An Introduction to Analytic Combinatorics

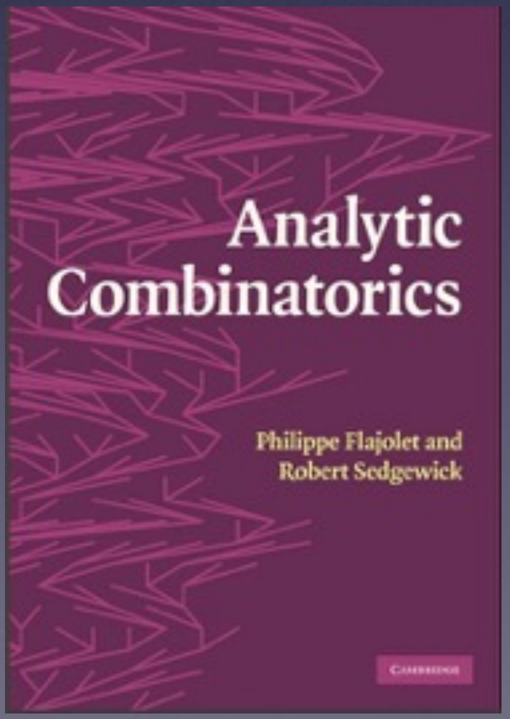
Meeting on Discrete Structures
Dec. 17, 2015

Bruno Salvy
@ AriC

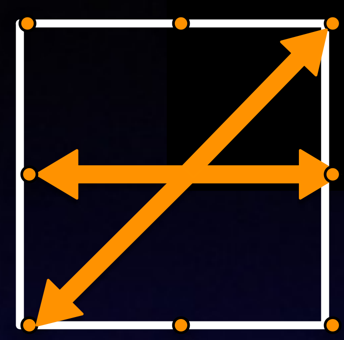
An Introduction to Analytic Combinatorics

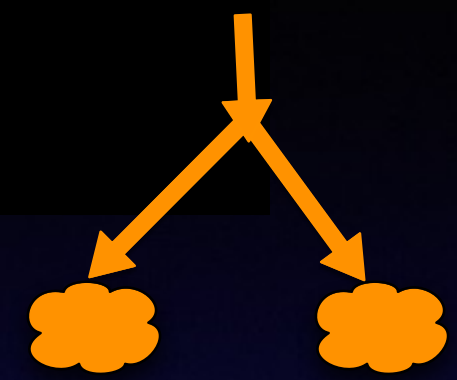
Meeting on Discrete Structures
Dec. 17, 2015

