field, we will mean "separated field of finite type over k". Also, the letter ℓ always stands for a prime number invertible over all the schemes considered.

Contents

2 Horizontal perverse sheaves

In this section, we recall the definition of ℓ-adic constructible complexes and ℓ-adic perverse sheaves, the construction of the realization functor from the bounded derived category of perverse sheaves to the category of constructible complexes, and finally the definition by A. Huber of horizontal ℓ-adic complexes and horizontal perverse sheaves, as well as the mixed versions.

2.1 ℓ-adic complexes

Let X be a scheme and E be an algebraic extension of $\mathbb{Q}_\ell$. If we want to stay in the familar framework of triangulated categories (and avoid ∞-categories), there are two approaches to the category of bounded constructible étale E-complexes on X that work at the level of generality that we need : Ekedahl's approach in [10] (see also Fargues's paper [11] for some complements) and the Bhatt-Scholze definition via the proétale site in [7]. The second works in a more general setting, and it is known to be equivalent to the first when they both apply. While we could make the constructions that we need work with both approaches, we will mostly stick to the Bhatt-Scholze approach, because it makes the homological algebra simpler.

Remark 2.1.1 In his article [10], Ekedahl makes the assumption that the scheme X is of finite type over a regular scheme of dimension ≤ 1. The reason for this is that the necessary theorems for torsion étale sheaves were only available in this setting at the time. Since then, Gabber has proved the finiteness theorem (see Exposé XIII of [18]), the absolute purity theorem (see [12] or III.3 of [18]) and the existence of a dualizing complex (see Exposé XVII of [18]) in the more general setting considered here, so the results of [10] extend to this setting.

Let us review quickly the construction of Bhatt and Scholze via the proétale site $X_{\mathrm{proét}}$ of X (see [7]) : The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ is defined as a the full subcategory of the category $\mathrm{D}(X_{\mathrm{proét}},E)$ of sheaves of E-modules on $X_{\mathrm{proét}}$ (definition 4.1.1 of [7]) whose objects are bounded complexes with constructible cohomology sheaves, where a proétale sheaf $\mathscr{F}$ of E-vector spaces is called *constructible* if X has a finite stratification $(Z_i)_{i\in I}$ by locally closed subschemes such that each $\mathscr{F}_{|Z_i}$ is lisse, i.e. locally (in the proétale topology) free of finite rank; see definitions 6.8.6 and 6.8.8 of [7]. By propositions 5.5.4, 6.6.11, 6.8.11 and 6.8.14 of [7], this category is canonically equivalent to the one defined by Ekedahl if X satisfies condition (A) or (B) of [7] 5.5.1. [2] This point of view is conceptually simpler and has the advantage that direct images, tensor products and internal Homs are the restriction of actual derived functors on the categories $\mathrm{D}(X_{\mathrm{proét}},E)$.

The six operations on the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ (direct and inverses images, direct images with proper support, exceptional inverse images, derived tensor products and derived internal Homs) are constructed in sections 6.7 and 6.8 of [7]. Suppose that we are given a dimension function δ on X (see Définition XVII.2.1.1 of [18]), and let K_X be the corresponding dualizing complex on X. By this, we mean a potential dualizing complex on X for the dimension fucntion δ (see Définition XVII.2.1.2 of [18]); this is known to be unique up to unique isomorphism (Théorème XVII.5.1.1 of [18]) and to be a dualizing complex (Théorème XVII.6.1.1 of [18]). We then denote by $D_X = \underline{\mathrm{Hom}}_X(.,K_X)$ the duality functor defined by K_X; it satisfies all the usual properties, see Lemma 6.7.20 of [7].

The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ has a canonical t-structure, whose heart is the category $\mathrm{Sh}_c(X,E)$ of constructible sheaves (this is automatic if we use definition 6.8.8 of [7] for $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$). This category has a full abelian subcategory stable by extensions $\mathscr{L}(X,E)$, the category of lisse sheaves (or locally constant sheaves, or local systems), see definition 6.8.3 of [7]. We will only use the category $\mathscr{L}(X,E)$ if X is connected regular; in that case (and more generally if X is geometrically unibranch), this category is equivalent to the category of continuous representations of the étale fundamental group $\pi^{\mathrm{ét}}(X)$ of X on finite-dimensional E-vector spaces (see lemmas

[2]Technically, Ekedahl only defines the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\mathscr{O}_E)$ for a finite extension E of $\mathbb{Q}_\ell$, so to be precise, we should say that the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ of Bhatt-Scholze is canonically equivalent to the inverse 2-limit over all finite subextensions of E of the tensor product over $\mathscr{O}_E$ of E and of the category of constructible $\mathscr{O}_E$-complexes defined by Ekedahl.

7.4.7 and 7.4.10 and remark 7.4.8 of [7]; the equivalence is given by taking stalks at a geometric point of X). In particular, if X is smooth of relative dimension d over a field k and if we use the dimension function $\delta : x \longmapsto \dim(\overline{\{x\}})$ (see the beginning of section 2.2), then for every $\mathscr{L} \in \mathrm{Ob}(\mathscr{L}(X,E))$ corresponding to a representation of $\pi^{\text{ét}}(X)$, if we denote by $\mathscr{L}^\vee$ the lisse sheaf corresponding to the dual representation, then $D_X(\mathscr{L}) \simeq \mathscr{L}^\vee(-d)[-2d]$. Indeed, we have $K_X = \underline{E}_X(-d)[-2d]$, hence $\mathrm{H}^0\,\underline{\mathrm{Hom}}_X(\mathscr{L}, K_X) = \mathscr{L}^\vee(-d)[-2d]$, and all the $\mathrm{H}^i\,\underline{\mathrm{Hom}}_X(\mathscr{L}, K_X)$ for $i \geq 1$ vanish by exercise III.1.31 in [20] (and lemma 6.7.13 of [7]).

2.2 Perverse sheaves

In this section, we assume that X satisfies the conditions of Corollaire XIV.2.4.4 of [18] (for example, X is of finite type over $\mathbb{Z}$ or over a field) and we fix the dimension function δ on X defined by $\delta(x) = \dim(\overline{\{x\}})$. As explained in 2.1, it determines a dualizing complex K_X and a duality functor D_X. We define two full subcategories ${}^p\mathrm{D}^{\leq 0}$ and $Dp^{\geq 0}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ by the following formulas :

$$ {}^p\mathrm{D}^{\leq 0} = \{K \in \mathrm{Ob}\,\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E) | \forall x \in X,\ \forall i > -\delta(x),\ \mathrm{H}^i(i_x^* K) = 0\} $$

and

$$ {}^p\mathrm{D}^{\geq 0} = \{K \in \mathrm{Ob}\,\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E) | \forall x \in X,\ \forall i < -\delta(x),\ \mathrm{H}^i(i_x^! K) = 0\}, $$

where, for every point x of X (not necessarily closed), we denote the inclusion $x \longrightarrow X$ by i_x. This is a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ for the same reasons as in [6] 2.2.9-2.2.19 : We consider couples $(\mathscr{S}, \mathscr{L})$, where where $\mathscr{S}$ is a finite stratification of X by locally closed connected regular subschemes, and $\mathscr{L}$ is the data, for each stratum Z of $\mathscr{S}$, of a finite set $\mathscr{L}(Z)$ of lisse sheaves on Z, such that condition (c) of [6] 2.2.9 is satisfied. We denote by $\mathrm{D}_{(\mathscr{S},\mathscr{L})}(X,E)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ whose objects are the complexes K such that, for each stratum Z of $\mathscr{S}$ and each $i \in \mathbb{Z}$, $\mathrm{H}^i K_{|Z}$ is isomorphic to an element of $\mathscr{L}(Z)$. The categories $({}^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$ induces a t-structure on $\mathrm{D}_{(\mathscr{S},\mathscr{L})}(X,E)$ by gluing, as in [6] 1.4. Then we note that the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ is the filtered inductive limit of its subcategories $\mathrm{D}_{(\mathscr{S},\mathscr{L})}(X,E)$, and that the t-structures are compatible thanks to the purity theorem ([12] or XVI.3 of [18]).

We will call the t-structure $({}^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$ the *perverse t-structure* on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$, and denote its heart by $\mathrm{Perv}(X,E)$. This is the category of *perverse sheaves* on X (with coefficients in E). We denote the associated cohomology functor by ${}^p\mathrm{H}^i : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E) \longrightarrow \mathrm{Perv}(X,E)$.

Let us list the exactness properties of the (derived) sheaf operations for this t-structure.

Suppose that we have a flat morphism of finite type $X \longrightarrow S$. The following proposition is an immediate consequence of the definitions.

Proposition 2.2.1 *Let $u : T \longrightarrow S$ be an étale map (resp. the inclusion of the generic point of S), and consider the functor $u^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X \times_S T, E)$.*

Then u^* (resp. $u^*[-\dim S]$) is t-exact.

Then we recall the properties proved in sections 4.1 and 4.2 of [6].

Proposition 2.2.2 *Let $f : X \longrightarrow Y$ be a finite type morphism. Then :*

(i) The functors D_X and D_Y are t-exact.

(ii) If f is affine, then f_* is right t-exact and $f_!$ is left t-exact.

(iii) If the dimension of the fibers of f is $\leq d$, then f_* (resp. $f_!$, resp. f^*, resp. $f^!$) is of perverse cohomological amplitude $\geq -d$ (resp. $\leq d$, resp. $\leq d$, resp. $\geq -d$).

(iv) If f is quasi-finite and affine, then f_* and $f_!$ are t-exact.

(v) If f is smooth of relative dimension d, then $f^! \simeq f^*[2d](d)$, and $f^*[d]$ and $f^![-d]$ are t-exact. In particular, if f is étale, then $f^* = f^!$ is t-exact.

(vi) The external tensor product $\boxtimes : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y, E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X \times Y, E)$ is t-exact.

(vii) The Tate twist functor $K \longmapsto K(1)$ is t-exact.

Remember that the external tensor product of $K \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E)$ and $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y, E)$ is the object $K \boxtimes L$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X \times Y, E)$ defined by

$$K \boxtimes L = (pr_X^* K) \otimes (pr_Y^* L),$$

where $pr_X : X \times Y \longrightarrow X$ and $pr_Y : X \times Y \longrightarrow Y$ are the two projections (see SGA 5 III 1.5, where $K \boxtimes L$ is denoted $K \otimes_{\operatorname{Spec} k} L$).

Proof. For (i), note that, by Theorem 6.3(iii) of [10], for all $K, L \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E)$ and every $x \in X$, we have a canonical isomorphism

$$i_x^! \underline{\operatorname{Hom}}_X(K, L) \simeq \underline{\operatorname{Hom}}_x(i_x^* K, i_x^! L).$$

Applying this to $L = K_X$ and using the isomorphisms $i_x^! K_X \simeq E(\delta(x))[2\delta(x)]$ that are part of the definition of a potential dualizing complex (see Définition XVII.2.1.2 of [18]), we see that $K \in \operatorname{Ob}(Dp^{\leq 0})$ if and only if $D_X(K) \in \operatorname{Ob}({}^p\,\mathrm{D}^{\geq 0})$. As $D_X^2 \simeq \mathrm{id}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)}$, this also implies that $K \in \operatorname{Ob}({}^p\,\mathrm{D}^{\geq 0})$ if and only if $D_X(K) \in \operatorname{Ob}({}^p\,\mathrm{D}^{\leq 0})$, and we are done.

Point (ii) is proved exactly as Théorème 4.1.1 of [6], as soon as we have the analogue of Artin's vanishing theorem, which is proved in Exposé XV of [18]. Point (iii) is proved exactly as 4.2.4 of [6], and (iv) follows from (ii) and (iii). To prove point (v), it suffices to prove the isomorphism $f^! \simeq f^*[2d](d)$; but this is SGA 4 XVIII 3.2.5. Point (vi) is proved as in Proposition 4.2.8 of [6]. Finally, point(vii) follows from (vi), because $K(1)$ is the exterior tensor product of the complex K on X and of the perverse sheaf $\mathbb{Q}_\ell(1)$ on $\operatorname{Spec} k$.

$\square$

We define the intermediate extension functor as in [6] : if $j : U \longrightarrow X$ is a locally closed immersion and K is an object of $\mathrm{Perv}(U, E)$, then

$$j_{!*}K = \mathrm{Im}({}^p\mathrm{H}^0 j_! K \longrightarrow {}^p\mathrm{H}^0 j_* K).$$

The methods of section 4.3 of [6] adapt immediately to our case and give the following result :

Theorem 2.2.3 *The category* $\mathrm{Perv}(X, E)$ *is Artinian and Noetherian, that is, all its objects have finite length. Moreover, the simple objects are of the form* $j_{!*}L[d]$*, where* $j : Z \longrightarrow X$ *is a locally closed immersion, the subscheme* Z *is connected regular of dimension* d*, and* L *is a lisse sheaf on* Z *corresponding to an irreducible representation of* $\pi_1^{\text{ét}}(Z)$ *(we call such a* L *a* simple *lisse sheaf).*

2.3 Filtered derived categories

Before we can define the realization functors, we need to recall the formalism of filtered derived categories. Let $\mathscr{A}$ be an abelian category. We denote by $\mathrm{F}(\mathscr{A})$ the category of filtered objects of $\mathscr{A}$, where filtrations are assumed to be decreasing. The category $\mathrm{F}(\mathscr{A})$ is a quasi-abelian category, and we denote by $\mathrm{DF}(\mathscr{A})$ its derived category (see chapter V of Illusie's [16], section 3.1 of [6] or sections 2 and 3 of Schapira and Schneiders's [25]). We will denote objects of $\mathrm{DF}(\mathscr{A})$ by $(K, F^\bullet)$ (or often just K), where K is a complex of objects of $\mathscr{A}$ and $F^\bullet$ is a decreasing filtration on this complex. We have triangulated functors $\sigma_{\leq i}, \sigma_{\geq i} : \mathrm{DF}(\mathscr{A}) \longrightarrow \mathrm{DF}(\mathscr{A})$, $\mathrm{Gr}^i : \mathrm{DF}(\mathscr{A}) \longrightarrow \mathrm{D}(\mathscr{A})$ for each $i \in \mathbb{Z}$ and $\omega : \mathrm{DF}(\mathscr{A}) \longrightarrow \mathrm{D}(\mathscr{A})$, that are the derived functors of the three functors $\mathrm{F}^i : \mathrm{F}(\mathscr{A}) \longrightarrow \mathrm{F}(\mathscr{A})$, $(.)/\,\mathrm{F}^{i-1} : \mathrm{F}(\mathscr{A}) \longrightarrow \mathrm{F}(\mathscr{A})$, $\mathrm{Gr}^i : \mathrm{F}(\mathscr{A}) \longrightarrow \mathscr{A}$, $(K, \mathrm{F}^\bullet) \longmapsto \mathrm{F}^i K/\,\mathrm{F}^{i-1} K$ and of the forgetful functor $\omega : \mathrm{F}(\mathscr{A}) \longrightarrow \mathscr{A}$, $(K, F^\bullet) \longmapsto K$. For every $i \in \mathbb{Z}$, we have a natural transformation $\mathrm{Gr}^i \longrightarrow \mathrm{Gr}^{i+1}[1]$ (coming from the triangle $\mathrm{Gr}^{i+1} K \longrightarrow F^i K/F^{i+2} K \longrightarrow \mathrm{Gr}^i K$, if $(K, F^\bullet)$ is an object of $\mathrm{DF}(\mathscr{A})$).

In the language of Definition A.1 of [4], the category $\mathrm{DF}(\mathscr{A})$ is a *f-category* over $\mathrm{D}(\mathscr{A})$. This says in particular that $\mathrm{D}(\mathscr{A})$ is equivalent (via ω) to the full subcategory of objects $(K, F^\bullet)$ in $\mathrm{DF}(\mathscr{A})$ such that $\mathrm{Gr}^i K = 0$ for $i \neq 0$.

We review the construction of the realization functor from [6] 3.1. Let $\mathscr{D}$ be a full triangulated subcategory of $\mathrm{D^b}(\mathscr{A})$, let $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ be a t-structure on $\mathscr{D}$, and denote its heart by $\mathscr{C}$. Let DF be the full subcategory of K in $\mathrm{DF}(\mathscr{A})$ such that $\mathrm{Gr}^i K \in \mathrm{Ob}\,\mathscr{D}$ for every $i \in \mathbb{Z}$. This is a triangulated subcategory of $\mathrm{DF}(\mathscr{A})$ stable by the functors $\sigma_{\leq i}$ and $\sigma_{\geq i}$, $i \in \mathbb{Z}$, hence a f-category over $\mathscr{D}$. The t-structure of $\mathscr{D}$ also lifts to a compatible t-structure $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ of DF (see Definition A.4 and Proposition A.5 of [4]), given by

$$\mathrm{DF}^{\leq 0} = \{K \in \mathrm{Ob}\,\mathrm{DF}\,|\forall i \in \mathbb{Z},\ \mathrm{Gr}^i K \in \mathrm{Ob}\,\mathscr{D}^{\leq i}\}$$

and

$$\mathrm{DF}^{\geq 0} = \{K \in \mathrm{Ob}\,\mathrm{DF}\,|\forall i \in \mathbb{Z},\ \mathrm{Gr}^i K \in \mathrm{Ob}\,\mathscr{D}^{\geq i}\}.$$

The heart of this t-structure is the abelian category $\mathrm{DF}_{\text{bête}}$ with objects

$$\{K \in \operatorname{Ob}\mathrm{DF}(\mathscr{A}) | \forall i \in \mathbb{Z},\ \operatorname{Gr}^i K[i] \in \operatorname{Ob}\mathscr{C}\}.$$

It is the category called "$\mathscr{D}F_{\text{bête}}$" in [6] 3.1.7. If $(K, F^\bullet)$ is an object of $\mathrm{DF}_{\text{bête}}$, then the sequence

$$\ldots \longrightarrow \operatorname{Gr}^i K[i] \longrightarrow \operatorname{Gr}^{i+1} K[i+1] \longrightarrow \operatorname{Gr}^{i+2} K[i+2] \longrightarrow \ldots$$

is a bounded complex of objects of $\mathscr{C}$. We get in this way a functor G from $\mathrm{DF}_{\text{bête}}$ to the category $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ of bounded complexes of objects of $\mathscr{C}$.

Theorem 2.3.1 *(Propositions 3.1.6 and 3.1.10 of [6].) The functor G is an equivalence of categories, and the functor $\omega \circ G^{-1} : \mathrm{C}^{\mathrm{b}}(\mathscr{C}) \longrightarrow \mathscr{D}$ factors through $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$.*

So we get a functor $\mathrm{D}^{\mathrm{b}}(\mathscr{C}) \longrightarrow \mathscr{D}$, that we will denote by real and call the *realization functor*.

We denote by $\mathrm{Fct}(\mathbb{Z}, \mathscr{A})$ the category of functors from $\mathbb{Z}$ to $\mathscr{A}$. This is an abelian category, and the category $\mathrm{F}(\mathscr{A})$ can be identified with the full subcategory of $\mathrm{Fct}(\mathbb{Z}, \mathscr{A})$ whose objects are the functors sending morphisms to monomorphisms (see definition 3.1 of [25]). We have the following result :

Theorem 2.3.2 *(Theorem 3.16 of [25].) Assume that $\mathscr{A}$ admits small filtrant inductive limits, and that these limits are exact. Then the inclusion functor induces an equivalence of categories*

$$\mathrm{D}^*(\mathrm{F}(\mathscr{A})) \longrightarrow \mathrm{D}^*(\mathrm{Fct}(\mathbb{Z}, \mathscr{A})),$$

where $* = \varnothing, +, -$ or b.

Corollary 2.3.3 *Let $\mathscr{B}$ be another abelian category and $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a left exact functor. Suppose that $\mathscr{A}$ has enough injectives, or more generally that it has a F-injective subcategory in the sense of Definition 13.3.4 of [19].*

Then we have a functor $RF : \mathrm{DF}(\mathscr{A}) \longrightarrow \mathrm{DF}(\mathscr{B})$ such that the squares

$$\begin{tikzcd} \mathrm{DF}(\mathscr{A}) \arrow[r, "RF"] \arrow[d, "\omega"'] & \mathrm{DF}(\mathscr{B}) \arrow[d, "\omega"] \\ \mathrm{D}(A) \arrow[r, "RF"'] & \mathrm{D}(\mathscr{B}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{DF}(\mathscr{A}) \arrow[r, "RF"] \arrow[d, "\operatorname{Gr}^i"'] & \mathrm{DF}(\mathscr{B}) \arrow[d, "\operatorname{Gr}^i"] \\ \mathrm{D}(\mathscr{A}) \arrow[r, "RF"'] & \mathrm{D}(\mathscr{B}) \end{tikzcd}$$

commute up to natural isomorphism.

Indeed, the functor F induces a left exact functor $\mathrm{Fct}(\mathbb{Z}, \mathscr{A}) \longrightarrow \mathrm{Fct}(\mathbb{Z}, \mathscr{B})$, and the functor RF of the corollary is its right derived functor. Of course, we have a similar statement for right exact functors.

Now suppose that we have two abelian categories $\mathscr{A}$ and $\mathscr{A}'$, subcategories $\mathscr{D}$ resp. $\mathscr{D}'$ of $\mathrm{DF}(\mathscr{A})$ resp. $\mathrm{DF}(\mathscr{A}')$ with t-structures whose hearts we denote by $\mathscr{C}$ and $\mathscr{C}'$. We denote by DF' (resp. $\mathrm{DF}_{\text{bête}}{}'$) the subcategory of $\mathrm{DF}(\mathscr{A}')$ defined similarly to DF (resp. $\mathrm{DF}_{\text{bête}}$). The following proposition is proved in appendix A of [4] .

Proposition 2.3.4 *Let $F : \mathscr{D} \longrightarrow \mathscr{D}'$ be a triangulated functor. Suppose that F is right exact for the t-structures on $\mathscr{D}$ and $\mathscr{D}'$, so that it induces a right exact functor $\mathscr{C} \longrightarrow \mathscr{C}'$, that we will denote by F^0. Suppose that F lifts to a triangulated functor $F' : \mathrm{DF} \longrightarrow \mathrm{DF}'$ such that the squares*

$$\begin{tikzcd} \mathrm{DF} \arrow[r, "F'"] \arrow[d, "\omega"'] & \mathrm{DF}' \arrow[d, "\omega"] \\ \mathscr{D} \arrow[r, "F"'] & \mathscr{D}' \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{DF} \arrow[r, "F'"] \arrow[d, "\mathrm{Gr}^i"'] & \mathrm{DF}' \arrow[d, "\mathrm{Gr}^i"] \\ \mathscr{D} \arrow[r, "F"'] & \mathscr{D}' \end{tikzcd}$$

commute up to natural isomorphisms. Finally, suppose that $\mathscr{C}$ has a F^0-projective subcategory, in the sense of Definition 13.3.4 of [19]. (Informally, this means that there exist enough F^0-acyclic objects.)

Then the left derived functor LF^0 of F^0 exists, and the diagram

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(\mathscr{C}) \arrow[r, "LF^0"] \arrow[d, "\mathrm{real}"'] & \mathrm{D}^{\mathrm{b}}(\mathscr{C}') \arrow[d, "\mathrm{real}"] \\ \mathscr{D} \arrow[r, "F"'] & \mathscr{D}' \end{tikzcd}$$

commutes up to canonical natural isomorphism.

Proof. The existence of LF^0 is proved in Proposition 13.3.5 of [19]. The remark after Lemma A.7.1 of [4] shows that there exists a canonical morphism of functors $F \circ \mathrm{real} \longrightarrow \mathrm{real} \circ LF^0$. That this morphism is an isomorphism follows from the existence of F^0-acyclic resolutions, as in the proof of Theorem 3.2 of [4].

□

2.4 Perverse t-structures and the realization functor

Fix a scheme X as in section 2.2. Then we can apply Theorem 2.3.1 to the triangulated subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E)$ of the derived category $\mathrm{D}(X_{\text{proét}}, E)$ of proétale E-modules on X and its perverse t-structure. It gives an exact functor $\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}(X, E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E)$, that we denote by real and call the *realization functor*.

If $f : X \longrightarrow Y$ is a finite type morphism such that (a shift of) f^* or f_* is t-exact for the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E)$, we would like to know that real intertwines the trivial derived functor on the derived categories of perverse sheaves and the original functor on the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$. We would also like to have similar statements for the exterior tensor product and duality functors. For this, we need to extend all these functors to the appropriate filtered derived categories.

We have triangulated functors f_*, $\otimes$ and $\underline{\mathrm{Hom}}_X$ on $\mathrm{D}(X_{\text{proét}}, E)$, and they are all derived functors, so we can extend them to triangulated functors on $\mathrm{DF}(X_{\text{proét}}, E)$ (the filtered derived category of proétale E-modules on X) using Corollary 2.3.3. [3] Next, as $\mathrm{D}(X_{\text{proét}}, E)$ is equivalent to the full subcategory of $\mathrm{DF}(X_{\text{proét}}, E)$ with objects the K such that $\mathrm{Gr}^i K = 0$ for $i \neq 0$, we can see the dualizing complex $\widehat{K}_X$ as an object of $\mathrm{DF}(X_{\text{proét}}, E)$, and so we can define D_X on $\mathrm{DF}(X_{\text{proét}}, E)$ by $D_X(K) = \underline{\mathrm{Hom}}_X(K, K_X)$. Finally, we extend the inverse image functor. The functor $f_* : \mathrm{D}(X_{\text{proét}}, E) \longrightarrow \mathrm{D}(Y_{\text{proét}}, E)$ has a left adjoint f^*, given by $f^*K = f^*_{\text{naive}}K \otimes_{f^*_{\text{naive}}E_X} E_Y$, where f^*_{naive} is the regular pullback functor (see Remark 6.8.15 of [7]). The functor f^*_{naive} is exact and so extends to $\mathrm{DF}(Y_{\text{proét}}, E)$ by Corollary 2.3.3, and we can see $f^*_{\text{naive}}E_Y$ and E_X as objects of $\mathrm{DF}(X_{\text{proét}}, E)$, so f^* also extends.

We get the following result :

Proposition 2.4.1 *Let $f : X \longrightarrow Y$ be a finite type morphism.*

(i) If f is quasi-finite and affine, then we have commutative diagrams (up to natural isomorphism)

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{Perv}(X,E)) \arrow[r, "f_*"] \arrow[d, "\mathrm{real}"'] & \mathrm{D^b}(\mathrm{Perv}(Y,E)) \arrow[d, "\mathrm{real}"] \\ \mathrm{D^b_c}(X,E) \arrow[r, "f_*"'] & \mathrm{D^b_c}(Y,E) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{D^b}(\mathrm{Perv}(X,E)) \arrow[r, "f_!"] \arrow[d, "\mathrm{real}"'] & \mathrm{D^b}(\mathrm{Perv}(Y,E)) \arrow[d, "\mathrm{real}"] \\ \mathrm{D^b_c}(X,E) \arrow[r, "f_!"'] & \mathrm{D^b_c}(Y,E) \end{tikzcd}$$

(ii) If f is smooth and of relative dimension d, then we have a commutative diagram (up to natural isomorphism)

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{Perv}(Y,E)) \arrow[r, "{f^*[d]}"] \arrow[d, "\mathrm{real}"'] & \mathrm{D^b}(\mathrm{Perv}(X,E)) \arrow[d, "\mathrm{real}"] \\ \mathrm{D^b_c}(Y,E) \arrow[r, "{f^*[d]}"'] & \mathrm{D^b_c}(X,E) \end{tikzcd}$$

(iii) We have a commutative diagram (up to natural isomorphism)

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{Perv}(X,E))^{\mathrm{op}} \arrow[r, "D_X"] \arrow[d, "\mathrm{real}"'] & \mathrm{D^b}(\mathrm{Perv}(X,E)) \arrow[d, "\mathrm{real}"] \\ \mathrm{D^b_c}(X,E)^{\mathrm{op}} \arrow[r, "D_X"'] & \mathrm{D^b_c}(X,E) \end{tikzcd}$$

Proof. We need to check the hypothesis of Proposition 2.3.4. But, if we have triangulated functors F' on $\mathrm{DF}(X_{\text{proét}}, E)$ and F on $\mathrm{D}(X_{\text{proét}}, E)$ that are compatible as in the statement of Corollary 2.3.3, then it is clear that F' will preserve the appropriate categories $\mathrm{DF}_{\text{bête}}$.