Bruno Salvy
@ AriC



Combinatorics, Randomness and Analysis

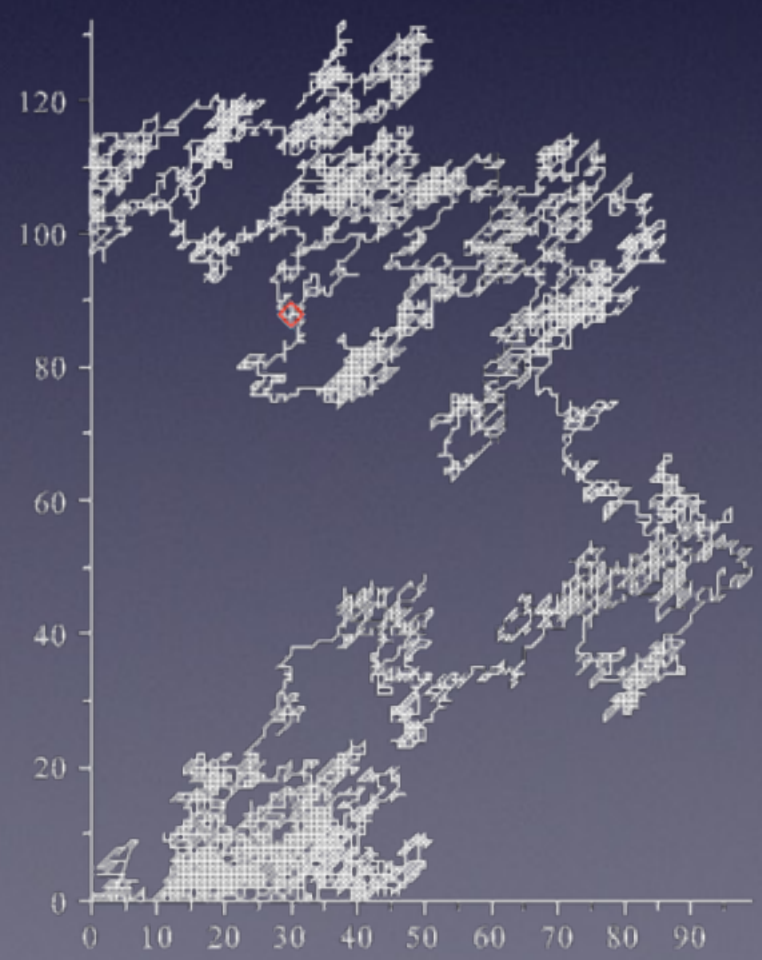


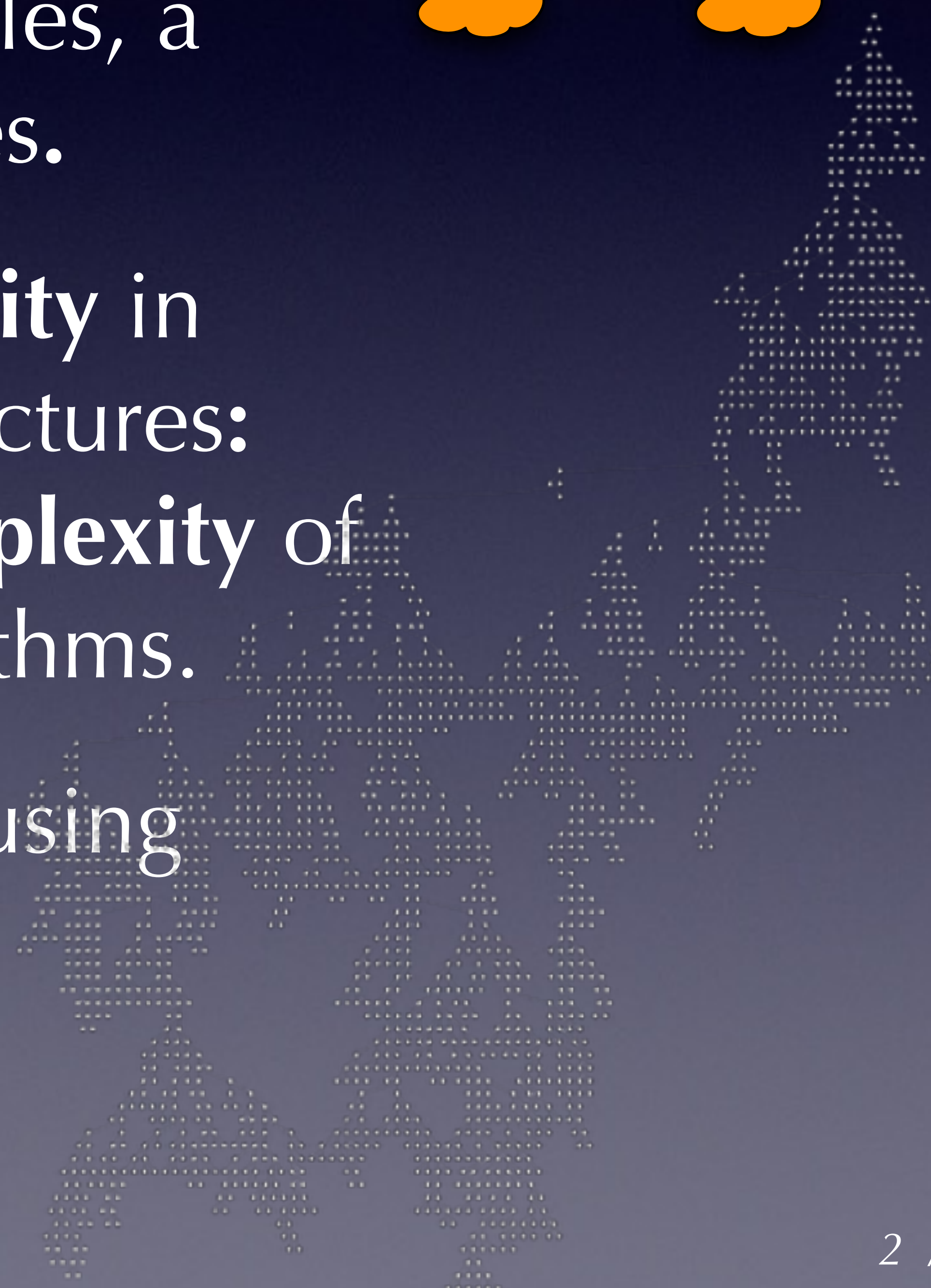


From **simple local** rules, a **global** structure arises**.**

A quest for **universality** in random discrete structures**:**
➡ **probabilistic complexity** of structures and algorithms.