[3] For $\underline{\mathrm{Hom}}_X$ and $\otimes$, we could also use V.2 of [16].

$\square$

We will need one last compatibility. Suppose that we have a flat morphism of finite type $X \longrightarrow S$. If $T \longrightarrow S$ is étale (resp. the inclusion of the generic point of S), $u : X_T \longrightarrow X$ is its base change to X and $u^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_T, E)$ is the restriction functor, then u^* (resp. $u^*[-\dim S]$) is t-exact by Proposition 2.2.1. The following result is proved exactly as Proposition 2.4.1.

Proposition 2.4.2 *In the situation above, we have a commutative diagram (up to natural isomorphism)*

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\mathrm{Perv}(X, E)) \arrow[r, "u^*[a]"] \arrow[d, "\mathrm{real}"'] & \mathrm{D}^{\mathrm{b}}(\mathrm{Perv}(X_T, E)) \arrow[d, "\mathrm{real}"] \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E) \arrow[r, "u^*[a]"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_T, E)
\end{tikzcd}$$

where $a = 0$ is $T \longrightarrow S$ is étale and $a = -\dim S$ if $T \longrightarrow S$ is the inclusion of the generic point of S.

2.5 Horizontal constructible complexes

From now on, we fix a field k of finite type over its prime field (in other words, k is the field of fractions of an integral scheme of finite type over $\mathbb{Z}$) and an algebraic extension E of $\mathbb{Q}_\ell$. We will consider separated schemes of finite type over k and denote by them by capital Roman letters such as X, Y, U etc.

We will recall some constructions and results of sections 1-3 of [14]. In this article, Huber assumes that k is a number field, but, as she notes herself in the remark after proposition 2.3, this is not really necessary and all her constructions extend to the more general situation considered here, either by Deligne's generic constructibility theorem (SGA 4 1/2 [Th. finitude]) or by Gabber's finiteness results ([18]).

Let $\mathscr{U}$ be the set (ordered by inclusion) of $\mathbb{Z}$-subalgebras $A \subset k$ that are regular and of finite type over $\mathbb{Z}$ and such that k is the field of fractions of A. By a theorem of Nagata (see EGA IV 6.12.6), if $\mathscr{B}$ is an integral scheme of finite type over $\mathbb{Z}$, then the regular locus of $\mathscr{B}$ is open in $\mathscr{B}$. Hence $k = \varinjlim_{A \in \mathscr{U}} A$. So we are in the situation of EGA IV 8 and can use the results of this reference.

If $A \in \mathscr{U}$, we say that a scheme over $\operatorname{Spec} A$ is *horizontal* if it is flat and of finite type over A. Let X be a scheme over k. We denote by $\mathscr{U}X$ the category of triples $(A, \mathscr{X}, u)$, where $A \in \mathscr{U}$, $\mathscr{X}$ is a horizontal scheme over A and u is an isomorphism of k-schemes $X \xrightarrow{\sim} \mathscr{X} \otimes_A k$; we will often omit u from the notation. A morphism $(A, \mathscr{X}, u) \longrightarrow (A', \mathscr{X}', u')$ is an inclusion $A \subset A'$ and an open embedding $f : \mathscr{X}' \longrightarrow \mathscr{X} \otimes_A A'$ such that $u' = u \circ f$. Then we have a canonical isomorphism (given by the entry u of the triples)

$$X \xrightarrow{\sim} \varinjlim_{(A,\mathscr{X})\in \mathrm{Ob}\,\mathscr{U}X} \mathscr{X}\otimes_A k.$$

If $f:(A,\mathscr{X})\longrightarrow(A',\mathscr{X}')$ is a morphism in $\mathscr{U}X$, then it induces an exact functor

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)\longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\otimes_A A',E)\xrightarrow{f^*}\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}',E),$$

where the first functor is the restriction functor along the open embedding $\mathscr{X}\otimes_A A'\longrightarrow\mathscr{X}$.

Definition 2.5.1 (See [14], Definition 1.2.) Let X be a scheme over k. We define the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$ by

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)=2-\varinjlim_{(A,\mathscr{X})\in \mathrm{Ob}\,\mathscr{U}X}\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E).$$

We call this category the *category of bounded constructible horizontal E-complexes of sheaves on X.*

Note that we could also define versions of these categories with coefficients in $\mathscr{O}_E$ (if E is a finite extension of $\mathbb{Q}_\ell$), as Huber does. But we will only be interested in this article in the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$.

As in the remark following Definition 1.2 of [14], we see that these categories are triangulated and have a tautological t-structure (induced by the tautological t-structure on the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)$), and that all the properties of Theorem 6.3 of [10] carry over. The heart of the canonical t-structure will be denoted by $\mathrm{Sh}_h(X,E)$, and we will call its objects *horizontal constructible sheaves* on X.

We denote by $\eta^*:\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)\longrightarrow\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ the exact functor induced by the restriction functors $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)\longrightarrow\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\otimes_A k,E)\xrightarrow{u^*}\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$, for $(A,\mathscr{X},u)\in\mathrm{Ob}\,\mathscr{U}X$.

Proposition 2.5.2 *(i) The functor η^* is fully faithful on the heart of the tautological t-structure.*

(ii) If $\mathscr{F},\mathscr{G}\in\mathrm{Ob}(\mathrm{Sh}_h(X,E))$, then

$$\eta^*:\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)}(\mathscr{F},\mathscr{G})\longrightarrow\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)}(\eta^*\mathscr{F},\eta^*\mathscr{G})$$

is injective.

Proof. The first point is proposition 1.3 of [14]. We prove the second point. Let $(A,\mathscr{X},u)\in\mathrm{Ob}(\mathscr{U}X)$ such that $\mathscr{F}$ and $\mathscr{G}$ come from objects K and L of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)$. We use u to identify X and $\mathscr{X}\otimes_A k$. The constructible sheaves $\bigoplus_{i\neq0}\mathrm{H}^iK$ and $\bigoplus_{i\neq0}\mathrm{H}^iL$ on $\mathscr{X}$

are supported on a closed subset disjoint from X, so, after shrinking $\mathscr{X}$, we may assume that K and L are constructible sheaves on $\mathscr{X}$. By definition of $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$, we have

$$\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)}(\mathscr{F},\mathscr{G}) = \varinjlim_{A\subset A'\in\mathscr{U}} \mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\otimes_A A',E)}(K_{|\mathscr{X}\otimes_A A'}, L_{|\mathscr{X}\otimes_A A'}).$$

Let $A' \supset A$ be an element of $\mathscr{U}$. The category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\otimes_A A', E)$ is a full subcategory of $\mathrm{D}((\mathscr{X}\otimes_A A')_{\mathrm{pro\acute{e}t}}, E)$, the derived category of the category of E-modules on the proétale site of $\mathscr{X}\otimes_A A'$, so the groups $\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\otimes_A A',E)}$ parametrize extensions in this category of E-modules (see section 3.2 of chapter III of Verdier's book [29]). But $\mathrm{Sh}_c(\mathscr{X}\otimes_A A', E)$ is a Serre subcategory of the category of all sheaves of E-modules (see proposition 6.8.11 of [7]), so $\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\otimes_A A',E)}(K_{|\mathscr{X}\otimes_A A'}, L_{|\mathscr{X}\otimes_A A'})$ is the group of equivalence classes of extensions of $K_{|\mathscr{X}\otimes_A A'}$ by $L_{|\mathscr{X}\otimes_A A'}$ in $\mathrm{Sh}_c(\mathscr{X}\otimes_A A', E)$. We have a similar statement for $\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)}(\eta^*\mathscr{F},\eta^*\mathscr{G})$.

Now let $c \in \mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)}(\mathscr{F},\mathscr{G})$, and suppose that its image in $\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)}(\eta^*\mathscr{F},\eta^*\mathscr{G})$ is 0. There exists an element $A'\supset A$ of $\mathscr{U}$ and an extension

$$0 \longrightarrow L_{|\mathscr{X}\otimes_A A'} \longrightarrow M \longrightarrow K_{|\mathscr{X}\otimes_A A'} \longrightarrow 0$$

in $\mathrm{Sh}_c(\mathscr{X}\otimes_A A', E)$ whose class is c. The hypothesis on c says that the restriction of this extension to X is split. But, by point (i), this implies that there exists an element $A''\supset A'$ of $\mathscr{U}$ such that the restriction of the extension to $\mathscr{X}\otimes_A A''$ already splits, which means that $c = 0$.

□

2.6 Horizontal perverse sheaves

In this section, we define the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$. Note that, by Proposition 1.4 of Giral's article [13], all the rings A in $\mathscr{U}$ have the same Krull dimension c, which is the transcendence degree of k over its prime field if k is of positive characteristic, and 1 plus this transcendence degree if k is of characteristic 0.

If $(A,\mathscr{X},u) \in \mathrm{Ob}\,\mathscr{U}X$, then we consider the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)$ defined in section 2.2. Then the functor $u^*[-c] : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ is t-exact by Proposition 2.2.1. Also, for every morphism $f : (A,\mathscr{X},u) \longrightarrow (A',\mathscr{X}',u')$ in $\mathscr{U}X$, the restriction functor $f^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}',E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)$ is t-exact by the same proposition. By taking the limit of the shift by c of the t-structures on the $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X},E)$, we get a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$ such that $\eta^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ is t-exact. We denote the heart of this t-structure by $\mathrm{Perv}_h(X,E)$ and call it the category of *horizontal perverse sheaves* on X. We still denote the perverse cohomology functors by ${}^p\mathrm{H}^i : \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E) \longrightarrow \mathrm{Perv}_h(X,E)$.

By Proposition 2.4.2, we get a realization functor $\mathrm{real} : \mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_h(X,E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$.

Remark 2.6.1 This is not Huber's construction. Let us recall her construction and compare it with ours. Let $A \in \mathscr{U}$ and let $\mathscr{X}$ be a horizontal scheme over A. As in [14] 2.1, we say that a stratification of $\mathscr{X}$ is *horizontal* if all its strata are smooth over A. Suppose that $E/\mathbb{Q}_\ell$ is finite. If S is a horizontal stratification of $\mathscr{X}$ and L is the data of a set of irreducible lisse $\mathscr{O}_E$-sheaves on every stratum of S satisfying condition (c) of [14] 2.2, we get as in Definition 2.2 and Lemma 2.4 of [14] a full subcategory $D^b_{(S,L)}(\mathscr{X}, \mathscr{O}_E)$ of (S, L)-constructible objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}, \mathscr{O}_E)$, and it has a self-dual perverse t-structure, whose heart we will denote by $\mathrm{Perv}_{(S,L)}(\mathscr{X}, \mathscr{O}_E)$. Because the strata S are smooth over $\operatorname{Spec} A$, lisse sheaves on them are perverse for our t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}, \mathscr{O}_E)$ when placed in degree $-\dim(A) = -c$, so Huber's t-structure is the shift by c of the one induced by our perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}, \mathscr{O}_E)$.

By Proposition 2.3 of [14], the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X, \mathscr{O}_E)$ is the 2-colimit of the categories $D^b_{(S,L)}(\mathscr{X}, \mathscr{O}_E)$ over all $(A, \mathscr{X}) \in \operatorname{Ob} \mathscr{U} X$ and couples (S, L) as before, and by Theorem 2.5 of [14], the perverse t-structure goes to the limit and induces a t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X, \mathscr{O}_E)$. This is the t-structure that is used in [14], and, by the observation of the previous paragraph, it coincides with the one that we defined at the beginning of this section.

Huber's definition has the advantage that we can apply Deligne's generic base change theorem (from SGA 4 1/2 [Th. finitude]) to deduce statements for horizontal perverse sheaves from statements for perverse sheaves on schemes over finite fields proved in sections 4 and 5 of [6].

For example, we see as in [14] 2.7 that the six operations have the usual exactness properties with respect to the perverse t-structure (which means the properties of [6] 4.1 and 4.2), that the category $\mathrm{Perv}_h(X, E)$ is Artinian and Noetherian, and that we have the same description of its simple objects as in Theorem 4.3.1 of [6]. [4]

The following result is a slight generalization of the first part of Lemma 2.12 of [14].

Proposition 2.6.2 *1. The functor $\eta^* : \mathrm{Perv}_h(X, E) \longrightarrow \mathrm{Perv}(X, E)$ is fully faithful, and its essential image is the full category of perverse sheaves on X that extend to a constructible complex on some $\mathscr{X}$, for $(A, \mathscr{X}) \in \operatorname{Ob} \mathscr{U} X$.*

2. For every $K, L \in \operatorname{Ob} \mathrm{Perv}(X, E)$, the morphism

$$\mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)}(K, L) \longrightarrow \mathrm{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)}(\eta^* K, \eta^* L)$$

induced by η is injective.

Proof. If the category where we take the Ext^i is clear from context, we omit it in this proof. Also, we omit the coefficients E in the notation.

We prove both points by Noetherian induction on X. If $\dim X = 0$, then the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, E)$ is the usual t-structure, so both points follow from Proposition 2.5.2 (which is an easy consequence of Proposition 1.3 of [14]).

[4] But we could also have proved all these statements from our definition.

Suppose that $\dim X > 0$, and let $K, L \in \operatorname{Ob}\operatorname{Perv}_h(X)$. Lemma 2.12 of [14] says that the map $\operatorname{Hom}(K, L) \longrightarrow \operatorname{Hom}(\eta^*K, \eta^*L)$ is injective, and we want to show that it is also surjective. We show this by induction on the sum of the lengths of K and L.

Suppose first that K and L are both simple. Then we have smooth connected locally closed subschemes $k_1 : Y_1 \longrightarrow X$ and $k_2 : Y_2 \longrightarrow X$ and horizontal locally constant sheaves $\mathscr{L}_1$ on Y_1 and $\mathscr{L}_2$ on Y_2 such that $K = k_{1!*}\mathscr{L}_1[\dim Y_1]$ and $L = k_{2!*}\mathscr{L}_2[\dim Y_2]$. We have $\operatorname{Hom}(K, L) = 0$ if $K \not\simeq L$, and $\operatorname{Hom}(K, K) = \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(\mathscr{L}_1, \mathscr{L}_1)$. In particular, by Proposition 1.3 of [14], η^*K and η^*L are also simple, and $\operatorname{Hom}(K, L) \xrightarrow{\sim} \operatorname{Hom}(\eta^*K, \eta^*L)$, proving the first point.

We prove the second point. Let $Z = \overline{Y}_1 \cap \overline{Y}_2$, and denote by $i : Z \longrightarrow X$ and $j : X - Z \longrightarrow X$ the inclusions. We have an exact triangle

$$R\operatorname{Hom}(i^*K, i^!L) \longrightarrow R\operatorname{Hom}(K, L) \longrightarrow R\operatorname{Hom}(j^*K, j^*L) \xrightarrow{+1}.$$

As j^*K and j^*L are perverse with disjoint supports on $X - Z$, $R\operatorname{Hom}(j^*K, j^*L) = 0$, so we get isomorphisms $\operatorname{Ext}^i(i^*K, i^!L) \xrightarrow{\sim} \operatorname{Ext}^i(K, L)$ for every $i \in \mathbb{Z}$. We have a similar result for η^*K and η^*L.

If $\overline{Y}_1$ is not contained is $\overline{Y}_2$, then Z is a proper closed subset of $\overline{Y}_1$, so i^*K and $i^*\eta^*K$ are concentrated in perverse degree ≤ -1. As $i^!L$ and $i^!\eta^*L$ are concentrated in perverse degree ≥ 0, we get

$$\operatorname{Ext}^1(K, L) = \operatorname{Hom}({}^p\mathrm{H}^{-1}i^*K, {}^p\mathrm{H}^0 i^!L)$$

and

$$\operatorname{Ext}^1(\eta^*K, \eta^*L) = \operatorname{Hom}({}^p\mathrm{H}^{-1}i^*\eta^*K, {}^p\mathrm{H}^0 i^!\eta^*L),$$

so the second point follows from the induction hypothesis applied to Z.

If $\overline{Y}_2$ is not contained is $\overline{Y}_1$, then Z is a proper closed subset of $\overline{Y}_2$, so $i^!L$ and $i^!\eta^*L$ are concentrated in perverse degree ≥ 1. As i^*K and $i^*\eta^*K$ are concentrated in perverse degree ≤ 0, we get

$$\operatorname{Ext}^1(K, L) = \operatorname{Hom}({}^p\mathrm{H}^0 i^*K, {}^p\mathrm{H}^1 i^!L)$$

and

$$\operatorname{Ext}^1(\eta^*K, \eta^*L) = \operatorname{Hom}({}^p\mathrm{H}^0 i^*\eta^*K, {}^p\mathrm{H}^1 i^!\eta^*L),$$

so the second point again follows from the induction hypothesis applied to Z.

Finally, suppose that $\overline{Y}_1 = \overline{Y}_2$. Then i^*K and $i^!L$ are perverse and simple, and we may assume that $Y_1 = Y_2$. Let b be the inclusion of the open subscheme Y_1 of Z, and a be the inclusion of its complement. As before, we have an exact triangle

$$R\operatorname{Hom}(a^*i^*K, a^!i^!L) \longrightarrow R\operatorname{Hom}(i^*K, i^!L) \longrightarrow R\operatorname{Hom}(b^*i^*K, b^*i^!L) \xrightarrow{+1}.$$

As i^*K and $i^!L$ are simple of support Z, we know that a^*i^*K is concentrated in perverse degree ≤ -1 and that $a^!i^!L$ is concentrated in perverse degree ≥ 1, so we get an injective map

$$\operatorname{Ext}^1(i^*K, i^!L) \longrightarrow \operatorname{Ext}^1(b^*i^*K, b^*i^!L) = \operatorname{Ext}^1_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(\mathscr{L}_1, \mathscr{L}_2).$$

We have a similar calculation for $i^*\eta^*K$ and $i^!\eta^*L$, and so the second point follows from Proposition 2.5.2. This finishes the proof in the case where K and L are both simple.

Now suppose that we have an exact sequence $0 \longrightarrow K_1 \longrightarrow K \longrightarrow K_2 \longrightarrow 0$, and that we know the result for the pairs (K_1, L) and (K_2, L). We show it for (K, L). Write $K' = \eta^*K$, $L' = \eta^*L$ etc. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(K_2, L) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}(K, L) \arrow[r] \arrow[d] & \mathrm{Hom}(K_1, L) \arrow[r] \arrow[d, "\wr"] & \mathrm{Ext}^1(K_2, L) \arrow[r] \arrow[d, hook] & \mathrm{Ext}^1(K, L) \arrow[r] \arrow[d] & \mathrm{Ext}^1(K_1, L) \arrow[d, hook] \\
0 \arrow[r] & \mathrm{Hom}(K_2', L') \arrow[r] & \mathrm{Hom}(K', L') \arrow[r] & \mathrm{Hom}(K_1', L') \arrow[r] & \mathrm{Ext}^1(K_2', L') \arrow[r] & \mathrm{Ext}^1(K', L') \arrow[r] & \mathrm{Ext}^1(K_1', L')
\end{tikzcd}$$

so both points follow from the five lemma.

The case where we have an exact sequence $0 \longrightarrow L_1 \longrightarrow L \longrightarrow L_2 \longrightarrow 0$ such that the result is known for (K, L_1) and (K, L_2) is treated in the same way.

$\square$

2.7 Mixed perverse sheaves

The key point is that, if $A \in \mathscr{U}$, then, as A is a $\mathbb{Z}$-algebra of finite type, the residue fields of closed points of $\operatorname{Spec} A$ are finite, so we can use the theory of [6] chapter 5 as in section 3 of [14] to define categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ of mixed horizontal complexes. Once we have defined what it means for a horizontal constructible sheaf to be punctually pure of a certain weight, the definition proceeds as in [6] 5.1.5. If $\mathscr{F} \in \operatorname{Ob}\mathrm{Sh}_h(X, E)$ and $w \in \mathbb{Z}$, we say that $\mathscr{F}$ is *punctually pure of weight* w if there exists $(A, \mathscr{X}) \in \operatorname{Ob}\mathscr{U}X$ and $\mathscr{F}' \in \operatorname{Ob}\mathrm{Sh}_c(\mathscr{X}, E)$ a constructible sheaf extending $\mathscr{F}$ such that, for every closed point x of $\operatorname{Spec} A$, $\mathscr{F}'_{|\mathscr{X}_x}$ is punctually pure of weight w in the sense of [6] 5.1.5 (that is, of Deligne's [9]). We say that $\mathscr{F}$ is *mixed* if it has a filtration whose graded pieces are punctually pure of some weight.

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ the full subcategory of *mixed complexes* in $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$; the objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ are complexes K such that all the (usual) cohomology sheaves of K are mixed.

By Proposition 3.2 of [14], these subcategories are stable by the 6 operations and inherit a perverse t-structure from $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$. We denote the heart of this t-structure by $\mathrm{Perv}_m(X, E)$; it is a full subcategory of $\mathrm{Perv}_h(X, E)$, stable by subquotients and extensions. All the compatibilities between the six operations (and the intermediate extension functor) and weights that are proved in [9] and [6] chapter 5 remain true, see [14] 3.3-3.6. Also, the functor $\mathrm{real} : D^b\,\mathrm{Perv}_h(X, E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$ restricts to a functor $\mathrm{real} : D^b\,\mathrm{Perv}_m(X, E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X, E)$, whose essential image is contained in $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$ by definition of $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X, E)$.

Let us introduce weight filtrations, following Definition 3.7 of [14].

Definition 2.7.1 Let $K \in \operatorname{Ob}\operatorname{Perv}_m(X,E)$. A *weight filtration* on K is a separated exhaustive ascending filtration W on K (in the abelian category $\operatorname{Perv}_m(X,E)$) such that $\operatorname{Gr}_k^W K$ is pure of weight k for every $k \in \mathbb{Z}$.

As the abelian category $\operatorname{Perv}_m(X,E)$ is Artinian and Noetherian, such a filtration is automatically finite. Note also that morphisms in $\operatorname{Perv}_m(X,E)$ are strictly compatible with weight filtrations (Lemma of 3.8 [14]), so in particular a weight filtration is unique if it exists.

Definition 2.7.2 Let $\operatorname{Perv}_{mf}(X,E)$ be the full subcategory of $\operatorname{Perv}_m(X,E)$ whose objects are mixed horizontal perverse sheaves admitting a weight filtration.

This subcategory is clearly stable by subquotients in $\operatorname{Perv}_m(X,E)$, but it is not stable by extensions (even if $X = \operatorname{Spec} k$), see the warning before Proposition 3.4 of [14].

Finally, the following conservativity result will be very useful.

Proposition 2.7.3 *(i) The functor* $\eta^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$ *is conservative.*

(ii) The realization functor $\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_h(X,E) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$ *is conservative.*

(iii) The obvious functor $\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X,E) \longrightarrow \mathrm{D}^{\mathrm{b}}\operatorname{Perv}_h(X,E)$ *is conservative.*

Proof. In all three cases, we have t-structures on the source and target for which the functors are t-exact and such that the family of cohomology functors for the t-structure is conservative (the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X,E)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,E)$, and the canonical t-structure on the derived categories). So it suffices to check that the functors on the hearts are conservative. But these functors are all faithful and exact (in fact, they are all fully faithful), so they are conservative.

$\square$

3 Main theorems

From now on, we will fix the algebraic extension E of $\mathbb{Q}_\ell$ and omit it in the notation.

3.1 Informal statement

Informally, the main theorems say that the sheaves operations (f_*, f^*, $f_!$ and $f^!$, $\underline{\operatorname{Hom}}$, $\otimes$, Poincaré-Verdier duality, unipotent nearby and vanishing cycles) lift to the categories $\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)$ in a way that is compatible with the realization functors $\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$, and that all the relations between these functors that are true in the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ are still true in the categories $\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)$.

A convenient way to say this is to use the formalism introduced in Ayoub's thesis [1] (and in his article [2]). Then Theorem 3.2.4 says that the four operations f_*, f^*, $f_!$ and $f^!$ exist and satisfy all the expected adjunctions and compatibilities, and Theorem 3.2.12 asserts the existence and properties of the derived internal Homs and derived tensor products. The stability of the categories Perv_{mf} under the perverse direct image functors is proved in section 6.3, and the unipotent vanishing cycles are constructed in section 5.2 (see Corollary 6.3.3).

3.2 Formal statement

We denote by $\mathbf{Sch}/k$ the category of schemes over k (always assumed to be separated of finite type, as before) and by $\mathfrak{TR}$ the 2-category of triangulated categories.

The notion of a formalism of the four operations (f^*, f_*, $f_!$, $f^!$) has been axiomatized by Deligne, Voevodsky and Ayoub, under the name of "foncteur croisé". [5] We will follow Ayoub's presentation.

Definition 3.2.1 (See Definition 1.2.12 of [1].) [6] A *crossed functor* ("foncteur croisé") on $\mathbf{Sch}/k$ with values in $\mathfrak{TR}$ (relatively to the class of cartesian squares) is a quadruple of 2-functors $H = (H^*, H_*, H_!, H^!) : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$, such that :

(0) for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, we have $H_*(X) = H_!(X) = H^*(X) = H^!(X)$ (we denote this triangulated category by $H(X)$);

(1) the functors $H_*, H_!$ are covariant, and the functors $H^*, H^!$ are contravariant;

(2) the functor H^* is a global left adjoint of H_*;

(3) the functor $H^!$ is a global right adjoint of $H_!$;

together with the data of exchange structures of type $\swarrow$ on the couples $(H_*, H_!)$ and $(H^*, H^!)$ (see Definition 1.2.1 of [1]), i.e., for every cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathbf{Sch}/k$, we have morphisms of functors $H_!(f) \circ H_*(g') \longrightarrow H_*(g) \circ H_!(f')$ and $H^*(g') \circ H^!(f) \longrightarrow H^!(f') \circ H^*(g)$ compatible with horizontal and vertical composition of squares.

---

[5] There are other approaches, but this particular one seems better suited to our situation. For example, using derivators is complicated by the fact that it is difficult to make sense of the notion of "perverse sheaf over a diagram of schemes", because inverse image functors typically do not preserve perverse sheaves.

[6] Note that we take the two categories $\mathscr{C}_1$ and $\mathscr{C}_2$ of this reference to be equal to $\mathbf{Sch}/k$.

This data is moreover required to satisfy the following condition : For every cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathbf{Sch}/k$, the morphisms $H^*(g) \circ H_!(f) \longrightarrow H_!(f') \circ H^*(g')$ and $H_!(f') \circ H^*(g') \longrightarrow H^*(g) \circ H_!(f)$ formally constructed used the exchange structures and adjunctions (see the beginning of [1] 1.2.4) are isomorphisms and inverses of each other; equivalently, we could required that the morphisms $H^!(f) \circ H_*(g') \longrightarrow H_*(g) \circ H^!(f')$ and $H_*(g) \circ H^!(f') \longrightarrow H^!(f) \circ H_*(g')$ are isomorphisms and inverses of each other.

Definition 3.2.2 (See Definition 3.1 and Theorem 3.4 of [2].) Suppose that we have two crossed functors $H_1, H_2 : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$. A *morphism of crossed functors* $R : H_1 \longrightarrow H_2$ is the following data :

(1) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, a triangulated functor $R_X : H_1(X) \longrightarrow H_2(X)$.

(2) For every $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, invertible natural transformations

$$\theta_f : H_2^*(f) \circ R_Y \xrightarrow{\sim} R_X \circ H_1^*(f)$$

$$\gamma_f : R_Y \circ H_{1,*}(f) \xrightarrow{\sim} H_{2,*}(f) \circ R_X$$

$$\rho_f : H_{2,!}(f) \circ R_X \xrightarrow{\sim} R_Y \circ H_{2,!}(f)$$

$$\xi_f : R_X \circ H_2^!(f) \xrightarrow{\sim} H_2^!(f) \circ R_Y.$$

We require these transformations to satisfy the compatibility conditions spelled out in section 3 of Ayoub's paper [2].

Example 3.2.3 For $a \in \{c, h, m\}$, we have a crossed functor $H_a = (H_a^*, H_{a,*}, H_{a,!}, H_a^!) : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ defined in the following way :

- For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$,

$$H_a^*(X) = H_{a,*}(X) = H_{a,!} = (X) = H_a^!(X) = \mathrm{D}_{\mathrm{a}}^{\mathrm{b}}(X).$$

- For every $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, we have $H_a^*(f) = f^*$, $H_{a,*}(f) = f_*$, $H_{a,!}(f) = f_!$ and $H_a^!(f) = f^!$.

Moreover, we have morphisms of crossed functors $H_m \longrightarrow H_h \longrightarrow H_c$.

Then our first main result is the following theorem.

Theorem 3.2.4 *There exists a crossed functor* $H_{mf} = (H_{mf}^*, H_{mf,*}, H_{mf,!}, H_{mf}^!) : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ *and a morphism of crossed functors* $R : H_{mf} \longrightarrow H_m$ *such that, for every* $X \in \mathrm{Ob}(\mathbf{Sch}/k)$*,* $H_{mf}(X) = \mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$ *and* $R_X : H_{mf}(X) = \mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \longrightarrow H_m(X) = \mathrm{D^b_m}(X)$ *is the composition of the obvious functor* $\mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{D^b}\,\mathrm{Perv}_m(X)$ *and of the realization functor of section 2.6.*

Moreover, the functor R_X *is conservative for every* k*-scheme* X*, and we have for every morphism* f *in* $\mathbf{Sch}/k$ *a natural transformation* $H_{mf,!}(f) \longrightarrow H_{mf,*}(f)$*, which is an isomorphism if* f *is proper.*

To prove this, we will follow the same strategy as in chapter 1 of [1] and section 3 of [2], and deduce the existence of the crossed functor and of the natural transformation $f_! \longrightarrow f_*$ from that of a stable homotopic 2-functor (see Definition 3.2.5).

We note that the conservativity of R_X follows immediately from Proposition 2.7.3, and then the fact that $f_! \longrightarrow f_*$ is an isomorphism for f proper follows from the conservativity of the functors R_X.

Definition 3.2.5 (See [1] 1.4.1.) Let $H^* : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ be a contravariant 2-functor. For $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, we write $H^*(X) = H(X)$, and for f a morphism of $\mathbf{Sch}/k$, we also denote the 1-functor $H^*(f)$ by f^*. We assume that H^* is strictly unital, i.e., for every morphism $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, the connection isomorphisms $(f \circ \mathrm{id}_X)^* \simeq f^*$ and $(\mathrm{id}_Y \circ f)^* \simeq f^*$ are the identity.

We say that H^* is a *stable homotopic* 2*-functor* if it satisfies the following conditions :

(1) $H(\varnothing) = 0$.

(2) For every $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, the functor $f^* : H(Y) \longrightarrow H(X)$ admits a right adjoint f_*. Moreover, if f is a locally closed immersion, then the counit $f^*f_* \longrightarrow \mathrm{id}_{H(X)}$ is an isomorphism.

(3) If $f : X \longrightarrow Y$ is a smooth morphism in $\mathbf{Sch}/k$, then the functor f^* admits a left adjoint $f_\sharp$. Moreover, if we have a cartesian square :

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

with f smooth, then the exchange morphism $f'_\sharp g'^* \longrightarrow g^* f_\sharp$ (defined formally using the adjonctions, see [1] 1.4.5) is an isomorphism.

(4) If $j : U \longrightarrow X$ and $i : Z \longrightarrow X$ are complementary open and closed immersions in $\mathbf{Sch}/k$, then the pair (j^*, i^*) is conservative.

(5) If $X \in \mathrm{Ob}(\mathbf{Sch}/R)$ and $p : \mathbb{A}^1_X \longrightarrow X$ is the canonical projection, then the unit $\mathrm{id}_X \longrightarrow p_* p^*$ is an isomorphism.

(6) With the notation of (5), if $s : X \longrightarrow \mathbb{A}^1_X$ is the zero section, then $p_\sharp s_* : H(X) \longrightarrow H(X)$ is an equivalence of categories.

Definition 3.2.6 (See Definition 3.1 of [2].) Let $H_1^*, H_2^* : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ be two stable homotopic 2-functors. A *morphism of stable homotopic* 2*-functors* $R : H_1^* \longrightarrow H_2^*$ is the data of :

(1) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, a triangulated functor $R_X : H_1(X) \longrightarrow H_2(X)$.

(2) For every $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, an invertible natural transformation

$$\theta_f : f^* \circ R_Y \xrightarrow{\sim} R_X \circ f^*.$$

We require that this data satisfy the following compatibility conditions :

(A) The natural transformations are compatible with the composition of morphisms in $\mathbf{Sch}/k$.

(B) If f is smooth, then the natural transformation $f_\sharp \circ R_X \longrightarrow R_Y \circ f_\sharp$ (obtained using the adjonction and θ_f^{-1}) is invertible.

Example 3.2.7 The crossed functors of Example 3.2.3 define (by forgetting part of the data) three stable homotopic 2-functors H_m^*, H_h^* and H_c^*, and morphisms $H_m^* \longrightarrow H_h^* \longrightarrow H_c^*$.

Theorem 3.2.4 now follows immediately from the following two results (the first one is a consequence of several theorems of Ayoub and is also used to construct the four operations on the triangulated categories of Voevodsky motives, and the second one is the main technical result of this paper).

Theorem 3.2.8 *(i) (See Scholie 1.4.2 of [1].) Let $H^* : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ be a stable homotopic 2-functor. Then H^* extends to a crossed functor $\mathbf{Sch}/k \longrightarrow \mathfrak{TR}$.*

(ii) (See Theorems 3.4 and 3.7 of [2].) Let $H_1^*, H_2^* : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ be two stable homotopic 2-functors and $R : H_1^* \longrightarrow H_2^*$ be a morphism. Let $H_1, H_2 : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ be crossed functors extending H_1^*, H_2^* as in (i). Then R extends to a morphism of crossed functors from H_1 to H_2.

Theorem 3.2.9 *There exists a stable homotopic 2-functor $H_{mf}^* : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ and a morphism of stable homotopic 2-functors $R : H_{mf}^* \longrightarrow H_m^*$ such that, for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, $H_{mf}(X) = \mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$ and $R_X : H_{mf}(X) \longrightarrow H_m(X)$ is the same functor as in Theorem 3.2.4.*

Proof. The construction of the 2-functor H_{mf}^* is given in Corollary 7.2.4. Let's check that it is a stable homotopic 2-functor. Property (1) is obvious. The fact that $H_{mf}^*(f)$ admits a right adjoint for every f follows from the definition of H_{mf}^* as a global left adjoint, and the last part of property (2) follows from Corollary 6.4.2 and Proposition 7.1.7. The fact that f^* admits a left adjoint for f smooth is proved in Proposition 7.3.2(iv), and the last part of property (3) as well as properties (4) and (5) follow from the conservativity of the realization functor. Finally, let Y be a k-scheme, let $p : \mathbb{A}^1_Y \longrightarrow Y$ be the canonical projection and $s : Y \longrightarrow \mathbb{A}^1_Y$ be the zero section. By Proposition 7.3.2(v), we have a natural isomorphism $s_! \xrightarrow{\sim} s_*$. So we get a natural isomorphism

$$p_\sharp s_* = p_![2](1)s_* \xrightarrow{\sim} p_!s_![2](1) \simeq (ps)_![2](1) = \mathrm{id}[2](1),$$

which shows that $p_\sharp s_* : \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y) \longrightarrow \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y)$ is an equivalence of categories.

$\square$

Finally, we show the existence of tensor products and internal Homs on the categories $\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$.

Definition 3.2.10 (i) (See Definition 2.3.1 of [1].) A *unitary symmetric monoïdal* stable homotopic 2-functor is a stable homotopic 2-functor H^* that takes its values in the 2-category of symmetric monoïdal unitary triangulated categories, that is, that associates to every $X \in \mathrm{Ob}\,\mathbf{Sch}/k$ a unitary symmetric monoïdal category $(H(X), \otimes_X, \mathbb{1}_X)$ and such that :

(a) For every morphism $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, the functor f^* is unitary monoïdal.

(b) (Projection formula.) If $f : X \longrightarrow Y$ is smooth, $K \in \mathrm{Ob}\,H(Y)$ and $L \in \mathrm{Ob}\,H(X)$, then the functorial map

$$p : f_\sharp(f^*(K) \otimes_Y L) \longrightarrow K \otimes_X f_\sharp(L)$$

constructed in Proposition 2.1.97 of [1] is an isomorphism.

(ii) (See Definition 3.2 of [2].) Let H_1^* and H_2^* be two symmetric monoïdal unitary stable homotopic 2-functors. Then a morphism of symmetric monoïdal unitary stable homotopic 2-functors from H_1^* to H_2^* is a morphism of stable homotopic 2-functors $R : H_1^* \longrightarrow H_2^*$ such that :

(a) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, the functor R_X is monoïdal unitary.

(b) For every morphism of k-schemes f, the natural transformation θ_f is a morphism of monoïdal unitary functors.

(iii) (See Definition 2.3.50 of [1].) If H^* is as in (i), we say that H^* is *closed* if, for every $X \in \mathrm{Ob}\,\mathbf{Sch}/k$, the symmetric monoïdal category $(H(X), \otimes_X)$ is closed; this means that, for every object K of $H(X)$, the endofunctor $K \otimes_X \cdot$ of $H(X)$ admits a right adjoint, that will be denoted by $\underline{\mathrm{Hom}}_X(K, \cdot)$.

Example 3.2.11 The stable homotopic 2-functors H_m^*, H_h^* and H_c^* are all closed symmetric monoïdal unitary (for the derived tensor product), and the morphisms $H_m^* \longrightarrow H_h^* \longrightarrow H_c^*$ are morphisms of symmetric monoïdal unitary stable homotopic 2-functors.

Our last result is the following :

Theorem 3.2.12 *There exists a structure of closed symmetric monoïdal unitary stable homotopic 2-functor on H^*_{mf} such that $R : H^*_{mf} \longrightarrow H^*_m$ is a morphism of symmetric monoïdal unitary stable homotopic 2-functors.*

Moreover, for every k-scheme X, the functorial map

$$R_X \underline{\mathrm{Hom}}_{\mathrm{D^b\,Perv}_{mf}(X)}(\cdot,\cdot) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{D^b_m}(X)}(R_X(\cdot), R_X(\cdot))$$

of [2] (3.1) is an isomorphism.

Proof. This theorem is proved in section 8. More precisely, the bifunctors $\otimes_X$ and $\underline{\mathrm{Hom}}$ are constructed in section 8, and all their properties are proved there except for condition (i)(b) of Definition 3.2.10. But this last condition follows from the fact that the functor R_X is conservative (and that the analogous result is true in $\mathrm{D^b_m}(X)$).

□

4 Easy stabilities

The proof of Theorem 3.2.9 will require us to show that the full subcategories $\mathrm{Perv}_{mf}(X) \subset \mathrm{Perv}_m(X)$ are preserved by a certain number of sheaf operations. Here we list the easier such results.

Proposition 4.1 *Let $f : X \longrightarrow Y$ be a morphism of k-schemes.*

(i) If f is smooth of relative dimension d, then the exact functor $f^*[d] : \mathrm{Perv}_m(Y) \longrightarrow \mathrm{Perv}_m(X)$ sends $\mathrm{Perv}_{mf}(Y)$ to $\mathrm{Perv}_{mf}(X)$.

(ii) If f is proper, then, for every $k \in \mathbb{Z}$, the functor ${}^p\mathrm{H}^k f_* : \mathrm{Perv}_m(X) \longrightarrow \mathrm{Perv}_m(Y)$ sends $\mathrm{Perv}_{mf}(X)$ to $\mathrm{Perv}_{mf}(Y)$.

Proof. Point (i) follows from the fact that the functor $f^*[d]$ is exact (see Proposition 2.2.2) and sends pure perverse sheaves to pure perverse sheaves (by [6] 5.1.14). Point (ii) is Proposition 3.9 of Huber's paper [14]. (This proposition is stated for f smooth, but its proof doesn't use the smoothness of f.)

□

Proposition 4.2 *Let $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$.*

(i) The Poincaré-Verdier duality functor $D_X : \mathrm{Perv}_m(X)^{\mathrm{op}} \longrightarrow \mathrm{Perv}_m(X)$ sends $\mathrm{Perv}_{mf}(X)^{\mathrm{op}}$ to $\mathrm{Perv}_{mf}(X)$.

(ii) The external tensor product functor $\boxtimes : \mathrm{Perv}_m(X) \times \mathrm{Perv}_m(Y) \longrightarrow \mathrm{Perv}_m(X \times Y)$ *sends* $\mathrm{Perv}_{mf}(X) \times \mathrm{Perv}_{mf}(Y)$ *to* $\mathrm{Perv}_{mf}(X \times Y)$.

(iii) The Tate twist functor $(1) : \mathrm{Perv}_m(X) \longrightarrow \mathrm{Perv}_m(X)$, $K \longmapsto K(1)$ *sends* $\mathrm{Perv}_{mf}(X)$ *to* $\mathrm{Perv}_{mf}(X)$.

Proof. This follows from the fact all these functors are exact (see Proposition 2.2.2) and send pure perverse sheaves to pure perverse sheaves (see [6] 5.1.14).

□

In particular, by deriving trivially the functors above, we get :

(i) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, an exact functor $D_X : \mathrm{D^b\,Perv}_{mf}^{\mathrm{op}}(X) \longrightarrow \mathrm{D^b\,Perv}_{mf}(X)$ and an isomorphism $D_X^2 \simeq \mathrm{id}$, and also an exact functor $\mathrm{D^b\,Perv}_{mf}(X) \longrightarrow \mathrm{D^b\,Perv}_{mf}(X)$, $K \longmapsto K(1)$.

(ii) For every $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$, an exact functor $\boxtimes : \mathrm{D^b\,Perv}_{mf}(X) \times \mathrm{D^b\,Perv}_{mf}(Y) \longrightarrow \mathrm{D^b\,Perv}_{mf}(X \times Y)$, satisfying the same properties of commutativity and associativity as the external tensor product on the categories $\mathrm{D^b_c}$.

Moreover, these functors correspond to the usual ones on $\mathrm{D^b_m}(X)$ by the realization functor (by Proposition 2.4.1).

Note that, by Proposition 3.2.2 and Theorem 3.2.4 of [6], the 2-functor $X \longmapsto \mathrm{Perv}(X)$ is a stack for the étale topology on X. We have the following easy result :

Proposition 4.3 *The categories* $\mathrm{Perv}_h(U)$ *(resp.* $\mathrm{Perv}_m(U)$*, resp.* $\mathrm{Perv}_{mf}(U)$*) define a substack of* $X \longmapsto \mathrm{Perv}(X)$.

Proof. As $\mathrm{Perv}_h(U)$ (resp. $\mathrm{Perv}_m(U)$, resp. $\mathrm{Perv}_{mf}(U)$) is a full subcategory of $\mathrm{Perv}(U)$ for every U, we only need to show the following fact : If K is an object of $\mathrm{Perv}(X)$ and if there exists an étale cover $(u_i : U_i \longrightarrow X)_{i \in I}$ of X such that $u_i^* K$ is in $\mathrm{Perv}_h(U_i)$ (resp. $\mathrm{Perv}_m(U_i)$, resp. $\mathrm{Perv}_{mf}(U_i)$) for every $i \in I$, then K is in $\mathrm{Perv}_h(X)$ (resp. $\mathrm{Perv}_m(X)$, resp. $\mathrm{Perv}_{mf}(X)$).

We first treat the case of Perv_h and Perv_m. We may assume that I is finite and that the U_i are affine. For all $i, j \in I$, we denote the fiber product of u_i and u_j by $u_{ij} : U_i \times_X U_j \longrightarrow X$. Then, as Perv is a stack, we have an exact sequence in $\mathrm{Perv}(X)$:

$$0 \longrightarrow K \longrightarrow \bigoplus_{i \in I} u_{i*} u_i^* K \longrightarrow \bigoplus_{i,j \in I} u_{ij*} u_{ij}^* K.$$

As the last two terms are in $\mathrm{Perv}_h(X)$ (resp. $\mathrm{Perv}_m(X)$) by assumption, and as $\mathrm{Perv}_h(X)$ (resp. $\mathrm{Perv}_m(X)$) is a full abelian subcategory of $\mathrm{Perv}(X)$ by Proposition 2.6.2, the perverse sheaf K is also an object of $\mathrm{Perv}_h(X)$ (resp. $\mathrm{Perv}_m(X)$).

We now treat the case of $\mathrm{Perv}_{mf}(X)$. Let $a \in \mathbb{Z}$. We need to construct a subobject L of K such that L is of weight $\leq a$ and K/L is of weight $> a$. For every $i \in I$, we set $L_i = W_a(u_i^* K)$, where W is the weight filtration on K_i. By the uniqueness of the weight filtration, the L_i glue to a subobject L of K. As we can test weights on an étale cover of X (for example by Theorem 5.2.1 and 5.1.14(iii) of [6]), this L satisfies the required conditions.

□

Lemma 4.4 *Let $i : Y \longrightarrow X$ be a closed immersion, and let $K \in \mathrm{Ob}\,\mathrm{Perv}_m(Y)$. Then K is in $\mathrm{Perv}_{mf}(Y)$ if and only if i_*K is in $\mathrm{Perv}_{mf}(X)$.*

Proof. If K is in $\mathrm{Perv}_{mf}(Y)$, then i_*K is in $\mathrm{Perv}_{mf}(X)$ by Proposition 4.1(ii).

Conversely, assume that i_*K is in $\mathrm{Perv}_{mf}(X)$. Let $a \in \mathbb{Z}$. We want to show that there exists a subobject K' of K (in $\mathrm{Perv}_m(Y)$) such that K' is of weight $\leq a$ and K/K' is of weight $\geq a+1$. By the assumption, there exists a subobject $L' \subset i_*K$ (in $\mathrm{Perv}_m(X)$) such that L' is of weight $\leq a$ and $L'' := (i_*K)/L'$ is of weight $\geq a+1$. Let j be the inclusion of the complement of Y in X. Then the functor j^* is t-exact, so, applying j^* to the exact sequence $0 \longrightarrow L' \longrightarrow i_*K \longrightarrow L'' \longrightarrow 0$, we get an exact sequence $0 \longrightarrow j^*L' \longrightarrow 0 \longrightarrow j^*L'' \longrightarrow 0$ of mixed perverse sheaves on $X - Y$. This implies that $j^*L' = j^*L'' = 0$, so the adjunction morphisms $i_*i^!L' \longrightarrow L' \longrightarrow i_*i^*L'$ and $i_*i^!L'' \longrightarrow L'' \longrightarrow i_*i^*L''$ are isomorphisms. In particular, the mixed complexes $i^*L' = i^!L'$ and $i^*L'' = i^!L''$ are perverse. Let $K' = i^*L'$. We have just seen that K' is perverse, and the weights of K' are $\leq a$ (see the remark after Definition 3.3 of [14]). Also, we have an exact triangle $K' = i^*L' \longrightarrow K \longrightarrow i^*L'' = i^!L'' \xrightarrow{+1}$, which is actually an exact sequence in $\mathrm{Perv}_m(Y)$, so the canonical map $K' \longrightarrow K$ is injective, and $K/K' \simeq i^!L''$, which is of weight $\geq a+1$ (by the same remark in [14]).

□

5 Beilinson's construction of unipotent nearby cycles

In this section, we review Beilinson's construction of the unipotent nearby and vanishing cycles functors from [3]. There are two reasons to do this :

(1) We will want to define nearby cycles for horizontal perverse sheaves, and to apply known theorems (about weights for example). The easiest way to do this is to use Deligne's generic base change theorem, but this might cause technical problems if we use the original construction of nearby cycles (from SGA 7 I and XIII), which involves direct images by morphisms that are not of finite type.

(2) We will need some of Beilinson's auxiliary functors anyway to construct a left adjoint of i_* for i a closed immersion.

All the proofs of the results in this section can be found in [3] (see also [22]).

5.1 Unipotent nearby cycles

Fix a base field k, let X be a k-scheme, and let $f : X \longrightarrow \mathbb{A}^1_k$ be a morphism. We write $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, $U = X \times_{\mathbb{A}^1} \mathbb{G}_m \xrightarrow{j} X$ and $Y = X \times_{\mathbb{A}^1} \{0\} \xrightarrow{i} X$.

We have an exact sequence

$$1 \longrightarrow \pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1) \longrightarrow \pi_1(\mathbb{G}_m, 1) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1,$$

which is split by the morphism coming from the unit section of $\mathbb{G}_m$. If k is of characteristic 0, then $\pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1) \simeq \widehat{\mathbb{Z}}(1)$; if k is of characteristic $p > 0$, then $\pi_1^{\mathrm{geom},(p)}(\mathbb{G}_m, 1) \simeq \widehat{\mathbb{Z}}^{(p)}(1)$. In both cases, we get a projection $t_\ell : \pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1) \longrightarrow \mathbb{Z}_\ell(1)$. We also denote by $\chi : \mathrm{Gal}(\overline{k}/k) \longrightarrow \widehat{\mathbb{Z}}_\ell$ the ℓ-adic cyclotomic character.

Let $\Psi_f : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y_{\overline{k}})$ and $\Phi_f : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y_{\overline{k}})$ be the nearby and vanishing cycles functors defined in SGA 7 Exposé XVIII, shifted by -1 so that they will be t-exact for the perverse t-structure. (See Corollary 4.5 of Illusie's [17], and note that the dimension function we use on U is shifted by $+1$ when compared with Illusie's dimension function.) We denote by T a topological generator of $\pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1)$ or $\pi_1^{\mathrm{geom},(p)}(\mathbb{G}_m)$ (depending on the characteristic of k). We have a functorial exact triangle $\Psi_f \xrightarrow{T-1} \Psi_f \longrightarrow i^* j_* \xrightarrow{+1}$.

Proposition 5.1.1 *There exists a functorial T-equivariant direct sum decomposition $\Psi_f = \Psi_f^u \oplus \Psi_f^{nu}$ such that, for every $K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)$, $T - 1$ acts nilpotently on $\Psi_f^u(K)$ and invertibly on $\Psi_f^{nu}(K)$.*

In particular, the functorial exact triangle $\Psi_f \xrightarrow{T-1} \Psi_f \longrightarrow i^* j_* \xrightarrow{+1}$ induces a functorial exact triangle $\Psi_f^u \xrightarrow{T-1} \Psi_f^u \longrightarrow i^* j_* \xrightarrow{+1}$.

The functor Ψ_f^u is called the *unipotent nearby cycles functor*.

Proof. It suffices to prove that, for every $K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)$, there exists a nonzero polynomial P (with coefficients in the coefficient field E that we are using for the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$) such that $P(T)$ acts by 0 on $\Psi_f(K)$. (The rest is standard linear algebra.) As we know that Ψ_f sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y_{\overline{k}})$ (i.e. preserves constructibility), this follows from the fact that, for every $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y_{\overline{k}})$, the ring of endomorphisms of L is finite-dimensional (over the same coefficient field E). To prove this fact, we use induction on the dimension of X to reduce to the case where the cohomology sheaves of L are local systems, and then it is trivial.

$\square$

Let $K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)$. Then $T : \Psi_f^u K \longrightarrow \Psi_f^u K$ is unipotent, so there exists a unique nilpotent $N : \Psi_f^u K \longrightarrow \Psi_f^u K(-1)$ such that $T = \exp(t_\ell(T)N)$ on $\Psi_f^u K$. The operator N is usually called the “logarithm of the unipotent part of the monodromy”. We get a functorial exact triangle $\Psi_f^u \xrightarrow{N} \Psi_f^u(-1) \longrightarrow i^*j_* \xrightarrow{+1}$.

5.2 Beilinson’s construction

Now we introduce the unipotent local systems that are used in Beilinson’s construction of Ψ_f^u.

Definition 5.2.1 For every $i \geq 0$, we define a E-local system $\mathscr{L}_i$ on $\mathbb{G}_m$ in the following way : the stalk $\mathscr{L}_{i,1}$ of $\mathscr{L}_i$ at $1 \in \mathbb{G}_m(k)$ is the E-vector space E^{i+1}, on which an element $u \rtimes \sigma$ of $\pi_1(\mathbb{G}_m, 1) \simeq \widehat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}(\overline{k}/k)$ acts by $\exp(t_\ell(u)N)\,\mathrm{diag}(1, \chi(\sigma)^{-1}, \ldots, \chi(\sigma)^{-i})$, where $\mathrm{diag}(x_0, \ldots, x_i)$ is the diagonal matrix with diagonal entries $x_0, \ldots, x_i$ and N is the Jordan block $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$.

If $i \leq j$, we have an obvious injection $\alpha_{i,j} : \mathscr{L}_i \longrightarrow \mathscr{L}_j$ and an obvious surjection $\beta_{j,i} : \mathscr{L}_j \longrightarrow \mathscr{L}_i(i-j)$.

Note that $\mathscr{L}_i^\vee \simeq \mathscr{L}_i(i)$, so (by the calculation at the end of section 2.1) we have $D_U(\mathscr{L}_i) \simeq \mathscr{L}_i(i-1)[-2]$, and $D_U(\alpha_{i,j})$ corresponds by this isomorphism to $\beta_{j,i}(j-1)[-2]$.

Notation 5.2.2 If $\mathscr{L}$ is a lisse sheaf on $\mathbb{G}_m$ and K is a perverse sheaf on U, then the complex $K \otimes f^*\mathscr{L}$ is also perverse. We denote it by $K \otimes \mathscr{L}$.

We start with the construction of Ψ_f^u.

Proposition 5.2.3 *Let $K \in \mathrm{Ob}\,\mathrm{Perv}(U)$.*

(i) For every $a \in \mathbb{N}$, we have a canonical isomorphism

$$i_* \operatorname{Ker}(N^{a+1}, \Psi_f^u K) \xrightarrow{\sim} \operatorname{Ker}(j_!(K \otimes \mathscr{L}_a) \longrightarrow j_*(K \otimes \mathscr{L}_a)) = {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathscr{L}_a).$$

In particular, if a is big enough, we get an isomorphism $i_*\Psi_f^u K \xrightarrow{\sim} {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathscr{L}_a)$.

(ii) For every $a \in \mathbb{N}$ such that $N^{a+1} = 0$ on $\Psi_f^u K$, the following diagram is commutative :

$$\begin{tikzcd}
0 \arrow[r] & i_*\Psi_f^u K \arrow[r] \arrow[d, equal] & j_!(K \otimes f^*\mathscr{L}_a) \arrow[r] \arrow[d, "\alpha_{a,a+1}"] & j_*(K \otimes f^*\mathscr{L}_a) \arrow[d, "\alpha_{a,a+1}"] \\
0 \arrow[r] & i_*\Psi_f^u K \arrow[r] \arrow[d, "N"] & j_!(K \otimes f^*\mathscr{L}_{a+1}) \arrow[r] \arrow[d, "\beta_{a,a+1}"] & j_*(K \otimes f^*\mathscr{L}_{a+1}) \arrow[d, "\beta_{a,a+1}"] \\
0 \arrow[r] & i_*\Psi_f^u K(-1) \arrow[r] & j_!(K \otimes f^*\mathscr{L}_a)(-1) \arrow[r] & j_*(K \otimes f^*\mathscr{L}_a)(-1)
\end{tikzcd}$$

(iii) Let $a, b \in \mathbb{N}$ be such that $N^{a+1} = N^{b+1} = 0$ on $\Psi_f^u K$. Then there is a canonical isomorphism

$$\mathrm{Ker}(j_!(K \otimes f^*\mathscr{L}_b) \longrightarrow j_*(K \otimes f^*\mathscr{L}_b))(-a-1) \xrightarrow{\sim} \mathrm{Coker}(j_!(K \otimes f^*\mathscr{L}_a) \longrightarrow j_*(K \otimes f^*\mathscr{L}_a))$$

induced by the connecting map coming from the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & j_!(K \otimes f^*\mathscr{L}_a) \arrow[r, "\alpha_{a,a+b+1}"] \arrow[d] & j_!(K \otimes f^*\mathscr{L}_{a+b+1}) \arrow[r, "\beta_{b,a+b+1}"] \arrow[d] & j_!(K \otimes f^*\mathscr{L}_b)(-a-1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & j_*(K \otimes f^*\mathscr{L}_a) \arrow[r, "\alpha_{a,a+b+1}"] & j_*(K \otimes f^*\mathscr{L}_{a+b+1}) \arrow[r, "\beta_{a,a+b+1}"] & j_*(K \otimes f^*\mathscr{L}_b)(-a-1) \arrow[r] & 0 \end{tikzcd}$$

Moreover, the morphism

$$^p\mathrm{H}^0 i^* j_*(K \otimes \mathscr{L}_a) \longrightarrow {}^p\mathrm{H}^0 i^* j_*(K \otimes \mathscr{L}_{a+b+1})$$

induced by $\alpha_{a,a+b+1}$ is zero.