Quantitative results using **complex analysis.**

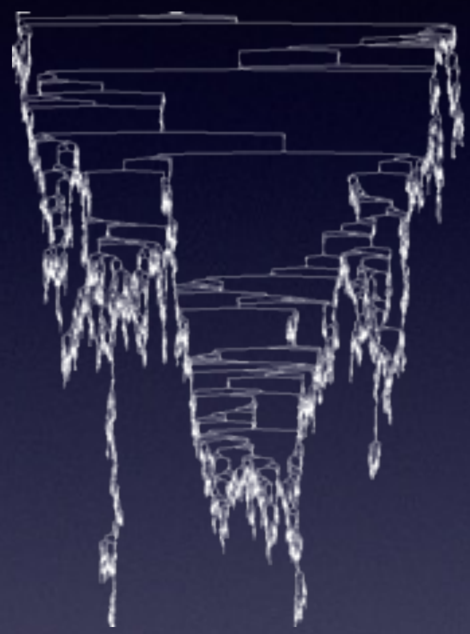




Overview (1/2)

Overview (1/2)

Ex.: binary trees



Overview (1/2)

Ex.: binary trees

- Equations over combinatorial structures

$$\mathcal{B} = \mathcal{Z} \cup \mathcal{B} \times \mathcal{B}$$

Overview (1/2)

Ex.: binary trees

- Equations over combinatorial structures
- Generating functions

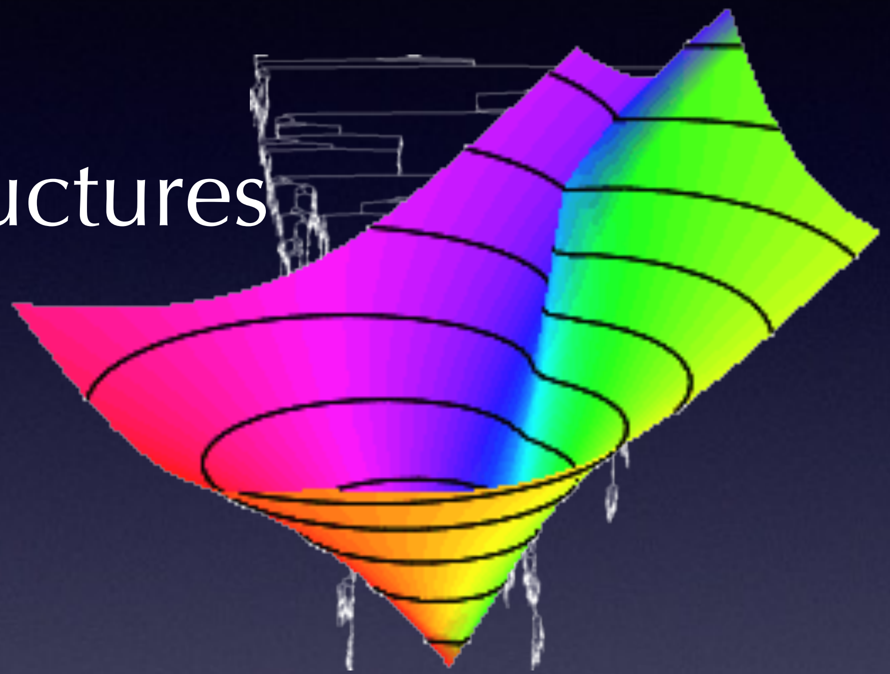
$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

$$\mathcal{B} = \mathcal{Z} \cup \mathcal{B} \times \mathcal{B}$$

$$B(z) = z + B(z)^2$$

Overview (1/2)