Note in particular that we can use this construction to see Ψ_f^u as a functor from $\mathrm{Perv}(U)$ to $\mathrm{Perv}(Y)$ (and not just to the category of $\mathrm{Gal}(\overline{k}/k)$-equivariant objects in $\mathrm{Perv}(Y)$).

Corollary 5.2.4 *For every $K \in \mathrm{Perv}(U)$, we have a canonical isomorphism $D(\Psi_f^u K) \simeq \Psi_f^u(DK)(-1)$.*

Corollary 5.2.5 *For every $a \in \mathbb{N}$, we define a functor $C_a^\bullet : \mathrm{Perv}(U) \longrightarrow C^{[0,1]}(\mathrm{Perv}(X))$ (where the second category is the category of complexes concentrated in degrees 0 and 1) by*

$$C_a^\bullet(K) = (j_!(K \otimes \mathscr{L}_a) \longrightarrow j_*(K \otimes \mathscr{L}_a)).$$

With the transition morphisms given by the $\alpha_{a,b}$, the family $(C_a)_{a \geq 0}$ becomes an inductive system of functors.

Then we have canonical isomorphisms

$$i_*\Psi_f^u \simeq \varinjlim_{a \in \mathbb{N}} \mathrm{H}^0(C_a^\bullet)$$

and

$$0 = \varinjlim_{a \in \mathbb{N}} \mathrm{H}^1(C_a^\bullet).$$

Remark 5.2.6 If we use the Ind-category $\mathrm{Ind}(\mathrm{Perv}(X))$ of $\mathrm{Perv}(X)$ (see for example Chapter 6 of Kashiwara and Schapira's book [19], and Theorem 8.6.5 of the same book for the fact that this category is abelian), then we can reformulate this corollary in the following way : We have a canonical isomorphism

$$i_*\Psi_f^u \xrightarrow{\sim} \varinjlim_{a \in \mathbb{N}} C_a^\bullet$$

of functors $\mathrm{Perv}(X) \longrightarrow \mathrm{D^b}\,\mathrm{Ind}(\mathrm{Perv}(X))$. Note that, by Theorem 15.3.1 of [19], the obvious functor $\mathrm{D^b}\,\mathrm{Perv}(X) \longrightarrow \mathrm{D^b}\,\mathrm{Ind}(\mathrm{Perv}(X))$ is fully faithful (and its essential image is the full subcategory of complexes with all their cohomology objects in $\mathrm{Perv}(X)$). So $\varinjlim_{a\in\mathbb{N}} C_a^\bullet$ factors through the category $\mathrm{D^b}\,\mathrm{Perv}(X)$.

We now give the definition of the maximal extension functor.

Let $K \in \mathrm{Ob}\,\mathrm{Perv}(U)$. For each $a \geq 1$, we have a commutative diagram :

$$\begin{tikzcd}
j_!(K\otimes f^*\mathscr{L}_a) \arrow[r] \arrow[d, "\beta_{a,a+1}"'] & j_*(K\otimes f^*\mathscr{L}_a) \arrow[d, "\beta_{a,a+1}"] \\
j_!(K\otimes f^*\mathscr{L}_{a-1})(-1) \arrow[r] & j_*(K\otimes f^*\mathscr{L}_{a-1})(-1)
\end{tikzcd}$$

We write $\gamma_{a,a-1} : j_!(K\otimes f^*\mathscr{L}_a) \longrightarrow j_*(K\otimes f^*\mathscr{L}_{a-1})(-1)$ for the diagonal map in this diagram.

Proposition 5.2.7 *(i) For $a \in \mathbb{N}$ big enough, the (injective) map $\mathrm{Ker}(\gamma_{a,a-1}) \longrightarrow \mathrm{Ker}(\gamma_{a+1,a})$ induced by $\alpha_{a,a+1} : j_!(K\otimes f^*\mathscr{L}_a) \longrightarrow j_!(K\otimes f^*\mathscr{L}_{a+1})$ is an isomorphism. We write $\Xi_f K$ for the direct limit of the $\mathrm{Ker}(\gamma_{a,a-1})$. This defines a left exact functor from $\mathrm{Perv}(U)$ to $\mathrm{Perv}(X)$, called the* maximal extension functor.

Moreover, if a and b are big enough, then the map $\mathrm{Coker}(\gamma_{a,a-1}) \longrightarrow \mathrm{Coker}(\gamma_{a+b,a+b-1})$ induced by $\alpha_{a-1,a+b-1}(-1)$ is zero. In particular, we have

$$\varinjlim_a \mathrm{Coker}(\gamma_{a-1,a}) = 0.$$

(ii) We have a functorial isomorphism $D_X \circ \Xi_f \simeq \Xi_f \circ D_U$ and two functorial exact sequences

$$0 \longrightarrow j_! \xrightarrow{\alpha} \Xi_f \longrightarrow i_*\Psi_f^u(-1) \longrightarrow 0$$

and

$$0 \longrightarrow i_*\Psi_f^u \longrightarrow \Xi_f \xrightarrow{\beta} j_* \longrightarrow 0,$$

dual of each other, in which the maps are the obvious ones. For example, in the first sequence, the map $j_!K \longrightarrow \Xi_f K$ is induced by the injection $\alpha_{0,a} : j_!K = j_!(K\otimes f^*\mathscr{L}_0) \longrightarrow j_!(K\otimes f^*\mathscr{L}_a)$, and the map $\Xi_f \longrightarrow i_*\Psi_f^u(-1)$ is induced by the commutative square

$$\begin{tikzcd}
j_!(\cdot\otimes f^*\mathscr{L}_a) \arrow[r, "\gamma_{a,a-1}"] \arrow[d, "\beta_{a,a-1}"'] & j_*(\cdot\otimes f^*\mathscr{L}_{a-1})(-1) \arrow[d, "\mathrm{id}"] \\
j_!(\cdot\otimes f^*\mathscr{L}_{a-1})(-1) \arrow[r] & j_*(\cdot\otimes f^*\mathscr{L}_{a-1})(-1)
\end{tikzcd}$$

Remark 5.2.8 As in Remark 5.2.6, we can deduce from (i) of the proposition a natural isomorphism

$$\Xi_f K \xrightarrow{\sim} \varinjlim_{a\in\mathbb{N}} (j_!(K\otimes f^*\mathscr{L}_a) \xrightarrow{\gamma_{a,a-1}} j_*(K\otimes\mathscr{L}_{a-1})(-1))$$

in $\mathrm{D^b}\,\mathrm{Ind}(\mathrm{Perv}(X))$.

The next functor that we construct is the *unipotent vanishing cycles functor* Φ_f^u. It is not very hard to show that this functor is isomorphic to the direct summand of the usual vanishing cycles functor on which the monodromy acts unipotently, but we will not need this, so we will just use the following proposition as the definition of Φ_f^u.

Proposition 5.2.9 *(i) The complex of exact endofunctors of* $\mathrm{Perv}(X)$ *defined by*

$$j_!j^* \xrightarrow{\alpha+\eta} \Xi_f j^* \oplus \mathrm{id} \xrightarrow{\beta-\varepsilon} j_*j^*$$

in degrees -1, 0 *and* 1, *where* $\eta : j_!j^* \longrightarrow \mathrm{id}$ *is the counit of the adjunction* $(j_!, j^*)$ *and* $\varepsilon : \mathrm{id} \longrightarrow j_*j^*$ *is the unit of the adjunction* (j^*, j_*), *has its cohomology concentrated in degree* 0 *and with support in* Y.

We define an exact functor $\Phi_f^u : \mathrm{Perv}(X) \longrightarrow \mathrm{Perv}(Y)$ *by setting* $i_*\Phi_f^u$ *to be the* H^0 *of this complex.*

(ii) We denote by $\mathrm{can} : \Psi_f^u j^*K \longrightarrow \Phi_f^u K$ *the functorial map defined by* $i_*\Psi_f^u j^*K \longrightarrow \Xi_f j^*K$, *and by* $\mathrm{var} : \Phi_f^u K \longrightarrow \Psi_f^u j^*K(-1)$ *the functorial map defined by* $\Xi_f j^*K \longrightarrow \Psi_f^u K(-1)$.

Then $\mathrm{var}\circ\mathrm{can} = N$ *and* $\mathrm{can}(-1)\circ\mathrm{var} = N$.

(iii) We have a functorial isomorphism $D\circ\Phi_f^u \simeq \Phi_f^u\circ D$, *and the duality exchanges* can *and* var.

(iv) There are canonical isomorphisms $\mathrm{Ker}(\mathrm{can}) = {}^p\mathrm{H}^{-1}i^*K$ *and* $\mathrm{Coker}(\mathrm{can}) = {}^p\mathrm{H}^0 i^*K$.

Dually, we have canonical isomorphisms $\mathrm{Ker}(\mathrm{var}) = {}^p\mathrm{H}^0 i^!K$ *and* $\mathrm{Coker}(\mathrm{var}) = {}^p\mathrm{H}^1 i^!K$.

Finally, we will need the functor that M. Saito calls Ω_f.

Proposition 5.2.10 *The functor* $\beta+\varepsilon : \Xi_f j^* \oplus \mathrm{id} \longrightarrow j_*j^*$ *is surjective. Its kernel* Ω_f *is an exact endofunctor of* $\mathrm{Perv}(X)$, *and we have functorial exact sequences*

$$0 \longrightarrow j_!j^* \xrightarrow{\alpha-\eta} \Omega_f \longrightarrow i_*\Phi_f^u \longrightarrow 0$$

and

$$0 \longrightarrow i_*\Psi_f^u j^* \longrightarrow \Omega_f \longrightarrow \mathrm{id} \longrightarrow 0,$$

in which the unmarked maps are the obvious ones.

6 Nearby cycles and mixed perverse sheaves

The goal of this section is to show that the functor of unipotent nearby cycles preserves the categories $\mathrm{Perv}_{mf}(X)$ and to deduce that these categories are also preserved by the functors ${}^p\mathrm{H}^k f_*$, for every morphism f of $\mathbf{Sch}/k$. The main tool is Deligne's weight-monodromy theorem from [9].

We will also give an application to the direct image functor by a closed immersion i, which then allows us to construct the functor i^* on the categories $\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}$.

6.1 Nearby cycles on horizontal perverse sheaves

We assume again that k is a field that is finitely generated over its prime field. Let X be a k-scheme and $f : X \longrightarrow \mathbb{A}^1$ be a morphism. We write $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, $U = X \times_{\mathbb{A}^1} \mathbb{G}_m \xrightarrow{j} X$ and $Y = X \times_{\mathbb{A}^1} \{0\} \xrightarrow{i} X$.

We will use the constructions of section 5.2 to *define* the functors Ψ^u_f, Φ^u_f, Ξ^u_f and Ω^u_f on the category $\mathrm{Perv}_h(U)$. As the lisse sheaves $\mathscr{L}_a$ on $\mathbb{G}_m$ are clearly horizontal and as we have the six operations on the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}$, this makes perfect sense, and it is compatible with the usual constructions via the functor $\eta^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{h}} \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}$. Note also that these functors respect the subcategories of mixed perverse sheaves, because all the functors used in their definition respect the categories of mixed complexes.

The point of doing this is that now only finite type schemes and constructible complexes are involved in the definition of Ψ^u_f, so we can use Deligne's generic base change to compare our situation with the situation over closed points of some ring $A \in \mathscr{U}$. We will see an example of this in the next section.

6.2 The relative monodromy filtration

We recall the definition of the relative monodromy filtration, due to Deligne.

Proposition 6.2.1 *(See Propositions 1.6.1 and 1.6.13 of [9].) Let K be an object in some abelian category, and suppose that we have a finite increasing filtration W on K and a nilpotent endomorphism N of K. Then there exists at most one finite increasing filtration M on K such that $N(M_i) \subset M_{i-2}$ for every $i \in \mathbb{Z}$ and that, for every $k \in \mathbb{N}$ and every $i \in \mathbb{Z}$, the morphism N^k induces isomorphisms*

$$\mathrm{Gr}^M_{i+k}\,\mathrm{Gr}^W_i\,K \xrightarrow{\sim} \mathrm{Gr}^M_{i-k}\,\mathrm{Gr}^W_i\,K.$$

Moreover, if W is trivial (that is, if there exists $i \in \mathbb{Z}$ such that $\mathrm{Gr}^W_i\,K = K$), then the filtration M always exists.

The filtration M is called the *monodromy filtration on K relative to the filtration W*. If W is trivial, it is simply called the *monodromy filtration on K*.

We will use the following theorem, which is a close relative of Theorem 1.8.4 of Deligne's Weil II paper [9].

Theorem 6.2.2 *Let $K \in \operatorname{Ob}\operatorname{Perv}_{mf}(U)$, and let W be the weight filtration on K. Then the monodromy filtration M on $\Psi_f^u K$ relative to the filtration $\Psi_f^u W$ exists, and $\operatorname{Gr}_i^M \Psi_f^u K$ is pure of weight $i-1$ for every $i \in \mathbb{Z}$. In particular, $\Psi_f^u K$ is an object of $\operatorname{Perv}_{mf}(Y)$.*

Lemma 6.2.3 *In the situation of the theorem, suppose that K is pure. Then the monodromy filtration M on $\Psi_f^u K$ (which always exists) is such that $\operatorname{Gr}_i^M \Psi_f^u K$ is pure of weight $i-1$ for every $i \in \mathbb{Z}$.*

Proof. Let w be the weight of K. Let $(A, \mathscr{X}, u)$ be an object of $\mathscr{U}X$ such that K comes by restriction from a shifted perverse sheaf $\mathscr{K}[-d]$ on $\mathscr{X}$, where $d = \dim \operatorname{Spec} A$. Fix $a \in \mathbb{N}$ such that $N^{a+1} = 0$ on $\Psi_f^u K$. After shrinking $\operatorname{Spec} A$ and $\mathscr{X}$ if necessary, we may assume that :

- The morphism $f : X \longrightarrow \mathbb{A}_k^1$ extends to a morphism $F : \mathscr{X} \longrightarrow \mathbb{A}_A^1$. We write $\mathscr{U} = \mathscr{X} \times_{\mathbb{A}_A^1} \mathbb{G}_{A,m} \xrightarrow{J} \mathscr{X}$ and $\mathscr{Y} = \mathscr{X} \times_{\mathbb{A}_A^1} \{0\} \xrightarrow{I} \mathscr{X}$.
- The lisse sheaves $\mathscr{L}_0, \dots, \mathscr{L}_{a+1}$ all extend to $\mathbb{G}_{m,A}$. (In fact we can get all the $\mathscr{L}_b$ as soon as we have $\mathscr{L}_1$, because they are the symmetric powers of $\mathscr{L}_1$.)
- For every closed point x of $\operatorname{Spec} A$, the restriction of $\mathscr{K}$ to $\mathscr{X}_x$ is still perverse, and it is pure of weight $w + d$.
- The formation of the complexes $J_!(\mathscr{K} \otimes \mathscr{L}_b)$ and $J_*(\mathscr{K} \otimes \mathscr{L}_b)$, for $b \in \{a, a+1\}$, is compatible with every base change $x \longrightarrow \operatorname{Spec} A$, where x is a closed point of $\operatorname{Spec} A$. Moreover, if $\mathscr{L}$ is any subquotient of ${}^p\mathrm{H}^{-1} I^* J_*(\mathscr{K} \otimes \mathscr{L}_a)$ (in the category $\operatorname{Perv}(\mathscr{X}, E)$), then its restrictions to the fibers of $\mathscr{X}$ above all the closed points of $\operatorname{Spec} A$ are still perverse.

Indeed, the first two points are standard, and the last two follow from Deligne's generic base change theorem (see SGA 4 1/2, [Th. finitude], Théorème 1.9) and from the purity theorem.

Let $\mathscr{K}' = {}^p\mathrm{H}^{-1} I^* J_*(\mathscr{K} \otimes \mathscr{L}_a)$, and let M the monodromy filtration on $\mathscr{K}'$ induced by N. By the conditions above (and (i) of Proposition 5.2.3), for every closed point x of $\operatorname{Spec} A$, the restriction of $\mathscr{K}'$ to $\mathscr{X}_x$ is a subobject of $\Psi_f^u \mathscr{K}_x$, and the restriction of M is the monodromy filtration. The result about the weights of the graded pieces of the monodromy filtration over the spectrum of a finite field (such as x) is known by Theorem 5.1.2 of Beilinson and Bernstein's paper [5] (where it is attributed to Gabber). So we get the conclusion by definition of the weights on horizontal sheaves.

□

Proof of the theorem. We reason by induction on the length of the filtration W. If K is pure (i.e., if W is trivial), then the conclusion of the theorem is proved in Lemma 6.2.3.

Now assume that W is of length ≥ 2, and that we know the result for every object of $\mathrm{Perv}_{mf}(U)$ with a shorter weight filtration. Let $a \in \mathbb{Z}$ be such that $W_a K = K$ and $\mathrm{Gr}_a^W K \neq 0$. By the induction hypothesis, we know the theorem for $W_{a-1}K$ and $\mathrm{Gr}_a^W K$. Write $L = \Psi_f^u K$, and let F be the filtration $\Psi_f^u W$ on L. By Theorem 2.20 of Steenbrink and Zucker's paper [28], the filtration M exists if and only if, for every $i \geq 1$, we have :

$$N^i(L) \cap F_{a-1}L(-i) \subset N^i(F_{a-1}L) + M_{a-i-1}F_{a-1}L(-i).$$

This is equivalent to saying that

$$(N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L) \subset [M_{a-i-1}F_{a-1}L/(M_{a-i-1}F_{a-1}L \cap N^i(F_{a-1}L))](-i).$$

As the filtration M on $F_{a-1}L$ is the weight filtration up to a shift, the inclusion above is also equivalent to the fact that $(N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L)$ is of weight $\leq a + i - 2$. Observe that $(N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L)$ is the kernel of the map $F_{a-1}L(-i)/N^i(F_{a-1}L) \longrightarrow L(-i)/N^i(L)$, so applying the snake lemma to the commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & F_{a-1}L \arrow[r] \arrow[d, "N^i"] & L \arrow[r] \arrow[d, "N^i"] & \mathrm{Gr}_a^F L \arrow[r] \arrow[d, "N^i"] & 0 \\
0 \arrow[r] & F_{a-1}L(-i) \arrow[r] & L(-i) \arrow[r] & \mathrm{Gr}_a^F L(-i) \arrow[r] & 0
\end{tikzcd}$$

gives a surjection

$$\mathrm{Ker}(N^i : \mathrm{Gr}_a^F L \longrightarrow \mathrm{Gr}_a^F L(-i)) \longrightarrow (N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L).$$

But as $\mathrm{Gr}_a^F L = \Psi_f^u \mathrm{Gr}_a^W K$, we know by (1.6.4) of [9] that $\mathrm{Ker}(N^i : \mathrm{Gr}_a^F L \longrightarrow \mathrm{Gr}_a^F L(-i))$ is of weight $\leq a+i-2$ (or more correctly, we can deduce this from the result we cited and Deligne's generic base change theorem, as in the proof of Lemma 6.2.1), and hence all its quotients are. This proves the existence of the filtration M on L.

Finally, we prove that $\mathrm{Gr}_i^M L$ is pure of weight $i - 1$ for every $i \in \mathbb{Z}$. The two properties defining M in Proposition 6.2.1 stay true if we intersect M with $F_{a-1}L$ or take the quotient filtration in $\mathrm{Gr}_a^F L$, so this gives the relative monodromy filtration on $F_{a-1}L$ and $\mathrm{Gr}_a^F L$ (by the uniqueness statement). Hence we get exact sequences

$$0 \longrightarrow \mathrm{Gr}_i^M F_{a-1}L \longrightarrow \mathrm{Gr}_i^M L \longrightarrow \mathrm{Gr}_i^M \mathrm{Gr}_a^F L \longrightarrow 0,$$

and so the fact that $\mathrm{Gr}_i^M L$ is pure of weight $i - 1$ follows from the induction hypothesis.

□

6.3 Cohomological direct image functors and weights

Corollary 6.3.1 *Let $f : X \longrightarrow Y$ be a morphism of k-schemes. Then the functors ${}^p\mathrm{H}^i f_*, {}^p\mathrm{H}^i f_! : \mathrm{Perv}_m(X) \longrightarrow \mathrm{Perv}_m(Y)$ send $\mathrm{Perv}_{mf}(X)$ to $\mathrm{Perv}_{mf}(Y)$.*

Proof. As Poincaré-Verdier duality exchanges ${}^p\mathrm{H}^i f_*$ and ${}^p\mathrm{H}^{-i} f_!$ and preserves the categories Perv_{mf}, it suffices to treat the case of ${}^p\mathrm{H}^i f_*$.

By Nagata's compactification theorem (see for example Conrad's paper [8]), we can write $f = gj$, with $j : X \longrightarrow X'$ an open embedding and $g : X' \longrightarrow Y$ proper. After replacing X' by the blowup of $X' - X$ in X', we may assume that the ideal of $X' - j(X)$ is invertible. Then j is affine, so j_* is t-exact, so we have ${}^p\mathrm{H}^i f_* = ({}^p\mathrm{H}^i g_*) \circ j_*$ for every $i \in \mathbb{Z}$. By Proposition 4.1(ii), it suffices to prove the corollary for j. By Proposition 4.3, we may assume that X' is affine, and hence that there exists $h \in \mathscr{O}(X')$ generating the ideal of $X' - j(X)$.

So we see that it is enough to prove the corollary in the following situation : there exists $h : Y \longrightarrow \mathbb{A}^1_k$ such that $f = j$ is the inclusion of $X := h^{-1}(\mathbb{G}_m)$ in Y. Let $i : Y - X \longrightarrow X$ be the inclusion of the complement. Let K be an object of $\mathrm{Perv}_{mf}(X)$, and denote by W its weight filtration. Let $a \in \mathbb{Z}$. We want to find a subobject L of j_*K such that L is of weight $\leq a$ and j_*K/L is of weight $> a$. (This clearly implies that j_*K has a weight filtration.)

If $W_aK = 0$, then K is of weight $> a$, so j_*K is of weight $> a$, and we take $L = 0$.

If $W_aK = K$, then K is of weight $\leq a$, so $j_{!*}K$ is of weight $\leq a$ by Corollary 5.4.3 of [6]. So it is enough to find a subobject L' of weight $\leq a$ of $j_*K/j_{!*}K$ such that $(j_*K/j_{!*}K)/L'$ is of weight $> a$. But we know that $j_*K/j_{!*}K = i_*{}^p\mathrm{H}^0 i^* j_*K$ (by (4.1.11.1) of [6]), which is a quotient of $i_*\Psi^u_f K(-1)$. As $\Psi^u_f K$ has a weight filtration by Theorem 6.2.2, so does $j_*K/j_{!*}K$, and we can find a L' with the desired properties.

Suppose that $0 \neq W_aK \neq K$, and let $K' = W_aK$ and $K'' = K/W_aK$. By the previous paragraph, there exists a subobject L' of weight $\leq a$ of j_*K' such that j_*K'/L' is of weight $> a$. As K'' is of weight $> a$, so is j_*K''. Using the exact sequence

$$0 \longrightarrow j_*K' \longrightarrow j_*K \longrightarrow j_*K'' \longrightarrow 0,$$

we see that j_*K/L' is also of weight $> a$, so we can take $L = L'$.

□

Corollary 6.3.2 *Let $j : U \longrightarrow X$ be an affine open embedding. Denote by $j^* : D^b\mathrm{Perv}_{mf}(X) \longrightarrow D^b\mathrm{Perv}_{mf}(U)$ and $j_* : D^b\mathrm{Perv}_{mf}(U) \longrightarrow D^b\mathrm{Perv}_{mf}(X)$ the derived functors of the exact functors $j^* : \mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{Perv}_{mf}(U)$ and $j_* : \mathrm{Perv}_{mf}(U) \longrightarrow \mathrm{Perv}_{mf}(X)$.*

Then this derived functors (j^*, j_*) form a pair of adjoint functors.

Proof. By Corollary 8.12 of [26], it suffices to prove that the underived functors form a pair of adjoint functors. But, once we know that both functors preserve the full subcategories $\mathrm{Perv}_{mf} \subset \mathrm{Perv}_m$, this follows from the adjunction for the categories Perv_m.

$\square$

Corollary 6.3.3 *The exact functors Ψ_f^u, Φ_f^u, Ξ_f and Ω_f of section 5.2 preserve the full subcategories of mixed perverse sheaves with weight filtrations.*

Proof. We already know the result for Ψ_f^u, by Theorem 6.2.2.

Suppose that $K \in \mathrm{Perv}_{mf}(U)$. Then $K \otimes f^*\mathscr{L}_i$ is in $\mathrm{Perv}_{mf}(U)$ for every $i \geq 0$. Indeed, if we denote by W the weight filtration on K, then we get a weight filtration on $K \otimes f^*\mathscr{L}_i$ by setting

$$W_a(K \otimes f^*\mathscr{L}_i) = \sum_{0 \leq j \leq i} (W_{a-2j}K) \otimes f^*\mathscr{L}_j.$$

By Corollary 6.3.1, we see that $j_!(K \otimes f^*\mathscr{L}_i)$ and $j_*(K \otimes f^*\mathscr{L}_i)$ are in $\mathrm{Perv}_{mf}(X)$ for every $i \geq 0$. By definition of Ξ_f, this implies that $\Xi_f K \in \mathrm{Perv}_{mf}(X)$. The conclusion for Ω_f then follows from its construction in Proposition 5.2.10. Finally, by the construction in Propositions 5.2.7, the functor $i_*\Phi_f$ is a subquotient of $\Xi_f j^* \oplus \mathrm{id}$. As $\mathrm{Perv}_{mf}(X)$ is stable by subquotients in $\mathrm{Perv}_m(X)$, the functor $i_*\Phi_f$ sends $\mathrm{Perv}_{mf}(X)$ to itself. By Lemma 4.4, this implies that Φ_f sends $\mathrm{Perv}_{mf}(X)$ to $\mathrm{Perv}_{mf}(Y)$.

$\square$

6.4 Direct and inverse image by a closed immersion

Let X be a k-scheme and $Y \xrightarrow{i} X$ be a closed subscheme of X. We denote by $D^b_Y \mathrm{Perv}_{mf}(X)$ the full subcategory of $D^b \mathrm{Perv}_{mf}(X)$ whose objects are the complexes K such that the support of $\mathrm{H}^i K \in \mathrm{Perv}_{mf}(X)$ is contained in Y for every $i \in \mathbb{Z}$. The exact functor $i_* : \mathrm{Perv}_{mf}(Y) \longrightarrow \mathrm{Perv}_{mf}(X)$ induces a functor $i_* : D^b \mathrm{Perv}_{mf}(Y) \longrightarrow D^b \mathrm{Perv}_{mf}(X)$, whose image is obviously in contained in $D^b_Y \mathrm{Perv}_{mf}(X)$.

Corollary 6.4.1 *With notation as above, the functor $i_* : D^b \mathrm{Perv}_{mf}(Y) \longrightarrow D^b_Y \mathrm{Perv}_{mf}(X)$ is an equivalence of categories.*

We have a similar equivalence $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Y) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{m},Y}(X)$, where $\mathrm{D}^{\mathrm{b}}_{\mathrm{m},Y}(X)$ is the full subcategory of objects K of $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ such that ${}^p\mathrm{H}^i K$ is in $i_* \mathrm{Perv}_m(Y)$ for every $i \in \mathbb{Z}$.

Moreover, we can choose inverses $(i_*)^{-1}$ of these equivalences such that the following diagram *commutes :*

$$\begin{tikzcd} D^b\operatorname{Perv}_{mf}(Y) \arrow[r, "i_*"] \arrow[d, "R_Y"'] & D^b_Y\operatorname{Perv}_{mf}(X) \arrow[r, "(i_*)^{-1}"] \arrow[d, "R_X"'] & D^b\operatorname{Perv}_{mf}(Y) \arrow[d, "R_Y"'] \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Y) \arrow[r, "i_*"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{m},Y}(X) \arrow[r, "(i_*)^{-1}"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(Y) \end{tikzcd}$$

where the functors R_X and R_Y are defined in Theorem 3.2.4.

Proof. We prove the first statement. It suffices to prove that, for all $K, L \in \operatorname{Ob}\operatorname{Perv}_{mf}(Y)$ and every $n \in \mathbb{Z}$, the functor i_* induces an isomorphism $\operatorname{Hom}_{\mathrm{D^b}\operatorname{Perv}_{mf}(Y)}(K, L[n]) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{D^b}\operatorname{Perv}_{mf}(X)}(i_*K, i_*L[n])$. Note that both of these Hom groups are 0 for $n < 0$, so we only need to consider the case $n \geq 0$. Fix $K \in \operatorname{Ob}\operatorname{Perv}_{mf}(Y)$. The families of functors $(L \longmapsto \operatorname{Hom}_{\mathrm{D^b}\operatorname{Perv}_{mf}(Y)}(K, L[n]))_{n\geq 0}$ and $(L \longmapsto \operatorname{Hom}_{\mathrm{D^b}\operatorname{Perv}_{mf}(X)}(i_*K, i_*L[n]))_{n\geq 0}$ are δ-functors from $\operatorname{Perv}_{mf}(Y)$ to the category of abelian groups (in the sense of Definition [27, Tag 010Q]), and i_* induces a morphism between these δ-functors (see Definition [27, Tag 010R]). We want to show that this morphism is an isomorphism. We know that $i_* : \operatorname{Hom}_{\operatorname{Perv}_{mf}(Y)}(K, L) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Perv}_{mf}(X)}(i_*K, i_*L)$ is an isomorphism for every $L \in \operatorname{Ob}\operatorname{Perv}_{mf}(Y)$ (because this is true in the categories Perv_m). Moreover, it follows easily from the Yoneda description of the extension groups in the derived category (see Section 3.2 of Chapter III of Verdier's book [29] or Lemma [27, Tag 06XU]) that the first of the two δ-functors introduced above is effacable, i.e. satisfies the hypothesis of Lemma [27, Tag 010T]), and hence is a universal δ-functor (see Definition [27, Tag 010S]). By Lemma [27, Tag 010U] (and Lemma [27, Tag 010T] again), it suffices to prove that the second δ-functor is also effacable. So we want to prove that, for all $K, L \in \operatorname{Ob}\operatorname{Perv}_{mf}(Y)$, every $n \geq 1$ and every $u \in \operatorname{Hom}_{\mathrm{D^b}\operatorname{Perv}_{mf}(Y)}(K, L[n])$, there exists an injective morphism $L \longrightarrow L'$ in $\operatorname{Perv}_{mf}(Y)$ such that the image of u in $\operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(X)}(i_*K, i_*L'[n])$ is 0.

Let $(U_a)_{a\in A}$ be a finite affine cover of X. For every $a \in A$, we have a cartesian diagram of immersions

$$\begin{tikzcd} Y \arrow[r, "i"] & X \\ Y \cap U_\alpha \arrow[u, "j'_a"] \arrow[r, "i_a"'] & U_\alpha \arrow[u, "j_a"'] \end{tikzcd}$$

As j'_α and j_α are affine, the functors $j'_{\alpha*}$ and $j_{\alpha*}$ are t-exact. Let $L \in \operatorname{Ob}\operatorname{Perv}_{mf}(Y)$. By Corollary 6.3.1, the isomorphisms $i_*j'_{a*}{j'_a}^*L \simeq j_{a*}j_a^*i_*L$ and $j_a^*i_*L \simeq i_{a*}{j'_a}^*L$ in $\operatorname{Perv}_m(X)$ and $\operatorname{Perv}_m(U_\alpha)$ are isomorphisms of objects of $\operatorname{Perv}_{mf}(X)$ and $\operatorname{Perv}_{mf}(U_\alpha)$. Using this and Corollary 6.3.2 we get for $K, L \in \operatorname{Ob}\operatorname{Perv}_{mf}(Y)$ and $n \in \mathbb{Z}$ a canonical isomorphism

$$\operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(X)}(i_*K, i_*j'_{a*}{j'_a}^*L[n]) = \operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(U_a)}(i_{a*}{j'_a}^*K, i_{a*}{j'_a}^*L[n]).$$

As we have an injective morphism $L \longrightarrow \bigoplus_{a\in A} j_{a*}j_a^*L$ in $\operatorname{Perv}_{mf}(Y)$ (by Corollary 6.3.1 again), this reduces the corollary to the case where X is affine.

Now suppose that X is affine. By an easy induction on the number of generators of the ideal of Y, we may assume that this ideal only has one generator, i.e., that there exists a function $f : X \longrightarrow \mathbb{A}^1$ such that $Y = X \times_{\mathbb{A}^1} \{0\}$. The exact functor $\Phi_f^u : \mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{Perv}_{mf}(Y)$ induces a functor $\Phi_f^u : D_Y^b \mathrm{Perv}_{mf}(X) \longrightarrow D^b \mathrm{Perv}_{mf}(Y)$, and we have $\Phi_f^u \circ i_* \simeq \mathrm{id}_{D^b \mathrm{Perv}_{mf}(Y)}$. Let's show that $i_* \circ \Phi_f^u \simeq \mathrm{id}_{D_Y^b \mathrm{Perv}_{mf}(X)}$, which will finish the proof. By Proposition 5.2.10 and Corollary 6.3.3 we have two exact sequences of exact endofunctors of $\mathrm{Perv}_{mf}(X)$:

$$0 \longrightarrow j_! j^* \longrightarrow \Omega_f \longrightarrow i_* \Phi_f^u \longrightarrow 0$$

and

$$0 \longrightarrow i_* \Psi_f^u j^* \longrightarrow \Omega_f \longrightarrow \mathrm{id} \longrightarrow 0,$$

where $j : X - Y \longrightarrow X$ is the inclusion. Note that the restriction of the functor $j^* : D^b \mathrm{Perv}_{mf}(X) \longrightarrow D^b \mathrm{Perv}_{mf}(U)$ to the full subcategory $D_Y^b \mathrm{Perv}_{mf}(X)$ is zero. Hence the exact sequences above induces isomorphisms of endofunctors of $D_Y^b \mathrm{Perv}_{mf}(X)$:

$$i_* \Phi_f^u \xleftarrow{\sim} \Omega_f \xrightarrow{\sim} \mathrm{id}.$$

The proof of the second equivalence of categories is similar, except that we don't need to use the Yoneda description to show that the Ext groups in $\mathrm{D}_{\mathrm{m}}^{\mathrm{b}}(Y)$ define a δ-functor.

The last statement of the Corollary follows from the fact that we have isomorphisms

$$R_Y \circ \Phi_f^u \simeq \Phi_f^u \circ R_X$$

and

$$R_X \circ \Omega_f \simeq \Omega_f \circ R_X.$$

□

Corollary 6.4.2 *Let $i : X \longrightarrow Y$ be a closed immersion. Denote by $i_* : D^b \mathrm{Perv}_{mf}(X) \longrightarrow D^b \mathrm{Perv}_{mf}(Y)$ the derived functor of the exact functor $i_* : \mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{Perv}_{mf}(Y)$.*

Then this functor i_* admits a left adjoint $i^* : D^b \mathrm{Perv}_{mf}(Y) \longrightarrow D^b \mathrm{Perv}_{mf}(X)$, and the counit $i^* i_* \longrightarrow \mathrm{id}$ of this adjunction is an isomorphism. Moreover, we have an invertible natural transformation $\theta_i : i^* \circ R_Y \xrightarrow{\sim} R_X \circ i^*$.

Finally, if $i' : Y \longrightarrow Z$ is another closed immersion, then the following diagram is commutative :

$$\begin{tikzcd}
R_X \circ i^* i'^* \arrow[r, "\theta_i"] \arrow[d, "\wr"] & i^* \circ R_Y \circ i' \arrow[r, "\theta_{i'}"] & i^* i'^* \circ R_Z \arrow[d, "\wr"] \\
R_X \circ (i'i)^* \arrow[rr, "\theta_{i'i}"'] & & (i'i)^* \circ R_Z
\end{tikzcd}$$

where the vertical maps come from the composition isomorphisms $i'_* i_* \simeq (i'i)_*$ and the uniqueness of the adjoint.

Proof. By Corollary 6.4.1, we have an equivalence of categories $i_* : D^b \operatorname{Perv}_{mf}(X) \xrightarrow{\sim} D^b_X \operatorname{Perv}_{mf}(Y)$, where $D^b_X \operatorname{Perv}_{mf}(Y)$ is the full subcategory of $D^b \operatorname{Perv}_{mf}(Y)$ whose objects are complexes K such that the support of $H^i K \in \operatorname{Perv}_{mf}(Y)$ is contained in X for every $i \in \mathbb{Z}$. So, to show that $i_* : D^b \operatorname{Perv}_{mf}(X) \longrightarrow D^b \operatorname{Perv}_{mf}(Y)$ admits a left adjoint, it suffices to show that the inclusion $\alpha : D^b_X \operatorname{Perv}_{mf}(Y) \longrightarrow D^b \operatorname{Perv}_{mf}(Y)$ admits a left adjoint. Let $j : Y - X \longrightarrow Y$ be the inclusion. Then we have an exact triangle $j_! j^* \longrightarrow \mathrm{id} \longrightarrow i_* i^* \xrightarrow{+1}$ of endofunctors of $D^b_m(Y)$, and we can make sense of the first two terms in $D^b \operatorname{Perv}_{mf}(Y)$, so we will try to construct the left adjoint of α as their cone.

More precisely, let $(U_i)_{i \in I}$ be a finite open affine cover of $U := Y - X$. For every $J \subset I$, we denote by $j_J : \bigcap_{i \in J} U_i \longrightarrow X$ the inclusion. As X is separated, all the finite intersections of U_i's are affine, so the morphism j_J is affine for every $J \subset I$. If $K \in \operatorname{Ob} \operatorname{Perv}_{mf}(X)$, we denote by $D^\bullet(K)$ the complex of $\operatorname{Perv}_{mf}(X)$ defined by $D^{-r}(K) = \bigoplus_{|J|=r} j_{J!} j_J^* K$ if $r \geq 1$, $D^0(K) = K$ and $D^r(K) = 0$ if $r \geq 1$, where the maps $D^{-r-1}(K) \longrightarrow D^{-r}(K)$, $r \geq 0$, are alternating sums of adjunction morphisms. Note that we have a morphism of complexes $K \longrightarrow D^\bullet(K)$, where K is in degree 0. Also, there is a canonical morphism $D^{-1}(K) \longrightarrow j_! j^* K$, which induces an isomorphism $D^{\leq -1}(K) \xrightarrow{\sim} j_! j^* K[1]$ in $D^b \operatorname{Perv}_{mf}(Y)$, so we get a quasi-isomorphism $R_Y(D^\bullet(K)) \xrightarrow{\sim} i_* i^* K$ in $D^b_m(Y)$. In particular, $D^\bullet(K)$ is in $D^b_X \operatorname{Perv}_{mf}(Y)$. Note that the construction of $D^\bullet(K)$ is functorial in K, so we can define a functor $\beta : D^b \operatorname{Perv}_{mf}(Y) \longrightarrow D^b_X \operatorname{Perv}_{mf}(Y)$ by sending a complex K to the total complex of the double complex $D^\bullet(K)$.

Let's show that β is left adjoint to α. For every complex K of objects of $\operatorname{Perv}_{mf}(Y)$, the morphism of double complexes $K \longrightarrow D^\bullet(K)$ induces a morphism $\varepsilon_K : K \longrightarrow \alpha\beta(K)$ in $D^b \operatorname{Perv}_{mf}(Y)$. If moreover K is in $D^b_X \operatorname{Perv}_{mf}(X)$, then $K \longrightarrow D^\bullet(K)$ is a quasi-isomorphism, so we get an isomorphism $\eta_K : \beta\alpha(K) \xrightarrow{\sim} K$. Moreover, the morphism $\alpha(K) \xrightarrow{\varepsilon_K \alpha} \alpha\beta\alpha(K) \xrightarrow{\alpha\eta_K} \alpha(K)$ is clearly the identity of $\alpha(K)$. So we have constructed the unit and counit of the adjunction, and shown that the counit is an isomorphism.

To construct the isomorphism θ_i, we use the isomorphism $R_Y \circ V \xrightarrow{\sim} i_* i^* \circ R_X$ constructed above and the last statement of Corollary 6.4.1. The last statement is also easy to check.

□

7 Construction of the stable homotopic 2-functor H_{mf}

7.1 Direct images

If $f : X \longrightarrow Y$ is a morphism of k-schemes, we write ${}^0 f_*$ for ${}^p\mathrm{H}^0 f_*$. Remember that if f is affine, then ${}^0 f_*$ is right t-exact for the perverse t-structure by [6] Theorem 4.1.1 (see also Proposition 2.2.2).

In this section, we want to prove the following result.

Proposition 7.1.1 *There exists a 2-functor $H_{mf,*} : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ with $H_{mf,*}(X) = D^b \operatorname{Perv}_{mf}(X)$ for every $X \in \operatorname{Ob}(\mathbf{Sch}/k)$ and a natural transformation $R : H_{mf,*} \longrightarrow H_{m,*}$ (with the notation of Example 3.2.3) such that :*

(a) for every $X \in \operatorname{Ob}(\mathbf{Sch}/k)$, the functor $R_X : \mathrm{D^b} \operatorname{Perv}_{mf}(X) \longrightarrow \mathrm{D^b_m}(X)$ is the composition of the obvious functor $\mathrm{D^b} \operatorname{Perv}_{mf}(X) \longrightarrow \mathrm{D^b} \operatorname{Perv}_m(X)$ and of the realization functor $\mathrm{D^b} \operatorname{Perv}_m(X) \longrightarrow \mathrm{D^b_m}(X)$ (see Section 2.7);

(b) for every morphism $f : X \longrightarrow Y$, the natural transformation $\gamma_f : R_Y \circ H_{mf,*}(f) \longrightarrow H_{m,*}(f) \circ R_X$ is an isomorphism.

The proof of the proposition will occupy most of this section. The main ingredients are :

(1) Beilinson's calculation of derived direct images in derived categories of perverse sheaves, and in particular his result that, if $f : X \longrightarrow Y$ with X affine, then $f_* : D^b_c(X) \longrightarrow D^b_c(Y)$ is the left derived functor of ${}^0 f_*$. (See Theorem 7.1.2.)

(2) The fact that the functors ${}^p\mathrm{H}^k f_*$ (and in particular ${}^0 f_*$) preserve the categories Perv_{mf}. (See Corollary 6.3.1.)

(3) Čech resolutions for finite open affine coverings.

We first review some of Beilinson's results.

Theorem 7.1.2 *Let $f : X \longrightarrow Y$ be a morphism of k-schemes, let $(U_i)_{i\in I}$ be a finite family of open affine subsets of X and let $K \in \operatorname{Ob}\operatorname{Perv}(X)$. Then there exists an object L of $\operatorname{Perv}(X)$ and a surjective morphism $L \longrightarrow K$ (in $\operatorname{Perv}(X)$) such that, for every $i \in I$, the complex $f_{|U_i *} L_{|U_i}$ is a perverse sheaf.*

Moreover, if K is an object of $\operatorname{Perv}_h(X)$ (resp. $\operatorname{Perv}_m(X)$, resp. $\operatorname{Perv}_{mf}(X)$), we can choose L in this same subcategory.

The first statement is Lemma 3.3 of [4]. Also, it is clear from the proof of this lemma that L is a direct sum of objects of the form $j_! j^* K$, where $j : V \longrightarrow X$ is the embedding of an open affine subset; as the categories Perv_h, Perv_m and Perv_{mf} are stable by the functors j^* and $j_!$ (by Corollary 6.3.1 for Perv_{mf}), we get the second statement.

Corollary 7.1.3 *Let $f : X \longrightarrow Y$ be a morphism of k-schemes, and suppose that X is affine. We denote by $f^0_* : \operatorname{Perv}_{mf}(X) \longrightarrow \operatorname{Perv}_{mf}(Y)$ the functor ${}^p\mathrm{H}^0 \circ f_*$ (this makes sense by Corollary 6.3.1).*

Then f^0_* is right exact, it admits a left derived functor Lf^0_*, and the following diagram com-

mutes up to canonical natural isomorphism :

$$\begin{tikzcd} \mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \arrow[r, "Lf_*^0"] \arrow[d, "R_X"'] & \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y) \arrow[d, "R_Y"] \\ \mathrm{D^b_m}(X) \arrow[r, "f_*"'] & \mathrm{D^b_m}(Y) \end{tikzcd}$$

Proof. We know that f_*^0 is right exact by Proposition 2.2.2. The rest follows from the previous theorem and from Proposition 2.3.4, whose hypothesis is satisfied by Corollary 2.3.3.

□

Following Section 3.4 of [4], we now explain how to reconstruct the functor f_* from ${}^p\mathrm{H}^0 \circ f_*$ when the source of $f : X \longrightarrow Y$ is not affine.

Let $\mathscr{U} = (U_i)_{i \in I}$ be a finite covering of X by open affine subschemes. We denote by $C^\bullet_{\mathscr{U}} : \mathrm{Perv}(X) \longrightarrow C^{\geq 0}\,\mathrm{Perv}(X)$ the associated Čech resolution functor, defined as follows : For every $J \subset I$, we denote by $j_J : \bigcap_{i \in J} U_i \longrightarrow X$ the inclusion. As X is separated, all the finite intersections of U_i's are affine, so the morphism j_J is affine for every $J \subset I$, hence j_{J*} is exact in the perverse sense. If $K \in \mathrm{Ob}\,\mathrm{Perv}(X)$, we have $C^r_{\mathscr{U}}(K) = \bigoplus_{|J|=r+1} j_{J*}j_J^*K$, and the maps $C^r(K) \longrightarrow C^{r+1}(K)$ are alternating sums of adjunction morphisms. Note that the functor $C^\bullet_{\mathscr{U}}$ is exact.

By Corollary 6.3.1, the functor $C^\bullet_{\mathscr{U}}$ sends $\mathrm{Perv}_{mf}(X)$ to $C^{\geq 0}\,\mathrm{Perv}_{mf}(X)$, and, by Théorème 3.2.4 of [6], the canonical morphism $K \longrightarrow C^\bullet_{\mathscr{U}}(K)$ is a quasi-isomorphism for every $K \in \mathrm{Ob}\,\mathrm{Perv}(X)$.

Let $K \in \mathrm{Ob}\,\mathrm{Perv}(X)$. By Theorem 7.1.2, we can find a left resolution $K^\bullet \longrightarrow K$ in $\mathrm{Perv}(X)$ such that each $C^r_{\mathscr{U}}(K^s)$ is f_*-acyclic, and then the total complex of the double complex $f_*C^\bullet_{\mathscr{U}}(K^\bullet)$ is a complex of objects of $\mathrm{Perv}(Y)$ whose image by $\mathrm{real} : \mathrm{D^b}\,\mathrm{Perv}(Y) \longrightarrow \mathrm{D^b_c}(Y)$ is isomorphic to f_*K.

Here is another way to think of this : The covering $\mathscr{U}$ defines in the usual way (see for example the beginning of the proof of Théorème 3.2.4 of [6]) a simplicial scheme $\varepsilon_{\mathscr{U}} : \mathscr{U}_\bullet \longrightarrow X$ over X. For every $K \in \mathrm{Ob}\,\mathrm{Perv}(X)$, the Čech complex $C^\bullet_{\mathscr{U}}(K)$ is a representative of the complex $\varepsilon_{\mathscr{U}*}\varepsilon^*_{\mathscr{U}}K$.

Let $Y_\bullet$ be the constant simplicial scheme with value Y and $\varepsilon : Y_\bullet \longrightarrow Y$ be the obvious map. We also denote by $f_{\mathscr{U}\bullet} : \mathscr{U}_\bullet \longrightarrow Y_\bullet$ the morphism of simplicial schemes induced by f. We have a canonical isomorphism of functors $\mathrm{D^b_c}(X) \longrightarrow \mathrm{D^b_c}(Y)$ (where the first map is the adjuction map) :

$$(*) \qquad f_* \longrightarrow f_*\varepsilon_{\mathscr{U}*}\varepsilon^*_{\mathscr{U}} \simeq \varepsilon_* f_{\mathscr{U}\bullet*}\varepsilon^*_{\mathscr{U}}.$$

As all the face and degeneracy maps in the simplicial schemes $\mathscr{U}_\bullet$ and $Y_\bullet$ are affine open embeddings, we can define perverse sheaves on $\mathscr{U}_\bullet$ and $Y_\bullet$; also, as these simplicial schemes both have the property that all their levels are finite disjoint unions of schemes taken among some finite family, we can also make sense of horizontal and mixed perverse sheaves on them. Now, as all the levels of $\mathscr{U}_\bullet$ are affine schemes, the functor $f_{\mathscr{U}\bullet*}$ is right t-exact. Theorem 7.1.2 implies that the functor ${}^p\mathrm{H}^0 \circ f_{\mathscr{U}\bullet*} \circ \varepsilon_{\mathscr{U}}^*$ admits a left derived functor, and that this left derived functor is canonically isomorphic to $f_{\mathscr{U}\bullet*} \circ \varepsilon_{\mathscr{U}}^*$; indeed, this theorem implies that every object of $\mathrm{Perv}(X)$ is a quotient of a $f_{\mathscr{U}\bullet*} \circ \varepsilon_{\mathscr{U}}^*$-acyclic object. Let $K \in \mathrm{Ob}\,\mathrm{Perv}(X)$, and let $K^\bullet \longrightarrow K$ be a left resolution such that each $\varepsilon_{\mathscr{U}}^* K^r$ is $f_{\mathscr{U}\bullet*}$-acyclic. Then the total complex of the double complex $f_* C_{\mathscr{U}}^\bullet(K^\bullet)$ is just the complex $\varepsilon_* f_{\mathscr{U}\bullet*}\varepsilon_{\mathscr{U}}^* K^\bullet$, and the fact that its image by real is isomorphic to $f_* K$ follows from the isomorphism $(*)$.

Next, thanks to Corollary 6.3.1 (and Theorem 7.1.2), we can replace Perv by Perv_{mf} in the construction of the previous paragraph, and we get a functor $\varepsilon_* f_{\mathscr{U}\bullet*}\varepsilon_{\mathscr{U}}^* K^\bullet : \mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y)$ that makes the following diagram commute up to canonical isomorphism :

$$(**) \qquad \begin{tikzcd} \mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \arrow[r, "\varepsilon_* f_{\mathscr{U}\bullet*}\varepsilon_{\mathscr{U}}^*"] \arrow[d, "R_X"'] & \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y) \arrow[d, "R_Y"] \\ \mathrm{D^b_m}(X) \arrow[r, "f_*"'] & \mathrm{D^b_m}(Y) \end{tikzcd}$$

We want to use this as the definition of $H_{mf,*}(f)$, but we have to get rid of the dependence on $\mathscr{U}$. The solution is of course to take a limit over all finite open affine coverings of X. As in Section 9.2.2 of [15], we will use the notion of rigidified covering, so that these coverings form a filtered partially ordered set. Remember that (see Definition 9.2.7 of [15]) :

- a *rigidified open affine covering* of X is an open affine covering $(U_i)_{i\in I}$ with a map $X \longrightarrow I$, $x \longmapsto i_x$ such that $x \in U_{i_x}$ for every $x \in X$;
- a *morphism of rigidified open affine coverings* from $(U_i)_{i\in I}$ to $(V_j)_{j\in J}$ is a map $\phi : I \longrightarrow J$ such that $U_i \subset V_{\phi(j)}$ for every $i \in I$ and $\phi(i_x) = j_x$ for every $x \in X$;
- more generally, if $(U_i)_{i\in I}$ is a rigidified open affine covering of X and $(V_j)_{j\in J}$ is a rigidified open affine covering of Y, then a *morphism of rigidified open affine coverings over* f is a map $\phi : I \longrightarrow J$ such that $f(U_i) \subset V_{\phi(j)}$ for every $i \in I$ and $\phi(i_x) = j_{f(x)}$ for every $x \in X$.

We denote by $\mathrm{Cov}(X)$ the category of rigidified finite open affine coverings of X. By Lemma 9.2.9 of [15], this is a filtered partially ordered set.

Let $\phi : \mathscr{U} \longrightarrow \mathscr{V}$ be a morphism in $\mathrm{Cov}(X)$. We get as before commutative squares

$$\begin{tikzcd} \mathscr{U}_\bullet \arrow[r, "f_{\mathscr{U}\bullet}"] \arrow[d, "\varepsilon_{\mathscr{U}}"'] & Y_\bullet \arrow[d, "\varepsilon"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{V}_\bullet \arrow[r, "f_{\mathscr{V}\bullet}"] \arrow[d, "\varepsilon_{\mathscr{V}}"'] & Y_\bullet \arrow[d, "\varepsilon"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Moreover, the morphism of rigidified coverings ϕ induces a morphism of simplicial schemes $\phi_\bullet : \mathscr{U}_\bullet \longrightarrow \mathscr{V}_\bullet$ such that $f_{\mathscr{U}\bullet} = f_{\mathscr{V}\bullet}\phi_\bullet$.

Using the adjunction morphism $\mathrm{id} \longrightarrow \phi_{\bullet *}\phi_\bullet^*$, we get a morphism of functors from $\mathrm{D}^\mathrm{b}\operatorname{Perv}_{mf}(X)$ to $\mathrm{D}^\mathrm{b}\operatorname{Perv}_{mf}(Y)$:

$$\varepsilon_* f_{\mathscr{U}\bullet *}\varepsilon_{\mathscr{U}}^* \longrightarrow \varepsilon_* f_{\mathscr{U}\bullet *}\phi_{\bullet *}\phi_\bullet^*\varepsilon_{\mathscr{U}}^* = \varepsilon_* f_{\mathscr{V}\bullet *}\varepsilon_{\mathscr{V}}^*.$$

By the conservativity of R_Y (see Proposition 2.7.3), this morphism is an isomorphism.

Definition 7.1.4 Let $f : X \longrightarrow Y$. We define the functor

$$H_{mf,*}(f) : \mathrm{D}^\mathrm{b}\operatorname{Perv}(X) \longrightarrow \mathrm{D}^\mathrm{b}\operatorname{Perv}(Y)$$

by

$$H_{mf,*}(f) = \varprojlim_{\mathscr{U}\in\operatorname{Cov}(X)} \varepsilon_* f_{\mathscr{U}\bullet *}\varepsilon_{\mathscr{U}}^*.$$

We will also denote the functor $H_{mf,*}(f)$ by f_* if there is no risk of confusion.