Ex.: binary trees



- Equations over combinatorial structures
- Generating functions

$$F(z) = \sum_{n=0}^{\infty} f_n z^n$$

$$\mathcal{B} = \mathcal{Z} \cup \mathcal{B} \times \mathcal{B}$$

- Complex analysis

$$B(z) = z + B(z)^2$$

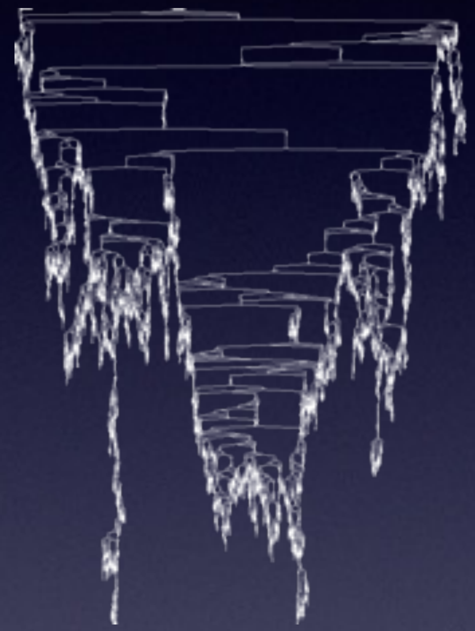
$$f_n \sim \cdots, n \to \infty$$

$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

Overview (2/2)

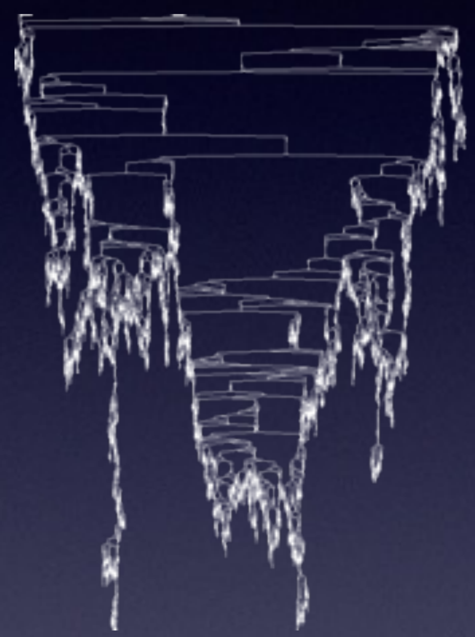
Overview (2/2)

Ex.: path length in binary trees



Overview (2/2)

Ex.: path length in binary trees



- Equations over combinatorial structures + parameters

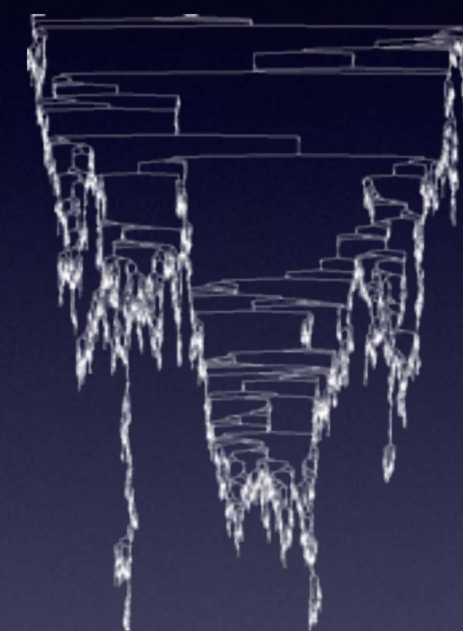
Overview (2/2)

- Equations over combinatorial structures + parameters

- Multivariate generating series

$$F(z,u) = \sum_{n,k} f_{n,k} u^k z^n$$

Ex.: path length in binary trees



$$B(z,u) = \sum_{t \in T} u^{\mathrm{pl}(t)} z^{|t|}$$

$$= z + B^2(zu, u)$$

$$P(z) := \left. \frac{\partial}{\partial u} B(z,u) \right|_{u=1}$$

Overview (2/2)