It remains to show that Definition 7.1.4 does give a 2-functor from $\mathbf{Sch}/k$ to $\mathfrak{TR}$. In a previous version of this article, we used Theorem 8.10 of Shulman's paper [26] to construct the connection morphisms, and then the conservativity of the realization functors (Proposition 2.7.3) to show that they are isomorphisms. This forced us to replace some of the categories by their Ind and/or Pro versions to ensure that the hypotheses of Theorem 8.10 of [26] hold (i.e. that the categories and functors are derivable), and it seems simpler to just construct the connection morphisms by hand.

So suppose that we have two morphisms of k-schemes $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$. We want to lift the connection isomorphism $g_* \circ f_* \simeq (g \circ f)_* : \mathrm{D}^\mathrm{b}_\mathrm{m}(X) \longrightarrow \mathrm{D}^\mathrm{b}_\mathrm{m}(Z)$ to a natural transformation $H_{mf,*}(g) \circ H_{mf,*}(f) \longrightarrow H_{mf,*}(g \circ f)$. Let $\mathscr{V}$ be a rigidified finite open affine covering of Y. By Lemma 9.2.9 of [15], there exists a rigidified finite open affine covering $\mathscr{U}$ of X and a morphism $\phi : \mathscr{U} \longrightarrow \mathscr{V}$ over f. As in the discussion before Definition 7.1.4, we have commutative squares

$$\begin{tikzcd} \mathscr{U}_\bullet \arrow[r, "f_{\mathscr{U}\bullet}"] \arrow[d, "\varepsilon_{\mathscr{U}}"'] & Y_\bullet \arrow[d, "\varepsilon_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{V}_\bullet \arrow[r, "g_{\mathscr{V}\bullet}"] \arrow[d, "\varepsilon_{\mathscr{V}}"'] & Z_\bullet \arrow[d, "\varepsilon_2"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{U}_\bullet \arrow[r, "(gf)_{\mathscr{U}\bullet}"] \arrow[d, "\varepsilon_{\mathscr{U}}"'] & Z_\bullet \arrow[d, "\varepsilon_2"] \\ X \arrow[r, "gf"'] & Z \end{tikzcd}$$

Also, the morphism ϕ induces a morphism of simplicial schemes $\phi_\bullet : \mathscr{U}_\bullet \longrightarrow \mathscr{V}_\bullet$ such that $\varepsilon_1 f_{\mathscr{U}\bullet} = \varepsilon_{\mathscr{V}}\phi_\bullet$ and $g_{\mathscr{V}\bullet}\phi_\bullet = (gf)_{\mathscr{U}\bullet}$.

Using the adjunction morphism $\varepsilon_{\mathscr{V}}^*\varepsilon_{\mathscr{V}*} \longrightarrow \mathrm{id}$, we get a functorial morphism from

$$H_{mf,*}(g) \circ H_{mf,*}(f) \simeq \varepsilon_{2*}g_{\mathscr{V}\bullet *}\varepsilon_{\mathscr{V}}^*\varepsilon_{1*}f_{\mathscr{U}\bullet *}\varepsilon_{\mathscr{U}}^* \simeq \varepsilon_{2*}g_{\mathscr{V}\bullet *}\varepsilon_{\mathscr{V}}^*\varepsilon_{\mathscr{V}*}\phi_{\bullet *}\varepsilon_{\mathscr{U}}^*$$

to

$$\varepsilon_{2*} g_{\mathscr{V}\bullet *} \phi_{\bullet *} \varepsilon_{\mathscr{U}}^{*} \simeq \varepsilon_{2*} (gf)_{\mathscr{U}\bullet *} \varepsilon_{\mathscr{U}}^{*} \simeq \mathrm{H}_{mf,*}(gf).$$

It is straightforward to check that this morphism does not depend on the choices and that it lifts the connection isomorphism $g_* \circ f_* \simeq (gf)_*$. In particular, it is an isomorphism by Proposition 2.7.3. This finishes the proof of Proposition 7.1.1.

Remark 7.1.5 Let us relate our definition to the definition given in Section 3.4 of [4]. Let $\mathrm{Zar}(X)$ be the category of open affine subschemes of X (the maps are inclusions); for any category $\mathscr{C}$, let $\mathrm{PSh}(\mathrm{Zar}(X), \mathscr{C})$ be the category of presheaves on $\mathrm{Zar}(X)$ with values in $\mathscr{C}$. We denote by ε_X^* the exact functor $\mathrm{Perv}(X) \longrightarrow \mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(X))$ sending K to the presheaf $U \longmapsto K_{|U}$, and by $\widetilde{f}_*^0 : \mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(X)) \longrightarrow \mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(Y))$ the right exact functor sending a presheaf $\mathscr{P}$ to the presheaf $U \longmapsto {}^p\mathrm{H}^0 f_* \mathscr{P}(U)$. It is not clear to us whether Theorem 7.1.2 implies that $\widetilde{f}_*^0$ admits a left derived functor, but we can use Proposition 15.3.5 of [19] to get a left derived functor $L\widetilde{f}_*^0 : \mathrm{D^b}\,\mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(X)) \longrightarrow \mathrm{D^b}\,\mathrm{Pro}\,\mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(Y))$, and then Theorem 7.1.2 (along with Theorem 15.3.1 and Proposition 15.3.2 of [19]) says that, for every open affine subscheme U of X and every object $\mathscr{P}$ of $\mathrm{D^b}\,\mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(X))$, the object $L\widetilde{f}_*^0 \mathscr{P}(U)$ of $\mathrm{D^b}\,\mathrm{Pro}\,\mathrm{Perv}(Y)$ is actually in $\mathrm{D^b}\,\mathrm{Perv}(Y)$.

For every finite covering $\mathscr{U}$ of X by open affine subschemes, we have the 0th Čech cohomology functor $\check{\mathrm{H}}^0(\mathscr{U}, .) : \mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(Y)) \longrightarrow \mathrm{Perv}(Y)$ (see for example Definition [27, Tag 03OL]). Then Theorem [27, Tag 03OS] says that the Čech complex of an object of $\mathrm{PSh}(X, \mathrm{Perv}(Y))$ associated to the covering $\mathscr{U}$ calculates the right derived functor of $\check{\mathrm{H}}^0(\mathscr{U}, .)$. Also, by the end of the previous paragraph (and Proposition 6.4.1 of [19]), the functor $R\check{\mathrm{H}}^0(\mathscr{U}, .) \circ L\widetilde{f}_*^0 : \mathrm{D^b}\,\mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(X)) \longrightarrow \mathrm{D^b}\,\mathrm{Pro}\,\mathrm{Perv}(Y)$ factors through $\mathrm{D^b}\,\mathrm{Perv}(Y)$. So we can reformulate the calculation with Čech complexes that we did above by saying that the following square commutes up to canonical isomorphism

$$\begin{tikzcd}
\mathrm{D^b}\,\mathrm{Perv}(X) \arrow[r, "R\check{\mathrm{H}}^0(\mathscr{U}, .)\circ L\widetilde{f}_*^0 \circ \varepsilon_X^*"] \arrow[d, "\mathrm{real}"'] & \mathrm{D^b}\,\mathrm{Perv}(Y) \arrow[d, "\mathrm{real}"] \\
\mathrm{D^b_c}(X) \arrow[r, "f_*"'] & \mathrm{D^b_c}(Y)
\end{tikzcd}$$

The limit over all coverings $\mathscr{U}$ of the functors $\check{\mathrm{H}}^0(\mathscr{U}, .) : \mathrm{PSh}(\mathrm{Zar}(X), \mathrm{Perv}(Y)) \longrightarrow \mathrm{Perv}(Y)$ is the functor of global sections of the associated separated presheaf $\mathscr{P} \longmapsto \mathscr{P}^+$, defined for example in Section [27, Tag 00W1]. We denote this functor by H_0^+. Note that, as the category $\mathrm{Perv}(Y)$ does not have all inductive limits, the image of the functor H_+^0 is the category $\mathrm{Ind}\,\mathrm{Perv}(Y)$, and its right derived functor will have $\mathrm{D^b}\,\mathrm{Ind}\,\mathrm{Perv}(Y)$ as a target. Nevertheless, when we apply $R\mathrm{H}_+^0$ to an object of the form $L\widetilde{f}_*^0 \varepsilon_X^* K$ with $K \in \mathrm{Ob}\,\mathrm{D^b}\,\mathrm{Perv}(X)$, we will obtain an object in the essential image of $\mathrm{D^b}\,\mathrm{Perv}(Y) \longrightarrow \mathrm{D^b}\,\mathrm{Ind}\,\mathrm{Perv}(Y)$, because the transition maps between the Čech complexes of $L\widetilde{f}_*^0 \varepsilon_X^* K$ for different coverings are all quasi-isomorphisms. Using Theorem 15.3.1 of [19] again, we get a functor

$R\mathrm{H}^0_+ \circ L\widetilde{f}^0_* \circ \varepsilon^*_X : \mathrm{D^b}\,\mathrm{Perv}(X) \longrightarrow \mathrm{D^b}\,\mathrm{Perv}(Y)$, and the discussion of the previous paragraph shows that we have a canonical isomorphism

$$\mathrm{real} \circ R\mathrm{H}^0_+ \circ L\widetilde{f}^0_* \circ \varepsilon^*_X \simeq f_* \circ \mathrm{real} : \mathrm{D^b}\,\mathrm{Perv}(X) \longrightarrow \mathrm{D^b_c}(Y).$$

Finally, hanks to Corollary 6.3.1 (and Theorem 7.1.2), the construction of $R\mathrm{H}^0_+ \circ L\widetilde{f}^0_* \circ \varepsilon^*_X$ still makes sense when we replace Perv by Perv_{mf} everywhere, and the following square commutes up to canonical isomorphism :

$$\begin{tikzcd}
\mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \arrow[r, "R\mathrm{H}^0_+ \circ L\widetilde{f}^0_* \circ \varepsilon^*_X"] \arrow[d, "R_X"'] & \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y) \arrow[d, "R_Y"] \\
\mathrm{D^b_m}(X) \arrow[r, "f_*"'] & \mathrm{D^b_m}(Y)
\end{tikzcd}$$

So we coud have defined $H_{mf,*}(f)$ by the formula

$$H_{mf,*}(f) = R\mathrm{H}^0_+ \circ L\widetilde{f}^0_* \circ \varepsilon^*_X.$$

Note that, in Section 3.4 of [4], Beilinson uses the sheafification functor $\mathscr{P} \longmapsto \mathscr{P}^{++}$ instead of the functor $\mathscr{P} \longmapsto \mathscr{P}^+$. We were not able to see why the map $\mathscr{P}^+(X) \longrightarrow \mathscr{P}^{++}(X)$ is an isomorphism for $\mathscr{P}$ in the essential image of $L\widetilde{f}^0_*\varepsilon^*_X$. But it changes little in practice to work with $\mathscr{P}^+$ instead of $\mathscr{P}^{++}$.

Proposition 7.1.6 *The functor $\boxtimes$ from Proposition 4.2 induces a natural isomorphism between the 2-functors $H_{mf,*} \times H_{mf,*} : \mathbf{Sch}_k \times \mathbf{Sch}_k \longrightarrow \mathfrak{TR}$ and $\mathbf{Sch}_k \times \mathbf{Sch}_k \longrightarrow \mathbf{Sch}_k \xrightarrow{H_{mf,*}} \mathfrak{TR}$ (where the first arrow sends (X, Y) to $X \times Y$).*

In other words, if $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ are morphisms in $\mathbf{Sch}/k$ and $K_1 \in D^b\,\mathrm{Perv}_{mf}(X_1)$, $K_2 \in D^b\,\mathrm{Perv}_{mf}(X_2)$, then we have an isomorphism

$$(f_1 \times f_2)_*(K_1 \boxtimes K_2) \xrightarrow{\sim} (f_{1*}K_1) \boxtimes (f_{2*}K_2)$$

functorial in K_1 and K_2 and compatible with the composition of arrows in $\mathbf{Sch}/k$.

Proof. On the categories $\mathrm{D^b_c}$, we have canonical isomorphisms

$$(f_1 \times f_2)_*((-) \boxtimes (-)) \xrightarrow{\sim} (f_{1*}(-)) \boxtimes (f_{2*}(-))$$

and

$$(f_1 \times f_2)^*((-) \boxtimes (-)) \xrightarrow{\sim} (f_1^*(-)) \boxtimes (f_2^*(-))$$

(see SGA 5 III 1.6). These induce similar isomorphisms in the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}$.

By the construction of f_* (see Definition 7.1.4), we only need to show the statement of the proposition for the functors ${}^p\mathrm{H}^0 f_*$ with the source of f affine between the categories $\mathrm{Perv}_{mf}(X)$, and for the restriction functors to an open subscheme. But then it is an immediate consequence of the similar result for the categories $\mathrm{Perv}(X)$, which follows from the result recalled at the beginning of the proof and from the t-exactness of the external tensor product (see Proposition 2.2.2).

□

Proposition 7.1.7 *Let $j : U \longrightarrow X$ be an open embedding. Denote by $j^* : D^b \mathrm{Perv}_{mf}(X) \longrightarrow D^b \mathrm{Perv}_{mf}(U)$ the derived functor of the exact functor $j^* : \mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{Perv}_{mf}(U)$.*

Then this functor j^* is left adjoint to the functor $j_* : D^b \mathrm{Perv}_{mf}(U) \longrightarrow D^b \mathrm{Perv}_{mf}(X)$, and the counit map $j^*j_* \longrightarrow \mathrm{id}$ is an isomorphism.

Proof. Let $(U_i)_{i\in I}$ be a finite open affine cover of U. For every $J \subset I$, we denote by $j_J : \bigcap_{i\in J} U_i \longrightarrow U$ the inclusion. As U is separated, all the finite intersections of U_i's are affine, so the morphisms j_J and jj_J are affine for every $J \subset I$. If $K \in \mathrm{Ob}\,\mathrm{Perv}_m(U)$, we denote by $C^\bullet(K)$ the Čech complex of K associated to the covering $(U_i)_{i\in I}$, so that $C^r(K) = \bigoplus_{|J|=r+1} j_{J*}j_J^*K$ and the maps $C^r(K) \longrightarrow C^{r+1}(K)$ are alternating sums of adjunction morphisms. The canonical morphism $K \longrightarrow C^\bullet(K)$ is a quasi-isomorphism, and all the $C^r(K)$ are j_*-acyclic (indeed, as jj_J is affine for every $J \subset I$, the complex $j_*(j_{J*}j_J^*K)$ is perverse, and so $j_*C^r(K)$ is perverse), so we get a quasi-isomorphism $j_*K \longrightarrow j_*C^\bullet(K)$ (by Definition 7.1.4.

Moreover, by Corollary 6.3.1, if K is in $\mathrm{Perv}_{mf}(U)$, then $C^\bullet(K)$ is a complex of objects of $\mathrm{Perv}_{mf}(U)$, and $j_*C^\bullet(K)$ is a complex of objects of $\mathrm{Perv}_{mf}(X)$, which is quasi-isomorphic to j_*K by definition of the functor j_*. Note also that this construction is functorial in K.

Now we want to define a unit map $\varepsilon : \mathrm{id} \longrightarrow j_*j^*$ and a counit map $j^*j_* \longrightarrow \mathrm{id}$. If K is a complex of objects of $\mathrm{Perv}_{mf}(U)$, then j_*K is quasi-isomorphic to the total complex of the double complex $j_*C^\bullet(K)$, so j^*j_*K is quasi-isomorphic to $C^\bullet(K)$, and we can take for η the inverse of the canonical quasi-isomorphism $K \longrightarrow C^\bullet(K)$. (Note in particular that η is an isomorphism, which gives the last statement of the proposition.) If L is a complex of objects of $\mathrm{Perv}_{mf}(X)$, then j_*j^*L is quasi-isomorphic to the total complex of the double complex $j_*C^\bullet(j^*L)$. But we have a canonical morphism $L \longrightarrow j_*C^0(j^*L)$ (because it exists in $C^b \mathrm{Perv}_m(X)$, and $C^b \mathrm{Perv}_{mf}(X)$ is a full subcategory of $C^b \mathrm{Perv}_m(X)$), and it is easy to see that this induces a morphism $L \longrightarrow j_*C^\bullet(j^*L)$, which is the desired morphism ε. To finish the proof, it suffices to show that, for $K \in \mathrm{Ob}\,D^b \mathrm{Perv}_{mf}(U)$ and $L \in \mathrm{Ob}\,D^b \mathrm{Perv}_{mf}(X)$, the composition

$$j_*K \xrightarrow{\eta j_*} j_*j^*j_*K \xrightarrow{j_*\varepsilon} j_*K$$

is the identity and the composition

$$j^*L \xrightarrow{j^*\eta} j^*j_*j^*L \xrightarrow{\varepsilon j^*} j^*L$$

is an isomorphism. The first statement is clear from the explicit descriptions of j_*, ε and η, and the second statement follows from the conservativity of the functor R_U.

□

7.2 Inverse images

In this section, we construct the inverse images functors as the left adjoints of the direct image functors of Proposition 7.1.1.

First we treat a particular case. For every smooth equidimensional k-scheme X, we denote by $\mathbb{1}_X$ the constant sheaf on X, seen as an object of $\mathrm{Perv}_{mf}(X)[-\dim X]$.

Proposition 7.2.1 *Let $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$, and suppose that X is smooth equidimensional. Let $p : X \times Y \longrightarrow Y$ be the second projection.*

Then the functor $p_* : D^b \mathrm{Perv}_{mf}(X \times Y) \longrightarrow D^b \mathrm{Perv}_{mf}(Y)$ admits a left adjoint p^*, which is given by $K \longmapsto \mathbb{1}_X \boxtimes K$.

In particular, we get a natural isomorphism $\theta_p : p^* \circ R_Y \xrightarrow{\sim} R_{X\times Y} \circ p^*$.

Proof. Let p^* be as in the statement. It suffices to construct natural morphisms $\varepsilon : \mathrm{id} \longrightarrow p_*p^*$ and $\eta : p^*p_* \longrightarrow \mathrm{id}$ whose images by R are the unit and counit of the adjonction in the categories $D^b \mathrm{Perv}_m$, and such that $p_* \xrightarrow{\varepsilon p_*} p_*p^*p_* \xrightarrow{p_*\eta} p_*$ is the identity. (As $R_{X\times Y}$ is conservative, we'll automatically get the fact that $p^* \xrightarrow{\eta p^*} p^*p_*p^* \xrightarrow{p^*\varepsilon} p^*$ is an isomorphism.)

Let $a_X : X \longrightarrow \mathrm{Spec}\, k$ be the structural map. Note that, as $a_{X*}\mathbb{1}_X \in D^{\geq 0} \mathrm{Perv}_{mf}(\mathrm{Spec}\, k)$, we have

$$\begin{aligned} \mathrm{Hom}_{D^b \mathrm{Perv}_{mf}(\mathrm{Spec}\, k)}(\mathbb{1}_{\mathrm{Spec}\, k}, a_{X*}\mathbb{1}_X) &= \mathrm{Hom}_{\mathrm{Perv}_{mf}(\mathrm{Spec}\, k)}(\mathbb{1}_{\mathrm{Spec}\, k}, \mathrm{H}^0 a_{X*}\mathbb{1}_X) \\ &= \mathrm{Hom}_{\mathrm{Perv}(\mathrm{Spec}\, k)}(E, \mathrm{H}^0 a_{X*}\underline{E}_X). \end{aligned}$$

So the canonical morphism $E \longrightarrow \mathrm{H}^0 a_{X*}\underline{E}_X$ (coming from the unit of the adjunction (a_X^*, a_{X*}) gives a morphism $u_X : \mathbb{1}_{\mathrm{Spec}\, k} \longrightarrow a_{X*}\mathbb{1}_X$ in $D^b \mathrm{Perv}_{mf}(\mathrm{Spec}\, k)$.

If $K \in \mathrm{Ob}(D^b \mathrm{Perv}_{mf}(Y))$, then we have a morphism

$$K = \mathbb{1}_{\mathrm{Spec}\, k} \boxtimes K \xrightarrow{u_X} (a_{X*}\mathbb{1}_X) \boxtimes K \simeq p_*(\mathbb{1}_X \boxtimes K) = p_*p^*K,$$

where the third arrow is the isomorphism of Proposition 7.1.6. This morphism is an isomorphism because its image by R_Y is an isomorphism, and we denote it by ε.

Now we want to construct η. Consider the commutative diagram

$$\begin{tikzcd}
X \times Y \arrow[d, "p"'] & X \times X \times Y \arrow[l, "q_2"'] \arrow[d, "q_1"] & X \times Y \arrow[l, "i"'] \\
Y & X \times Y \arrow[l, "p"]
\end{tikzcd}$$

where $q_1 = \mathrm{id}_X \times p$, $q_2 = a_X \times \mathrm{id}_{X\times Y}$ and i is the product of the diagonal embedding of X and of id_Y. Note that $q_1 i = q_2 i = p$. Using Proposition 7.1.6, we get an isomorphism

$$p^* p_* K = \mathbb{1}_X \boxtimes (p_* K) \simeq q_{1*}(\mathbb{1}_X \boxtimes K) = q_{1*} q_2^* K.$$

As i is a closed immersion, we know (by Corollary 6.4.2) that the functor i_* has a left adjoint i^*. This and the functoriality of $H_{mf,*}$ gives a morphism

$$q_{1*} q_2^* K \longrightarrow q_{1*} i_* i^* q_2^* K \simeq q_{2*} i_* i^* q_2^* K.$$

Note also that using the unit of (i^*, i_*) and the analogue of the natural transformation ε for q_2 instead of p, we get a morphism

$$K \xrightarrow{\sim} q_{2*} q_2^* K \longrightarrow q_{2*} i_* i^* q_2^* K,$$

which is an isomorphism because its image by $R_{X\times Y}$ is an isomorphism. Putting all these together gives

$$\eta : p^* p_* K \xrightarrow{\sim} q_{1*} q_2^* K \longrightarrow q_{2*} i_* i^* q_2^* K \simeq K.$$

It is clear from the construction that the images of ε and η by R are the unit and the counit of the adjonction (p^*, p_*) in D^b_m. So we just need to show that $p_* \xrightarrow{\varepsilon p_*} p_* p^* p_* \xrightarrow{p_* \eta} p_*$ is the identity. This follows from the fact that we get this composition by following the outside of the following commutative diagram in the clockwise direction (where the two arrows marked "adj" come from the unit of the adjunction (i^*, i_*)) :

$$\begin{tikzcd}
 & & & p_*(\mathbb{1}_X \boxtimes (p_* K)) \arrow[r, equal] \arrow[d, dash, "\wr"] & p_* p^* p_* K \\
 & \mathbb{1}_{\mathrm{Spec}\, k} \boxtimes (p_* K) \arrow[r, "u_X"] & (a_{X*}\mathbb{1}_X) \boxtimes (p_* K) \arrow[r, dash, "\sim"] & p_* q_{1*}(\mathbb{1}_X \boxtimes K) \arrow[r, equal] \arrow[dddl, dash, "\sim"] & p_* q_{1*} q_2^* K \arrow[d, "\mathrm{adj}"] \\
p_* K \arrow[ur, dash, "\sim"] \arrow[r, dash, "\sim"'] & p_*(\mathbb{1}_{\mathrm{Spec}\, k} \boxtimes K) \arrow[d, "u_X"'] & & & p_* q_{1*} i_* i^* q_2^* K \arrow[d, "\wr"'] \\
 & p_*((a_{X*}\mathbb{1}_X) \boxtimes K) \arrow[r, "\sim"] & p_* q_{2*}(\mathbb{1}_X \boxtimes K) \arrow[r, equal] & p_* q_{2*} q_2^* K \arrow[r, "\mathrm{adj}"'] & p_* q_{2*} i_* i^* q_2^* K
\end{tikzcd}$$

$\square$

Having at our disposal the constant sheaf on X was very important when constructing the inverse image of the second projection $X \times Y \longrightarrow Y$. Now, in order to generalize this construction to the case when X is not necessarily smooth, we want to construct (and characterize) the analogue in $D^b \operatorname{Perv}_{mf}(X)$ of the constant sheaf $\underline{E}_X$. Note that this is not totally obvious in this context because, if X is not smooth, then the constant sheaf is not perverse (or shifted perverse) in general.

For every k-scheme X, we denote by $a_X : X \longrightarrow \operatorname{Spec} k$ the structural morphism. We also denote by $\mathbb{1}_{\operatorname{Spec} k}$ the constant sheaf with value E on $\operatorname{Spec} k$, seen as an object of $\operatorname{Perv}_{mf}(\operatorname{Spec} k)$.

Corollary 7.2.2 *For every k-scheme X, the functor $D^b \operatorname{Perv}_{mf}(X) \longrightarrow \mathrm{Sets}$ (where Sets is the category of sets), $K \longmapsto \operatorname{Hom}_{D^b \operatorname{Perv}_{mf}(\operatorname{Spec} k)}(\mathbb{1}_{\operatorname{Spec} k}, a_{X*}K)$, is representable.*

Moreover, if $(\mathbb{1}_X, u_X : \mathbb{1}_{\operatorname{Spec} k} \longrightarrow a_{X*}\mathbb{1}_X)$ represents this functor, then there is an isomorphism $R_X(\mathbb{1}_X) \simeq \underline{E}_X$ that makes the following diagram commute :

$$\begin{tikzcd} R_{\operatorname{Spec} k}(\mathbb{1}_{\operatorname{Spec} k}) \arrow[r, equal] \arrow[d, "R_{\operatorname{Spec} k}(u_X)"'] & \underline{E}_{\operatorname{Spec} k} \arrow[r, "\mathrm{adj}"] & a_{X*}\underline{E}_X \arrow[d, "\wr"] \\ R_{\operatorname{Spec} k}(a_{X*}\mathbb{1}_X) \arrow[rr, "\gamma_{a_X}"'] & & a_{X*}R_X(\mathbb{1}_X) \end{tikzcd}$$

where the arrow marked "adj" is the unit of the adjunction (a_X^*, a_{X*}).

Note that the couple $(\mathbb{1}_X, u_X)$ is unique up to unique isomorphism if it exists.

Proof. First note that, thanks to Corollary 6.4.2 and Proposition 7.1.7, if $h : Z \longrightarrow X$ is an open embedding or a closed embedding and the result is true for X, then it is also true for Z, and moreover we have a canonical isomorphism $\mathbb{1}_Z \simeq h^*\mathbb{1}_X$. Moreover, if X is smooth, then the result follows immediately from Proposition 7.2.1. In particular, we get the result for X affine, because in that case X is a closed subscheme of some $\mathbb{A}^n$.

For a general k-scheme X, we chose a finite open cover $X = \bigcup_{i=1}^n U_i$ such that the result is known for every U_i. (For example, we can take a finite affine open cover.) We want to show that this implies the result for X. We reduce to the case $n = 2$ by an easy induction on n. Let $j_1 : U_1 \longrightarrow X$, $j_2 : U_2 \longrightarrow X$ and $j_{12} : U_1 \cap U_2 \longrightarrow X$ be the inclusions. By the uniqueness statement of the corollary, we have canonical isomorphisms $\mathbb{1}_{U_i|U_1 \cap U_2} \simeq \mathbb{1}_{U_1 \cap U_2}$ for $i = 1, 2$ that identify u_{U_i} and $u_{U_1 \cap U_2}$, so, using Proposition 7.1.7, we get morphisms $v_i : j_{i*}\mathbb{1}_{U_i} \longrightarrow j_{12,*}\mathbb{1}_{U_1 \cap U_2}$, $i = 1, 2$. Complete $v := v_1 \oplus (-v_2)$ into an exact triangle

$$(*) \qquad K \longrightarrow j_{1*}\mathbb{1}_{U_1} \oplus j_{2*}\mathbb{1}_{U_2} \xrightarrow{v} j_{12*}\mathbb{1}_{U_1 \cap U_2} \xrightarrow{+1}$$

Applying a_{X*}, we get a triangle

$$(**) \qquad a_{X*}K \longrightarrow a_{U_1,*}\mathbb{1}_{U_1} \oplus a_{U_2,*}\mathbb{1}_{U_2} \xrightarrow{a_{X*}v} a_{U_1 \cap U_2,*}\mathbb{1}_{U_1 \cap U_2} \xrightarrow{+1}$$

Consider the morphism $u_{U_1}\oplus u_{U_2}:\mathbb{1}_{\operatorname{Spec}k}\longrightarrow a_{U_1,*}\mathbb{1}_{U_1}\oplus a_{U_2,*}\mathbb{1}_{U_2}$. Composing it by $a_{X*}v$ gives 0, by definition of v, so it comes from a map $u_X:\mathbb{1}_{\operatorname{Spec}k}\longrightarrow a_{X*}K$. Also, as $a_{U_1\cap U_2,*}\mathbb{1}_{U_1\cap U_2}$ is concentrated in degree ≥ 0, we have $\operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(\operatorname{Spec}k)}(\mathbb{1}_{\operatorname{Spec}k},a_{U_1\cap U_2,*}\mathbb{1}_{U_1\cap U_2}[-1])=0$, and so the map u_X is uniquely determined.

Now we show that (K,u_X) represents the functor of the statement. For every $L\in\operatorname{Ob}(D^b\operatorname{Perv}_{mf}(X))$, the map $u_X:\mathbb{1}_{\operatorname{Spec}k}\longrightarrow a_{X*}K$ induces a morphism

$$\operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(X)}(K,L)\to\operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(\operatorname{Spec}k)}(a_{X*}K,a_{X*}L)\to\operatorname{Hom}_{D^b\operatorname{Perv}_{mf}(\operatorname{Spec}k)}(\mathbb{1}_{\operatorname{Spec}k},a_{X*}L),$$

and we must show that this is an isomorphism. Suppose that we can prove this if one of the adjunction maps $L\longrightarrow j_{1*}j_1^*L$, $L\longrightarrow j_{2*}j_2^*L$ or $L\longrightarrow j_{12*}j_{12}^*L$ is an isomorphism, then we are done. Indeed, for a general L, we have an exact triangle $L\longrightarrow j_{1*}j_1^*L\oplus j_{2*}j_2^*L\longrightarrow j_{12*}j_{12}^*L\xrightarrow{+1}$, and we use the five lemma.

Suppose that the adjunction map $L\longrightarrow j_{1*}j_1^*L$ is an isomorphism. Applying j_1^* to the triangle $(*)$ and noting that $j_1^*j_{2*}\mathbb{1}_{U_2}\longrightarrow j_1^*j_{12*}\mathbb{1}_{U_1\cap U_2}$ is an isomorphism, we get an isomorphism $j_1^*K\xrightarrow{\sim}\mathbb{1}_{U_1}$. We denote by c the base change morphism $a_{X*}\longrightarrow a_{U_1*}j_1^*$. Applying c to the entries of the triangle $(**)$, we get a commutative diagram

$$\begin{tikzcd}
a_{U_1*}j_1^*K \arrow[r] & a_{U_1,*}j_1^*j_{1*}\mathbb{1}_{U_1}\oplus a_{U_1,*}j_1^*j_{2*}\mathbb{1}_{U_2} \arrow[r] & a_{U_1*}j_1^*j_{12*}\mathbb{1}_{U_1\cap U_2} \arrow[r, "+1"] & {} \\
a_{X*}K \arrow[r] \arrow[u] & a_{U_1,*}\mathbb{1}_{U_1}\oplus a_{U_2,*}\mathbb{1}_{U_2} \arrow[r] \arrow[u] & a_{U_1\cap U_2,*}\mathbb{1}_{U_1\cap U_2} \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

The morphism $a_{U_1,*}j_1^*j_{2*}\mathbb{1}_{U_2}\longrightarrow a_{U_1*}j_1^*j_{12*}\mathbb{1}_{U_1\cap U_2}$ in the first row of this diagram is an isomorphism, so we get an isomorphism $a_{U_1*}j_1^*K\longrightarrow a_{U_1,*}j_1^*j_{1*}\mathbb{1}_{U_1}\simeq a_{U_1*}\mathbb{1}_{U_1}$ (which is just the image by a_{U_1*} of the isomorphism $j_1^*K\xrightarrow{\sim}\mathbb{1}_{U_1}$ of the beginning of this paragraph. By this isomorphism, the map $c_K\circ u_X:\mathbb{1}_{\operatorname{Spec}k}\longrightarrow a_{U_1*}j_1^*K$ corresponds to the composition of

$$(c_{j_{1*}\mathbb{1}_{U_1}}\oplus c_{j_{2*}\mathbb{1}_{U_2}})\circ(u_{U_1}\oplus u_{U_2}):\mathbb{1}_{\operatorname{Spec}k}\longrightarrow a_{U_1,*}j_1^*j_{1*}\mathbb{1}_{U_1}\oplus a_{U_1,*}j_1^*j_{2*}\mathbb{1}_{U_2}$$

and of the first projection. In other words, we get a commutative diagram :

$$\begin{tikzcd}
\mathbb{1}_{\operatorname{Spec}k} \arrow[r, "u_X"] \arrow[d, "u_{U_1}"'] & a_{X*}K \arrow[d, "c_K"] \\
a_{U_1*}\mathbb{1}_{U_1} & a_{U_1*}j_1^*K \arrow[l, "\sim"]
\end{tikzcd}$$

Consider the following diagram (where all the Hom groups are taken in the appropriate

$\mathrm{D^b\,Perv}_{mf}$ category) :

$$\begin{tikzcd}
\mathrm{Hom}(K, j_{1*}j_1^*L) \arrow[r, "a_{X*}"] \arrow[d, "j_1^*", "\wr"'] & \mathrm{Hom}(a_{X*}K, a_{X*}j_{1*}j_1^*L) \arrow[r, "(-)\circ u_X"] & \mathrm{Hom}(\mathbb{1}_{\mathrm{Spec}\,k}, a_{U_1*}j_1^*L) \arrow[dd, equal] \\
\mathrm{Hom}(j_1^*, j_1^*L) \arrow[r, "a_{U_1*}"] & \mathrm{Hom}(a_{U_1*}j_1^*K, a_{U_1*}j_1^*L) \arrow[u, "(-)\circ c_K"'] & \\
\mathrm{Hom}(\mathbb{1}_{U_1}, j_1^*L) \arrow[r, "a_{U_1*}"'] \arrow[u, "\wr"] & \mathrm{Hom}(a_{U_1*}\mathbb{1}_{U_1}, a_{U_1*}j_1^*L) \arrow[r, "(-)\circ u_{U_1}"'] \arrow[u, "\wr"'] & \mathrm{Hom}(\mathbb{1}_{\mathrm{Spec}\,k}, a_{U_1*}j_1^*L)
\end{tikzcd}$$

We have just seen that the right rectangle of this diagram is commutative. It is also easy to see that the two squares on the left are commutative, so the whole diagram commutes. As the composition of the two bottom horizontal arrows is an isomorphism by assumption, the composition of the two top horizontal arrows is also an isomorphism, which is what we wanted to prove.

The case where $L \longrightarrow j_{2*}j_2^*L$ (resp. $L \longrightarrow j_{12*}j_{12}^*L$) is an isomorphism is similar. This finishes the proof of the first statement of the corollary. The second statement of the corollary follows easily from the explicit definition of u_X.

□

Now that we have the object $\mathbb{1}_X$, the proof of the following corollary is exactly the same as the proof of Proposition 7.2.1.

Corollary 7.2.3 *Let $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$, and let $p : X \times Y \longrightarrow Y$ be the second projection.*

Then the functor $p_* : D^b\,\mathrm{Perv}_{mf}(X \times Y) \longrightarrow \mathrm{D^b\,Perv}_{mf}(Y)$ admits a left adjoint p^*, which is given by $K \longmapsto \mathbb{1}_X \boxtimes K$.

Corollary 7.2.4 *The 2-functor $H_{mf,*} : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ of Proposition 7.1.1 admits a global left adjoint in the sense of Definition 1.1.18 of [1].*

In particular, we get a uniquely determined 2-functor $H_{mf}^* : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ such, for every morphism $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, the functor $H_{mf}^*(f) : D^b\,\mathrm{Perv}_{mf}(Y) \longrightarrow D^b\,\mathrm{Perv}_{mf}(X)$ is a left adjoint of $H_{mf,*}(f) : D^b\,\mathrm{Perv}_{mf}(X) \longrightarrow D^b\,\mathrm{Perv}_{mf}(Y)$.

Moreover, for every morphism of k-schemes $f : X \longrightarrow Y$, we have an invertible natural transformation $\theta_f : H_m^*(f) \circ R_Y \xrightarrow{\sim} R_X \circ H_{mf}^*(f)$, and this is compatible with the composition of morphisms in $\mathbf{Sch}/k$.

Proof. By Proposition 1.1.17 of [1], to show the first statement, it suffices to show that, for every $f : X \longrightarrow Y$ in $\mathbf{Sch}/k$, the functor $H_{mf,*}(f) : D^b\,\mathrm{Perv}_{mf}(X) \longrightarrow D^b\,\mathrm{Perv}_{mf}(Y)$ admits a left adjoint. We factor f as $X \xrightarrow{i} X \times Y \xrightarrow{p} Y$, where $i = \mathrm{id}_X \times f$ and p is the second projection. The first map is a closed embedding, so it admits a left adjoint by Corollary 6.4.2, and the second map admits a left adjoint by Corollary 7.2.3. The natural transformation θ_i and θ_p are also constructed in these corollaries, and we take θ_f equal to :

$$R_X \circ f^* = R_X \circ i^* p^* \xrightarrow{\theta_i} i^* \circ R_{X\times Y} \circ p^* \xrightarrow{\theta_p} i^* p^* \circ R_Y \simeq f^* \circ R_Y.$$

By a slight abuse of notation, we will write that $\theta_f = \theta_p \circ \theta_i$.

Suppose that we are given a second morphism $g : Y \longrightarrow Z$, and that we are trying to prove the compatibility between θ_f, θ_g and θ_{gf}. Consider the commutative diagram :

$$\begin{tikzcd}
 & & & X \times Z \arrow[dddl, "p'''"] \\
X \arrow[r, "i"'] \arrow[urrr, "i'''"] \arrow[dr, "f"'] & X \times Y \arrow[r, "i''"'] \arrow[d, "p"] & X \times Y \times Z \arrow[ur, "q"'] \arrow[d, "p''"'] & \\
 & Y \arrow[r, "i'"'] \arrow[dr, "g"'] & Y \times Z \arrow[d, "p'"'] & \\
 & & Z &
\end{tikzcd}$$

where $i' = \mathrm{id}_Y \times g$, $i''(x,y) = (x, y, g(y))$, $i''' = \mathrm{id}_X \times (gf)$ and p', p'', p''', q are the obvious projections. Then $\theta_g = \theta_{p'} \circ \theta_{i'}$ and $\theta_{gf} = \theta_{p'''} \circ \theta_{i'''}$. So it suffices to prove that :

(a) $\theta_q \circ \theta_{i''i} = \theta_{i'''}$;

(b) $\theta_{i''i} = \theta_{i''}\theta_i$;

(c) $\theta_{p''} \circ \theta_q = \theta_{p'} \circ \theta_{p''}$;

(d) $\theta_{p''} \circ \theta_{i''} = \theta_{i'} \circ \theta_p$.

Point (b) follows from Corollary 6.4.2 and point (c) from the explicit formula for the inverse image of a projection in Corollary 7.2.3. The other two compatibilities can easily be proved directly.

$\square$

Finally, we have :

Proposition 7.2.5 *The functor $\boxtimes$ of Proposition 4.2 induces a natural isomorphism between the 2-functors $H^*_{mf} \times H^*_{mf} : \mathbf{Sch}_k \times \mathbf{Sch}_k \longrightarrow \mathfrak{TR}$ and $\mathbf{Sch}_k \times \mathbf{Sch}_k \longrightarrow \mathbf{Sch}_k \xrightarrow{H^*_{mf}} \mathfrak{TR}$, where the first arrow sends (X, Y) to $X \times Y$.*

In other words, if $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ are morphisms in $\mathbf{Sch}/k$ and $L_1 \in D^b \operatorname{Perv}_{mf}(Y_1)$, $L_2 \in D^b \operatorname{Perv}_{mf}(Y_2)$, then we have an isomorphism

$$(f_1 \times f_2)^*(L_1 \boxtimes L_2) \xrightarrow{\sim} (f_1^* L_1) \boxtimes (f_2^* L_2)$$

functorial in L_1 and L_2 and compatible with the composition of arrows in $\mathbf{Sch}/k$.

Proof. By the construction of the functors f^* above, we only need to show the statement when f_1 and f_2 are both closed immersions, or when they are both projections. If f_1 and f_2 are both projections, the result is obvious. If they are both closed immersions, the result follows from the construction in the proof of Corollary 6.4.2 and from Proposition 7.1.6.

□

7.3 Poincaré-Verdier duality

Just as in sections 7.1 and 7.2, we can prove the following result.

Proposition 7.3.1 *There exists a 2-functor $H_{mf,!} : \mathbf{Sch}/k \longrightarrow \mathfrak{TR}$ with $H_{mf,!}(X) = \mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)$ for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$ and a natural transformation $R : H_{mf,!} \longrightarrow H_{m,!}$ (with the notation of Example 3.2.3) such that :*

(a) for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, the functor $R_X : \mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ is the functor of Theorem 3.2.4;

(b) for every morphism $f : X \longrightarrow Y$, the natural transformation $\rho_f : R_Y \circ H_{mf,!}(f) \longrightarrow H_{m,!}(f) \circ R_X$ is an isomorphism.

This functor satisfies the same compatibility with $\boxtimes$ as in Proposition 7.1.6, and it admits a global right adjoint $H^!_{mf}$.

Moreover, by Proposition 4.2, we have an exact contravariant endofunctor D_X of $\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)$ together with an isomorphism $D_X^2 \simeq \mathrm{id}$, for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$.

Proposition 7.3.2 *Let $f : X \longrightarrow Y$ be a morphism of k-schemes.*

(i) We have a natural isomorphism $\alpha_f : f_* \xrightarrow{\sim} D_Y \circ f_! \circ D_X$ such that, if $g : Y \longrightarrow Z$ is another morphism of k-schemes, then the isomorphism $\alpha_{gf} : (gf)_* \simeq D_Z(gf)_! D_X$ is equal to the isomorphism

$$(gf)_* \simeq g_* f_* \xrightarrow{\alpha_g \alpha_f} D_Z g_! D_Y D_Y f_! D_X \simeq D_Z g_! f_! D_X \simeq D_Z (gf)_! D_X$$

where the first and fourth arrows are given by the composition isomorphisms of the 2-functors $H_{mf,*}$ and $H_{mf,!}$, and the third arrow is given by the isomorphism $D_Y^2 \simeq \mathrm{id}$.

(ii) We have a natural isomorphism $\beta_f : f^* \xrightarrow{\sim} D_X \circ f^! \circ D_Y$, where $f^! = H^!_{mf}(f)$, such that, if $g : Y \longrightarrow Z$ is another morphism of k-schemes, then the isomorphism $\beta_{gf}(gf)^* \simeq D_X(gf)^! D_Z$ is equal to the isomorphism

$$(gf)^* \simeq f^* g^* \xrightarrow{\beta_f \beta_g} D_X f^! D_Y D_Y g^! D_Z \simeq D_X f^! g^! D_Z \simeq D_X (gf)^! D_Z$$

where the first and fourth arrows are given by the composition isomorphisms of the 2-functors H^*_{mf} and $H^!_{mf}$, and the third arrow is given by the isomorphism $D_Y^2 \simeq \mathrm{id}$.

(iii) If f is smooth and purely of relative dimension d, then we have an natural isomorphism $f^![-d] \simeq f^*[d](d)$ of functors $\mathrm{D^b}\,\mathrm{Perv}_{mf}(Y) \longrightarrow \mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$.

(iv) If f is smooth and purely of relative dimension d, then the functor $f^* : \mathrm{D^b}\,\mathrm{Perv}_{mf}(Y) \longrightarrow \mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$ admits a left adjoint $f_\sharp$.

(v) If $i : X \longrightarrow Y$ is a closed immersion, then we have a natural isomorphism $i_! \xrightarrow{\sim} i_*$.

Proof. Point (iii) follows from the fact that both functors are t-exact and that such an isomorphism exists in the category of functors $\mathrm{Perv}_{mf}(Y) \longrightarrow \mathrm{Perv}_{mf}(X)$ (because it does for mixed perverse sheaves and the categories Perv_{mf} are full subcategories of the categories of mixed perverse sheaves).

Point (iv) follows from point (iii) : take $f_\sharp = f_![2d](d)$.

Point (v) is proved like point (iii) : both functors are t-exact, and the natural isomorphism exists when we see $i_!$ and i_* as functors from $\mathrm{Perv}_m(X)$ to $\mathrm{Perv}_m(Y)$.

Let's prove (i). By the construction of f_* in section 7.1 (and point (iii) applied to inverse images by open immersions) it suffices to prove the analogous result for the functors ${}^p\mathrm{H}^0 f_*$ and ${}^p\mathrm{H}^0 f_!$ if f is affine. But then this follows from the case of the categories $\mathrm{D^b_c}(X)$.

Point (ii) now follows from (i) and from the uniqueness of adjoint functors.

□

8 Tensor products and internal Homs

Definition 8.1 Let X be a k-scheme. We denote by $\Delta_X : X \longrightarrow X \times X$ the diagonal embedding. We define a functor $\otimes_X : (\mathrm{D^b}\,\mathrm{Perv}_{mf}(X))^2 \longrightarrow \mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$ by $K \otimes_X L = \Delta_X^*(K \boxtimes L)$.

Note that it follows from Proposition 7.2.5 that, for every morphism of k-schemes $f : X \longrightarrow Y$ and all $K, L \in \mathrm{Ob}\,\mathrm{D^b}\,\mathrm{Perv}_{mf}(Y)$, we have a canonical isomorphism

$$f^*(K \otimes_Y L) \simeq (f^*K) \otimes_X (f^*L).$$

Proposition 8.2 *The operation $\otimes_X$ defined above makes $\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$ into a symmetric monoïdal triangulated category. Also, the object $\mathbb{1}_X$ constructed in Corollary 7.2.2 is a unit for $\otimes_X$, and the functor $R_X : \mathrm{D^b}\,\mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{D^b_m}(X)$ is symmetric monoïdal unitary.*

Proof. The first statement follows easily from the commutativity and associativity of $\boxtimes$ (which in turn follows from the similar statement in $\mathrm{D^b_m}(X)$, as $\boxtimes$ is exact). Moreover, for every

$K \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}} \operatorname{Perv}_{mf}(X)$, if $p : X \times X \longrightarrow X$ is the second projection, then :

$$K \otimes_X \mathbb{1}_X = \Delta_X^*(K \boxtimes \mathbb{1}_X) = \Delta_X^* p^* K \simeq (p\Delta_X)^* K \simeq K$$

because $p\Delta_X = \mathrm{id}_X$. This proves the second statement. Finally, the fact that R_X is monoïdal follows from the fact that it preserves $\boxtimes$, and the last statement of Corollary 7.2.2 (i.e., the isomorphism $R_X(\mathbb{1}_X) \simeq \underline{E}_X$) implies that R_X is unitary.

$\square$

The main result of this section in the following :

Proposition 8.3 *For every k-scheme X and every $K \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}} \operatorname{Perv}_{mf}(X)$, the endofunctor $K \otimes_X \cdot$ of $\mathrm{D}^{\mathrm{b}} \operatorname{Perv}_{mf}(X)$ has a right adjoint $\underline{\operatorname{Hom}}(K, \cdot)$, given by $L \longmapsto D_X(K \otimes_X D_X(L))$. Moreover, for all $K, L, M \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}} \operatorname{Perv}_{mf}(X)$, we have a commutative diagram*

$$\begin{tikzcd}
R\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)}(K \otimes_X L, M) \arrow[r, "\sim"] \arrow[d, "R_X"'] & R\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)}(L, D_X(K \otimes_X D_X M)) \arrow[d, "R_X"] \\
R\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)}(R_X(K) \otimes_X R_X(L), R_X(M)) \arrow[r, "\sim"'] & R\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)}(R_X(L), D_X(R_X(K) \otimes_X D_X R_X(M)))
\end{tikzcd}$$

where the horizontal arrows are the adjunction isomorphisms (see Lemma 8.4 for the identification $D_X(R_X(K) \otimes_X D_X R_X(M)) = \underline{\operatorname{Hom}}_X(R_X(K), R_X(M))$).

In the lemmas that follow, we will denote the structural morphism $X \longrightarrow \operatorname{Spec} k$ by a. Remember that we write $K_X = a^! \underline{E}_{\operatorname{Spec} k}$ for the dualizing complex in $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)$. This is an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ (because $\underline{E}_{\operatorname{Spec} k}$ clearly is a mixed complex, and $a^!$ preserves mixed complexes).

Lemma 8.4 *In the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)$, we have a canonical isomorphism, functorial in K and L :*

$$\underline{\operatorname{Hom}}_X(K \otimes_X L, K_X) \simeq \underline{\operatorname{Hom}}_X(K, D_X(L)).$$

Moreover, these complexes are concentrated in perverse degree ≥ 0 if K and L are perverse.

In particular, if we replace L by $D_X(L)$, we get a natural isomorphism

$$\underline{\operatorname{Hom}}_X(K, L) \simeq D_X(K \otimes_X D_X(L)),$$

which explains the definition of the internal Hom given in Proposition 8.3.

Proof. By Theorem 6.3(ii) of [10] (see also the remark following Definition 1.2 of [14] for the extension of this to the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)$), we have a natural isomorphism

$$\underline{\operatorname{Hom}}_X(K \otimes_X L, M) = \underline{\operatorname{Hom}}_X(K, \underline{\operatorname{Hom}}_X(L, M))$$

for all $K, L, M \in \operatorname{Ob} \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)$. Applying this to $M = K_X$ gives the desired isomorphism.

If K and L are perverse, then the complex $K \otimes_X L$ is concentrated in perverse degree ≤ 0 (because it is equal by definition to $\Delta_X^*(K \boxtimes L)$, where $\Delta_X : X \longrightarrow X \times X$ is the diagonal morphism, and Δ_X^* is right t-exact), so its dual $\underline{\mathrm{Hom}}_X(K \otimes_X L, K_X)$ is concentrated in perverse degree ≥ 0.

$\square$

Lemma 8.5 *If $K, L \in \mathrm{Ob}\,\mathrm{Perv}_h(X)$, then the complex $a_!(K \otimes_X L) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathrm{Spec}\, k)$ is concentrated in degree ≤ 0, and so the adjunction $(a_!, a^!)$ gives a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X L, K_X) \simeq \mathrm{Hom}_{\mathrm{Perv}_h(\mathrm{Spec}\, k)}(\mathrm{H}^0(a_!(K \otimes_X L)), \underline{E}_{\mathrm{Spec}\, k})$$

and equalities

$$\mathrm{Ext}^i_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X L, K_X) = 0$$

for every $i < 0$.

We could also have deduced the vanishing of $\mathrm{Ext}^i_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X L, K_X)$ for $i < 0$ from the adjunction isomorphism $\mathrm{Ext}^i_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X L, K_X) = \mathrm{Ext}^i(K, D_X(L))$. (But we won't be able to do this in the next lemma, which is the analogous statement in $\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)$.)

Proof. We have

$$a_!(K \otimes_X L) \simeq D_{\mathrm{Spec}\, k}(a_* D_X(K \otimes_X L)) \simeq D_{\mathrm{Spec}\, k}(a_* \underline{\mathrm{Hom}}_X(K, D_X(L))),$$

where the second isomorphism comes from Lemma 8.4. So it suffices to show that $a_* \underline{\mathrm{Hom}}_X(K, D_X(L))) = R\,\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K, D_X(L))$ is concentrated in degree ≥ 0. As K and $D_X(L)$ are perverse, this just follows from the definition of a t-structure.

Now, using the adjunction $(a_!, a^!)$ and the fact that $K_X = a^! \underline{E}_{\mathrm{Spec}\, k}$, we get a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X L, K_X) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(\mathrm{Spec}\, k)}(a_!(K \otimes_X L), \underline{E}_{\mathrm{Spec}\, k}).$$

The second statement follows from this and from the fact that $a_!(K \otimes_X L)$ is concentrated in degree ≤ 0.

$\square$

Lemma 8.6 *If $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(X)$, then the complex $a_!(K \otimes_X L) \in \mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(\mathrm{Spec}\, k)$ is concentrated in degree ≤ 0, and so the adjunction $(a_!, a^!)$ gives a canonical isomorphism*

$$\mathrm{Hom}_{Db\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbb{1}_{\mathrm{Spec}\, k}) \simeq \mathrm{Hom}_{\mathrm{Perv}_{mf}(\mathrm{Spec}\, k)}(\mathrm{H}^0(a_!(K \otimes_X L)), \mathbb{1}_{\mathrm{Spec}\, k})$$

and equalities

$$\mathrm{Ext}^i_{\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbb{1}_X) = 0$$

for every $i < 0$.

Proof. We have
$$R_X(a_!(K \otimes_X L)) \simeq a_!(R_X(K) \otimes_X R_X(L)),$$
so $R_X(a_!(K \otimes_X L))$ is concentrated in degree ≤ 0 by Lemma 8.5. The first statement follows from the conservativity of R_X. The second statement is proved exactly as the second statement of Lemma 8.5, using the adjunction $(a_!, a^!)$ in the categories $\mathrm{D^b}\,\mathrm{Perv}_{mf}$.

$\square$

Lemma 8.7 *Let $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(X)$, write $K' = R_X(K)$, $L' = R_X(L)$. Then the morphism*
$$R_X : \mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, D_X(\mathbb{1}_X)) \longrightarrow \mathrm{Hom}_{\mathrm{D^b_h}(X)}(K' \otimes_X L', K_X)$$
is an isomorphism. In particular, there exists a unique isomorphism $\alpha_{K,L} : \mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, D_X(\mathbb{1}_X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Perv}_{mf}(X)}(K, D_X(L))$, making the following diagram commute

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, D_X(\mathbb{1}_X)) \arrow[r, "\alpha_{K,L}", "\sim"'] \arrow[d, "R_X"'] & \mathrm{Hom}_{\mathrm{Perv}_{mf}(X)}(K, D_X(L)) \arrow[d, "R_X"] \\ \mathrm{Hom}_{\mathrm{D^b_h}(X)}(K' \otimes_X L', K_X) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathrm{Perv}_h(X)}(K', D_X(L')) \end{tikzcd}$$

where the bottom isomorphism comes from applying the functor $\mathrm{H}^0(X, .)$ to the isomorphism of Lemma 8.4.

Proof. As $\mathrm{Perv}_{mf}(X)$ is a full subcategory of $\mathrm{Perv}_h(X)$, the morphism $R_X : \mathrm{Hom}_{\mathrm{Perv}_{mf}(X)}(K, D_X(L)) \longrightarrow \mathrm{Hom}_{\mathrm{Perv}_h(X)}(K', D_X(L'))$ is an isomorphism. So we just need to show that
$$R_X : \mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, D_X(\mathbb{1}_X)) \longrightarrow \mathrm{Hom}_{\mathrm{D^b_h}(X)}(K' \otimes_X L', K_X)$$
is an isomorphism. By Lemmas 8.5 and 8.6, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbb{1}_{\mathrm{Spec}\,k}) \arrow[r, "\sim"] \arrow[d, "R_X"'] & \mathrm{Hom}_{\mathrm{Perv}_{mf}(\mathrm{Spec}\,k)}(\mathrm{H}^0(a_!(K \otimes_X L)), \mathbb{1}_{\mathrm{Spec}\,k}) \arrow[d, "R_{\mathrm{Spec}\,k}"] \\ \mathrm{Hom}_{\mathrm{D^b_h}(X)}(K' \otimes_X L', K_X) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathrm{Perv}_h(\mathrm{Spec}\,k)}(\mathrm{H}^0(a_!(K' \otimes_X L')), \underline{E}_{\mathrm{Spec}\,k}) \end{tikzcd}$$

The right vertical map in this diagram is an isomorphism because $\mathrm{Perv}_{mf}(\mathrm{Spec}\,k)$ is a full subcategory of $\mathrm{Perv}_h(\mathrm{Spec}\,k)$, so the left vertical map is also an isomorphism.