- Equations over combinatorial structures + parameters
- Multivariate generating series

$$F(z,u) = \sum_{n,k} f_{n,k} u^k z^n$$

- Complex analysis

$$f_{n,k} \sim \cdots, n \to \infty$$

Ex.: path length in binary trees

$$B(z,u) = \sum_{t \in T} u^{\mathrm{pl}(t)} z^{|t|}$$

$$= z + B^2(zu, u)$$

$$P(z) := \left.\frac{\partial}{\partial u} B(z,u)\right|_{u=1}$$

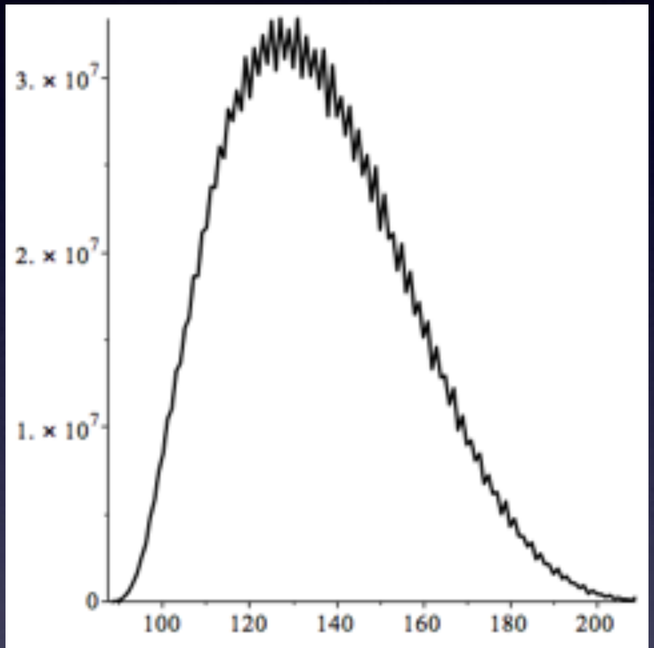
$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

$$P_n \sim 4^{n-1}$$

$$\frac{P_n}{nB_n} \sim \sqrt{\pi n}$$

Overview (2/2)

Ex.: path length in binary trees



- Equations over combinatorial structures + parameters

- Multivariate generating series

$$F(z,u) = \sum_{n,k} f_{n,k} u^k z^n$$

- Complex analysis

$$f_{n,k} \sim \cdots, n \to \infty$$

- Limiting distribution

$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

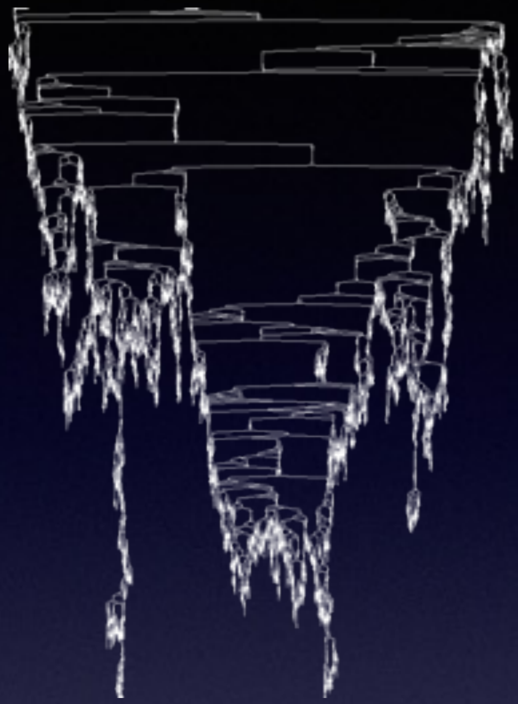
$$P_n \sim 4^{n-1}$$

$$\frac{P_n}{nB_n} \sim \sqrt{\pi n}$$

I. From Combinatorics to Generating Functions

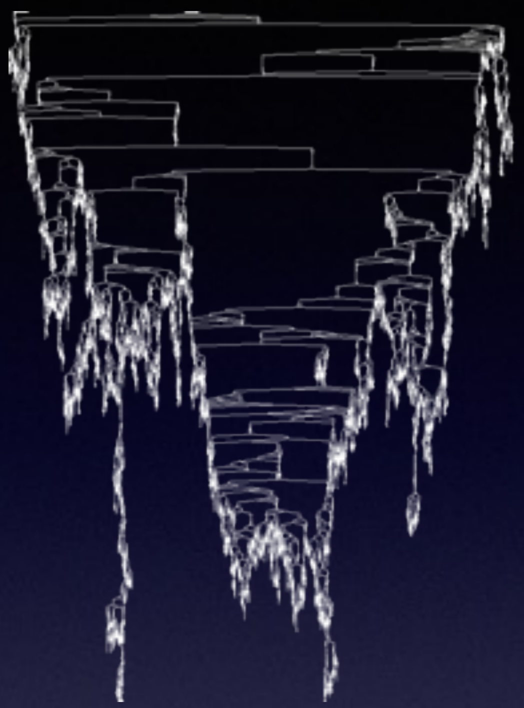
Combinatorial specifications

Language: *1*, $\mathcal{Z}$, +, ×, $\textsc{Seq}$, $\textsc{Set}$, $\textsc{Cyc}$ *and recursion.*