$\square$

As in section 2.3, we will use the filtered derived category of an abelian category. Let $\mathscr{A}$ be an abelian category, and let $\mathrm{DF}(\mathscr{A})$ be its filtered derived category. Let us recall the spectral sequence of [6] (3.1.3.4) : If K and L are two objects of $\mathrm{DF}(A)$ such that $\mathrm{Gr}^i_F K = \mathrm{Gr}^i_F L = 0$ for $|i|$ big enough (i.e. such that the filtrations are finite on K and L), then we have a spectral sequence

$$E_1^{pq} = \bigoplus_{j-i=p} \mathrm{Ext}^{p+q}_{\mathrm{D}(\mathscr{A})}(\mathrm{Gr}^i_F K, \mathrm{Gr}^j_F L) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}(\mathscr{A})}(\omega(K), \omega(L)).$$

Remember that $\omega : \mathrm{DF}(\mathscr{A}) \longrightarrow \mathrm{D}(\mathscr{A})$ is the functor that forgets the filtration.

Lemma 8.8 *Let $K^\bullet$ be a bounded complex of objects of $\mathrm{Perv}_h(X)$, and let K be its image by* $\mathrm{real} : \mathrm{D^b}\,\mathrm{Perv}_h(X) \longrightarrow \mathrm{D^b_h}(X)$*. Then, for every object L of $\mathrm{D^b_h}(X)$, we have a spectral sequence*

$$E_1^{pq} = \bigoplus_{a-b=-p} \mathrm{Ext}^{q}_{\mathrm{D^b_h}(X)}(K^a \otimes_X D_X(K^b), L) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathrm{D^b_h}(X)}(K \otimes_X D_X(K), L).$$

Proof. By definition of the category $\mathrm{D^b_h}(X)$, it suffices to prove the statement in $\mathrm{D^b_c}(\mathscr{X})$, where $(A, \mathscr{X}, u)$ is an object of $\mathscr{U}X$ such that all the K^i (resp. L) extend to shifts of objects of $\mathrm{Perv}(\mathscr{X})$ (resp. $\mathrm{D^b_c}(\mathscr{X})$), that we will denote by the same letters.

Remember the construction of the realization functor $\mathrm{D^b}\,\mathrm{Perv}(\mathscr{X}) \longrightarrow \mathrm{D^b_c}(\mathscr{X})$ at the beginning of section 2.4 : We consider the full subcategory $\mathrm{DF}_{\text{bête}}(\mathscr{X})$ of objects A of $\mathrm{DF}(\mathscr{X}_{\text{proét}})$ such that $\mathrm{Gr}^i_F A[i]$ is in $\mathrm{Perv}(\mathscr{X})$ for every $i \in \mathbb{Z}$ and 0 for $|i|$ big enough. We have a functor $G : \mathrm{DF}_{\text{bête}}(\mathscr{X}) \longrightarrow C^b(\mathrm{Perv}(\mathscr{X}))$ (see [6] 3.1.7 or section 2.3) that turns out to be an equivalence of categories, and real is induced by $\omega \circ G^{-1} : C^b(\mathrm{Perv}(\mathscr{X})) \longrightarrow \mathrm{D^b_c}(\mathscr{X})$.

Let $\Delta : X \longrightarrow X \times X$ be the diagonal morphism. As $K \otimes_X D_X(K) = \Delta^*(K \boxtimes D_X(K))$, we have a canonical isomorphism

$$R\,\mathrm{Hom}_{\mathrm{D^b_c}(\mathscr{X})}(K \otimes_X D_X(K), L) = R\,\mathrm{Hom}_{\mathrm{D^b_c}(\mathscr{X}\times\mathscr{X})}(K \boxtimes D_X(K), \Delta_* L).$$

Let $M = G^{-1}(K^\bullet \boxtimes D_X(K^\bullet)) \in \mathrm{Ob}\,\mathrm{DF}_{\text{bête}}(\mathscr{X} \times \mathscr{X})$. We can also see $\Delta_* L$ as an object of $\mathrm{DF}((\mathscr{X} \times \mathscr{X})_{\text{proét}})$ (because, for any abelian category $\mathscr{A}$, the category $D(\mathscr{A})$ is canonically equivalent to the full subcategory of $A \in \mathrm{Ob}\,\mathrm{DF}(\mathscr{A})$) such that $\mathrm{Gr}^i_F A = 0$ for $i \neq 0$). Using the spectral sequence recalled before the statement of the lemma (and (iii) of Proposition 2.4.1), we get a spectral sequence

$$E_1^{pq} = \bigoplus_{-i=p} \mathrm{Ext}^{p+q}_{\mathrm{D^b_c}(\mathscr{X}\times\mathscr{X})}(\mathrm{Gr}^i_F(M), \Delta_* L) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathrm{D^b_c}(\mathscr{X})}(K \otimes_X D_X(K), L).$$

For every $i \in \mathbb{Z}$, we have

$$\mathrm{Gr}^i_F M = \bigoplus_{a+b=i} K^a[-a] \boxtimes D_X(K^{-b})[-b] = \bigoplus_{a-b=i} (K^a \boxtimes D_X(K^b))[-i].$$

So

$$E_1^{pq} = \bigoplus_{a-b=-p} \operatorname{Ext}^{p+q}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}\times\mathscr{X})}(K^a \boxtimes D_X(K^b), \Delta_*[-p]) = \bigoplus_{a-b=-p} \operatorname{Ext}^{p}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X})}(K^a \otimes_X D_X(K^b), L).$$

The statement of the lemma now follows by taking the limit over A', with $A \subset A' \in \mathscr{U}$.

$\square$

Lemma 8.9 *Let $K^\bullet$ be a bounded complex of objects of $\operatorname{Perv}_{mf}(X)$, and let K be its image by the canonical functor $\mathrm{C}^{\mathrm{b}}\operatorname{Perv}_{mf}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)$. Then, for every object L of $\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)$, we have a spectral sequence*

$$E_1^{pq} = \bigoplus_{a-b=-p} \operatorname{Ext}^{q}_{\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)}(K^a \otimes_X D_X(K^b), L) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathrm{D}^{\mathrm{b}}\operatorname{Perv}_{mf}(X)}(K \otimes_X D_X(K), L).$$

Moreover, the functor R_X induces a morphism of spectral sequences from this spectral sequence to the one of Lemma 8.8.

Proof. The proof is exactly the same as for Lemma 8.8, except that we work in the filtered derived category $\mathrm{DF}(\operatorname{Perv}_{mf}(X \times X))$. The last statement is obvious.

$\square$

Notation 8.10 Let $K \in \operatorname{Ob}\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)$. We denote by ι_K the evaluation morphism

$$K \otimes_X D_X(K) = K \otimes_X \underline{\operatorname{Hom}}_X(K, K_X) \longrightarrow K_X.$$

This morphism is obviously functorial in K.

Lemma 8.11 *Let $K^\bullet$ be a bounded complex of objects of $\operatorname{Perv}_h(X)$, and let K be its image by the functor $\operatorname{real} : \mathrm{D}^{\mathrm{b}}\operatorname{Perv}_h(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)$. Let*

$$E_1^{pq} \Longrightarrow \operatorname{Ext}^{p+q}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes D_X(K), K_X)$$

be the spectral sequence of Lemma 8.8 for $L = K_X$.

Then $E_1^{pq} = 0$ if $q < 0$, the element $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ of E_1^{00} is in $\operatorname{Ker}(E_1^{00} \longrightarrow E_1^{10}) = E_2^{00}$, and the element ι_K of $\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X D_X(K), K_X) \supset E_\infty^{00}$ is the image of $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ by the map $E_2^{00} \longrightarrow E_\infty^{00}$.

Proof. We have

$$E_1^{pq} = \bigoplus_{a-b=-p} \operatorname{Ext}^{q}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K^a \otimes_X D_X(K^b), K_X).$$

As all the K^a and $D_X(K^b)$ are perverse, this is 0 for $q < 0$ by Lemma 8.5. This implies that $E_2^{00} = \mathrm{Ker}(E_1^{00} \longrightarrow E_1^{10})$ and that $E_r^{pq} = 0$ for any $r \geq 1$ and any $q < 0$, so $E_\infty^{pq} = 0$ for $q < 0$. In particular, we get that E_∞^{00} is a quotient of E_2^{00} and that $E_\infty^{00} \subset \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K \otimes_X D_X(K), K_X)$. The last statement now follows from the construction of the spectral sequence (and (iii) of Proposition 2.4.1).

□

Lemma 8.12 *Let* $K^\bullet$ *be a bounded complex of objects of* $\mathrm{Perv}_{mf}(X)$*, and let* $K \in \mathrm{Ob}\,\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)$ *be its image by the obvious functor* $C^b\,\mathrm{Perv}_{mf}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)$*. Then there exists a unique morphism* $\iota_K : K \otimes_X D_X(K) \longrightarrow a^! \mathbb{1}_{\mathrm{Spec}\,k}$ *satisfying the following conditions :*

(a) The image of ι_K *by* R_X *is the morphism* $\iota_{R_X(K)}$ *of 8.10.*

(b) The analogue of Lemma 8.11 holds if we use the spectral sequence of Lemma 8.9.

This morphism is functorial in K*.*

Proof. The functoriality of ι_K follows from the uniqueness statement.

Let $K'^\bullet = R_X(K^\bullet)$ and $K' = R_X(K)$. If $K^\bullet$ is concentrated in degree 0, then, by Lemma 8.7, the morphism

$$R_X : \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X D_X(K), D_X(\mathbb{1}_X)) \longrightarrow \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{h}}(X)}(K' \otimes_X D_X(K'), K_X)$$

is an isomorphism. So condition (a) forces us to take $\iota_K = R_X^{-1}(\iota_{K'})$, and condition (b) is trivial in this case.

We now treat the general case. The spectral sequence of Lemma 8.9 for $L = a^! \mathbb{1}_X$ is

$$E_1^{pq} = \mathrm{Ext}^q_{\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)}(K^a \otimes_X D_X(K^b), a^! \mathbb{1}_X) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X D_X(K), a^! \mathbb{1}_X).$$

We have $E_1^{pq} = 0$ for $q < 0$ by Lemma 8.6. As in the proof of Lemma 8.11, this implies that $E_2^{00} = \mathrm{Ker}(E_1^{00} \longrightarrow E_1^{10})$ surjects to E_∞^{00} and that $E_\infty^{00} \subset \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X D_X(K), a^! \mathbb{1}_X)$. By condition (b), the element $\iota_K \in \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X D_X(K), a^! \mathbb{1}_X)$ that we want to construct must be the image of $\sum_{a \in \mathbb{Z}} \iota_{K^a} \in E_1^{00}$. As ι_{K^a} exists and is uniquely determined by the first case, it suffices to show that $\sum_{a \in \mathbb{Z}} \iota_{K^a} \in \mathrm{Ker}(E_1^{00} \longrightarrow E_1^{10})$. Indeed, condition (a) will then follow from the fact that R_X induces a morphism between the spectral sequences of Lemmas 8.8 and 8.9 (and from Lemma 8.11). We denote by $E_1^{pq}(K')$ the spectral sequence of Lemma 8.8 for $K'^\bullet$. Then we have a commutative diagram

$$\begin{tikzcd} E_1^{00} \arrow[r] \arrow[d, "R_X"'] & E_1^{10} \arrow[d, "R_X"] \\ E_1^{00}(K') \arrow[r] & E_1^{10}(K') \end{tikzcd}$$

By Lemma 8.7, the vertical maps in this diagram are isomorphisms. By Lemma 8.11, the image by R_X of $\sum_{a\in\mathbb{Z}} \iota_{K^a} \in E_1^{00}$, which is $\sum_{a\in\mathbb{Z}} \iota_{K'^a}$ by construction of the ι_{K^a}, is in $\mathrm{Ker}(E_1^{00}(K') \longrightarrow E_1^{10}(K'))$. So $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ is in $\mathrm{Ker}(E_1^{00} \longrightarrow E_1^{10})$, and we are done.

□

Lemma 8.13 *For $K, L \in \mathrm{Ob}\,\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$, we define a morphism*

$$u_{K,L} : R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(L, D_X(K)) \longrightarrow R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^!\mathbb{1}_{\mathrm{Spec}\,k})$$

as the composition of

$$K \otimes_X (.) : R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(L, D_X(K)) \longrightarrow R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, K \otimes_X D_X(K))$$

and of

$$\iota_{K*} : R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, K \otimes_X D_X(K)) \longrightarrow R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^!\mathbb{1}_{\mathrm{Spec}\,k}).$$

Then this morphism is functorial in K and L, its image by R_X is the adjunction morphism

$$R\,\mathrm{Hom}_{\mathrm{D^b_h}(X)}(R_X(L), D_X(R_X(K))) = R\,\mathrm{Hom}_{\mathrm{D^b_h}(X)}(R_X(K) \otimes_X R_X(L), K_X),$$

and it is an isomorphism.

Proof. The first statement is obvious and the second statement follows from property (a) of Lemma 8.12.

We first prove the third statement (i.e. that $u_{K,L}$ is an isomorphism) in the case where X is smooth and connected and $K = \mathscr{L}$, $L = \mathscr{M}$ are lisse sheaves on X. Let $d = \dim(X)$. Then we have $a^!\mathbb{1}_{\mathrm{Spec}\,k} = \mathbb{1}_X[2d](d)$ (by Proposition 2.5.2(i)) and $D_X(\mathscr{L}) = \mathscr{L}^*[2d](d)$, where $\mathscr{L}^* = \underline{\mathrm{Hom}}(\mathscr{L}, \underline{E}_X)$ is the dual locally constant sheaf (by the calculation at the end of section 2.1 and Proposition 2.6.2). So $u_{\mathscr{L},\mathscr{M}}$ is a morphism

$$R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(\mathscr{M}, \mathscr{L}^*) \longrightarrow R\,\mathrm{Hom}_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(\mathscr{L} \otimes_X \mathscr{M}, \mathbb{1}_X),$$

and the morphism $\iota_{\mathscr{L}} : \mathscr{L} \otimes D_X(\mathscr{L}) \longrightarrow a^!\mathbb{1}_{\mathrm{Spec}\,k}$ of Lemma 8.12 is just the the canonical morphism $\mathscr{L} \otimes_X \mathscr{L}^* \longrightarrow \mathbb{1}_X$, shifted by $2d$ and twisted by d (we see this easily from conditions (a) and (b) of Lemma 8.12, as $\mathscr{L}$ is perverse up to a shift). We will use the Yoneda description of the Ext^k groups, as in section 3.2 of chapter III of Verdier's book [29]. The definition of $u_{\mathscr{L},\mathscr{M}}$ gives the following formula for the image of a class c in

$$\mathrm{Ext}^i_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(\mathscr{M}, \mathscr{L}^*) = \mathrm{Ext}^i_{\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)}(\mathscr{M}[d], \mathscr{L}^*[d]) \ :$$

Choose an exact sequence in $\mathrm{Perv}_{mf}(X)$ representing c, say :

$$0 \longrightarrow \mathscr{L}^*[d] \longrightarrow K_{i-1} \longrightarrow \ldots \longrightarrow K_0 \longrightarrow \mathscr{M}[d] \longrightarrow 0.$$

Tensoring this sequence by $\mathscr{L}$, we still get an exact sequence in $\mathrm{Perv}_{mf}(X)$:

$$0 \longrightarrow \mathscr{L} \otimes_X \mathscr{L}^*[d] \longrightarrow \mathscr{L} \otimes_X K_{i-1} \longrightarrow \ldots \longrightarrow \mathscr{L} \otimes_X K_0 \longrightarrow \mathscr{L} \otimes_X \mathscr{M}[d] \longrightarrow 0.$$

Then $u_{\mathscr{L},\mathscr{M}}(c)$ is represented by the exact sequence

$$0 \longrightarrow \mathbb{1}_X[d] \longrightarrow K'_{i-1} \longrightarrow \mathscr{L} \otimes_X K_{i-2} \longrightarrow \ldots \longrightarrow \mathscr{L} \otimes_X K_0 \longrightarrow \mathscr{L} \otimes_X \mathscr{M}[d] \longrightarrow 0,$$

where K'_{i-1} is the amalgamated sum

$$\mathbb{1}_X[d] \oplus_{\mathscr{L} \otimes_X \mathscr{L}^*[d]} (\mathscr{L} \otimes_X K_{i-1})$$

with the morphism $\mathscr{L} \otimes_X \mathscr{L}^*[d] \longrightarrow \mathbb{1}_X[d]$ being the shift of the obvious one. We want to show that $u_{\mathscr{L},\mathscr{M}}$ is bijective, so it suffices to construct its inverse. Suppose that c' is an element of

$$\mathrm{Ext}^i_{\mathrm{D^b\,Perv}_{mf}(X)}(\mathscr{L} \otimes_X \mathscr{M}, \mathbb{1}_X) = \mathrm{Ext}^i_{\mathrm{D^b\,Perv}_{mf}(X)}(\mathscr{L} \otimes_X \mathscr{M}[d], \mathbb{1}_X[d]),$$

and choose an exact sequence in $\mathrm{Perv}_{mf}(X)$ representing c', say :

$$0 \longrightarrow \mathbb{1}_X[d] \longrightarrow L_{i-1} \longrightarrow \ldots \longrightarrow L_0 \longrightarrow \mathscr{L} \otimes_X \mathscr{M}[d] \longrightarrow 0.$$

Tensoring this sequence by $\mathscr{L}^*$, we still get an exact sequence in $\mathrm{Perv}_{mf}(X)$:

$$0 \longrightarrow \mathscr{L}^*[d] \longrightarrow \mathscr{L}^* \otimes_X L_{i-1} \longrightarrow \ldots \longrightarrow \mathscr{L}^* \otimes_X L_0 \longrightarrow \mathscr{L}^* \otimes_X \mathscr{L} \otimes_X \mathscr{M}[d] \longrightarrow 0.$$

We send c' to the element of $\mathrm{Ext}^i_{\mathrm{Perv}_{mf}(X)}(\mathscr{M}[d], \mathscr{L}^*[d])$ represented by the exact sequence

$$0 \longrightarrow \mathscr{L}^*[d] \longrightarrow \mathscr{L}^* \otimes_X L_{i-1} \longrightarrow \ldots \longrightarrow \mathscr{L}^* \otimes_X L_1 \longrightarrow L'_0 \longrightarrow \mathscr{M}[d] \longrightarrow 0,$$

where L'_0 is the fiber product

$$(\mathscr{L}^* \otimes_X L_0) \times_{\mathscr{M}[d]} (\mathscr{L}^* \otimes_X \mathscr{L} \otimes_X \mathscr{M}[d])$$

with the morphism $\mathscr{L}^* \otimes_X \mathscr{L} \otimes_X \mathscr{M}[d] \longrightarrow \mathscr{M}[d]$ coming from $\mathscr{L}^* \otimes_X \mathscr{L} \longrightarrow \mathbb{1}_X$ by tensoring by $\mathscr{M}[d]$. This is clearly the inverse of $u_{\mathscr{L},\mathscr{M}}$.

Now we show that the morphism $u_{K,L}$ is an isomorphism for all $K, L \in \mathrm{Ob}\,\mathrm{D^b}\,\mathrm{Perv}_{mf}(X)$. Note the following two reductions : First, using the fact that all the functors are triangulated and the five lemma, we see that if we have an exact triangle

$$K' \longrightarrow K \longrightarrow K'' \xrightarrow{+1}$$

such that the result is true for (K', L) and (K'', L), then the result if true for (K, L). There is a similar statement for the second variable L. So it suffices to prove the result for K and L concentrated in perverse degree 0, and we may also assume that K and L are simple perverse sheaves. Second, suppose that we have a closed immersion $i : Y \longrightarrow X$, and let $j : U := X - Y \longrightarrow X$ be the complementary open immersion. Then we have a commutative diagram whose columns are distinguished triangles (all the $R\,\mathrm{Hom}$s are taken in the appropriate category $\mathrm{D^b\,Perv}_{mf}(Z)$, with $Z \in \{X, U, Y\}$) :

$$\begin{tikzcd}
R\,\mathrm{Hom}(i^*L, i^!D_X K) \arrow[r, "i^*u_{K,L}"] \arrow[d] & R\,\mathrm{Hom}(i^*(K \otimes_X L), i^!a^!\mathbb{1}_{\mathrm{Spec}\,k}) \arrow[d] \\
R\,\mathrm{Hom}(L, D_X K) \arrow[r, "u_{K,L}"] \arrow[d] & R\,\mathrm{Hom}(K \otimes_X L, a^!\mathbb{1}_{\mathrm{Spec}\,k}) \arrow[d] \\
R\,\mathrm{Hom}(j^*L, j^*D_X K) \arrow[r, "j^*u_{K,L}"] \arrow[d, "+1"'] & R\,\mathrm{Hom}(j^*(K \otimes_X L), j^*a^!\mathbb{1}_{\mathrm{Spec}\,k}) \arrow[d, "+1"'] \\
{} & {}
\end{tikzcd}$$

Moreover, using the compatibility of $\otimes_X$ with inverse images and point (ii) of Proposition 7.3.2, we get isomorphisms :

$$R\,\mathrm{Hom}(i^*(K \otimes_X L), i^!a^!\mathbb{1}_{\mathrm{Spec}\,k}) \simeq R\,\mathrm{Hom}((i^*K) \otimes_Y (i^*L), a_Y^!\mathbb{1}_{\mathrm{Spec}\,k})$$

and

$$R\,\mathrm{Hom}(j^*(K \otimes_X L), j^*a^!\mathbb{1}_{\mathrm{Spec}\,k}) \simeq R\,\mathrm{Hom}((j^*K) \otimes_U (j^*L), a_U^!\mathbb{1}_{\mathrm{Spec}\,k}),$$

where $a_Y = a \circ i$ and $a_U = a \circ j$. It is easy to see that these isomorphisms identify $i^*u_{K,L}$ (resp. $j^*u_{K,L}$) with u_{i^*K,i^*L} (resp. u_{j^*K,j^*L}). So the result for X follows from the result for Y and U.

Using the two reductions above and Noetherian induction on X, we can reduce to the case where X is smooth and K and L are both shifts of locally constant sheaves on X. But this case has already been treated in the first part of the proof.

□

Proof of Proposition 8.3. We have to construct an isomorphism

$$R\,\mathrm{Hom}_{\mathrm{D^b\,Perv}_{mf}(X)}(K \otimes_X L, M) \xrightarrow{\sim} R\,\mathrm{Hom}_{\mathrm{D^b\,Perv}_{mf}(X)}(L, D_X(K \otimes_X D_X M))$$

functorial in $K, L, M \in \mathrm{Ob\,D^b\,Perv}_{mf}(X)$ and compatible (via R_X) with the adjunction morphism in $\mathrm{D^b_h}(X)$. But such an isomorphism is given by

$$u_{L,K\otimes_X D_X M} \circ u^{-1}_{K\otimes_X L, D_X M},$$

where $u_{.,.}$ is constructed in Lemma 8.13.

□

9 Weight filtration on complexes

The goal of this section is to generalize the results of section 3 of [21], and in particular the formula for the intermediate extension of a pure perverse sheaf, to the categories $\mathrm{Perv}_{mf}(X)$ and their derived categories. This was the original motivation for considering the categories $D^b\,\mathrm{Perv}_{mf}(X)$.

Definition 9.1 Let X be a k-scheme. For every $a \in \mathbb{Z} \cup \{\pm\infty\}$, we denote by ${}^w\,\mathrm{D}^{\leq a}(X)$ (resp. ${}^w\,\mathrm{D}^{\geq a}(X)$) the full subcategory of $D^b\,\mathrm{Perv}_{mf}(X)$ whose objects are the complexes K such that, for every $i \in \mathbb{Z}$, $\mathrm{H}^i K \in \mathrm{Perv}_{mf}(X)$ is of weight $\leq a$ (resp. $\geq a$).

Note that ${}^w\,\mathrm{D}^{\leq a}(X)$ and ${}^w\,\mathrm{D}^{\geq a}(X)$ are triangulated subcategories of $D^b\,\mathrm{Perv}_{mf}(X)$.

Proposition 9.2 *Let $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(X)$. Suppose that there exists $a \in \mathbb{Z}$ such that K is of weight $\leq a$ and L is of weight $\geq a+1$. Then we have, for every $i \in \mathbb{Z}$,*

$$\mathrm{Ext}^i_{\mathrm{Perv}_{mf}(X)}(K, L) = 0.$$

For categories like that of mixed Hodge modules, this result follows from Lemma 6.9 of [24], but M. Saito assumes (and uses) the fact that pure objects are semisimple, which is false in our case.

Proof. We obviously have $\mathrm{Ext}^i_{\mathrm{Perv}_{mf}(X)}(K, L) = 0$ if $i < 0$, and $\mathrm{Hom}_{\mathrm{Perv}_{mf}(X)}(K, L) = 0$ because the weights of K and L are disjoint. We denote by W the weight filtration on objects of $\mathrm{Perv}_{mf}(X)$. For every $b \in \mathbb{Z}$, we get an endofunctor W_b of $\mathrm{Perv}_{mf}(X)$, which is exact because weight filtrations are strictly compatible with morphisms in $\mathrm{Perv}_{mf}(X)$ (by Lemma 3.8 of [14]).

As in the proof of Proposition 8.3, we will use the Yoneda description of the Ext^k groups (see section 3.2 of chapter III of Verdier's book [29]). Let $i \geq 1$ and let $\alpha \in \mathrm{Ext}^i_{\mathrm{Perv}_{mf}(X)}(K, L)$. Choose an exact sequence

$$0 \longrightarrow L \xrightarrow{u_i} M_{i-1} \xrightarrow{u_{i-1}} \dots \xrightarrow{u_1} M_0 \xrightarrow{u_0} K \longrightarrow 0$$

in $\mathrm{Perv}_{mf}(X)$ that represents α. Applying W_a to this exact sequence and using the fact that $W_a K = K$ and $W_a L = 0$, we get a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "u_i"] & M_{i-1} \arrow[r, "u_{i-1}"] & \dots \arrow[r, "u_1"] & M_0 \arrow[r, "u_0"] & K \arrow[r] & 0 \\
0 \arrow[r] & L \arrow[r, "\mathrm{id}_L+0"] \arrow[u, equal, "u+\mathrm{can}"] & L \oplus W_a M_{i-1} \arrow[r, "0+u_{i-1}"] \arrow[u, "\mathrm{can}"] & \dots \arrow[r, "u_1"] & W_a M_0 \arrow[r, "u_0"] \arrow[u, "\mathrm{can}"] & K \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $\mathrm{can} : W_a \longrightarrow \mathrm{id}$ is the canonical inclusion. So the class α is also represented by the second row of this diagram, hence it is trivial.

□

Corollary 9.3 *For every $a \in \mathbb{Z} \cup \{\pm\infty\}$, the pair $({}^w\mathrm{D}^{\leq a}, {}^w\mathrm{D}^{\geq a+1})$ is a t-structure on* $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$.

We denote by $w_{\leq a}$ and $w_{\geq a+1}$ the truncation functors for this t-structure. They extend the exact functors $K \longmapsto W_a K$ and $K \longmapsto K/W_a K$ on $\operatorname{Perv}_{mf}(X)$.

Proof. Once we have the vanishing result of Proposition 9.2, the proofs of Lemmas 3.2.1 and 3.2.2 of [21] apply without modification.

□

Corollary 9.4 *The results of sections 3 and 5.1 of [21] are still true in our situation. In particular, if $j : U \longrightarrow X$ is an open immersion of k-schemes and $K \in \operatorname{Ob}\operatorname{Perv}_{mf}(U)$ is pure of weight a, then the canonical morphisms*

$$w_{\geq a} j_! K \longrightarrow j_{!*} K \longrightarrow w_{\leq a} j_* K$$

are isomorphisms.

Proof. The proofs of [21] apply without modification.

□

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