Combinatorial specifications

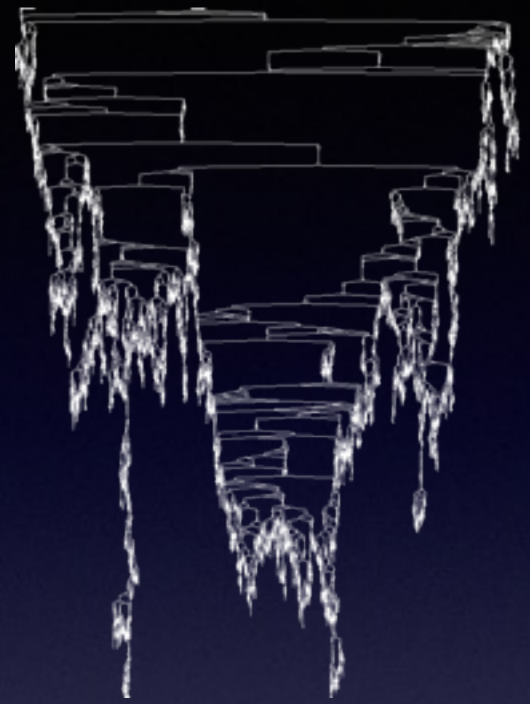
***Language**: 1,$\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.*



- Binary trees: $\mathcal{B}=\mathcal{Z}+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$

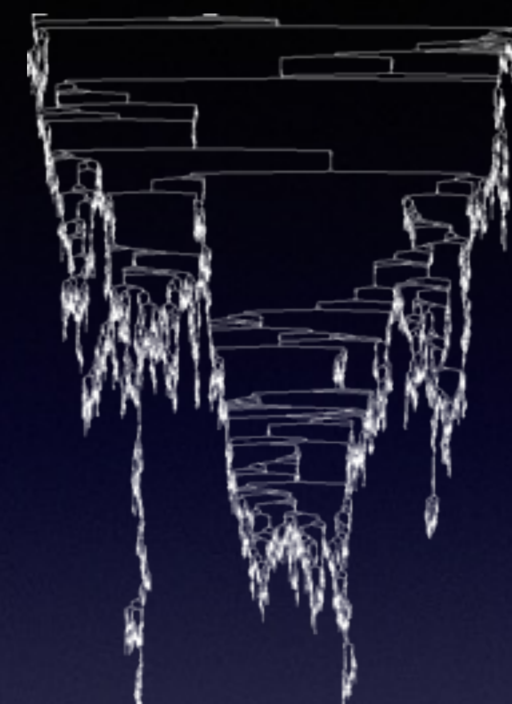
Combinatorial specifications

***Language**: 1, $\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.*



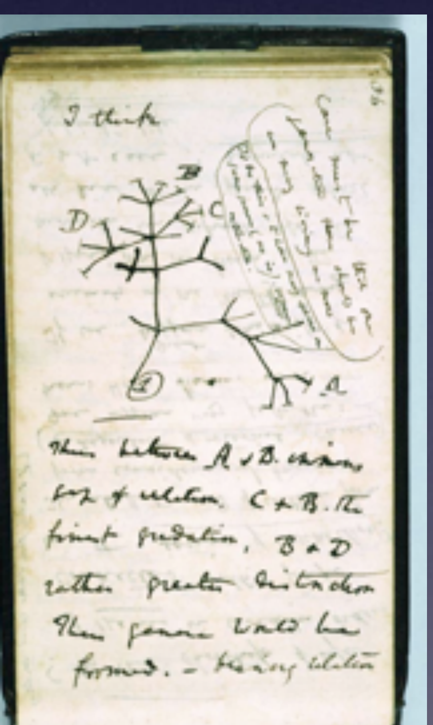
- Binary trees: $\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$
- Permutations: Perm=SET(CYC($\mathcal{Z}$));

Combinatorial specifications



***Language**: 1, $\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.*

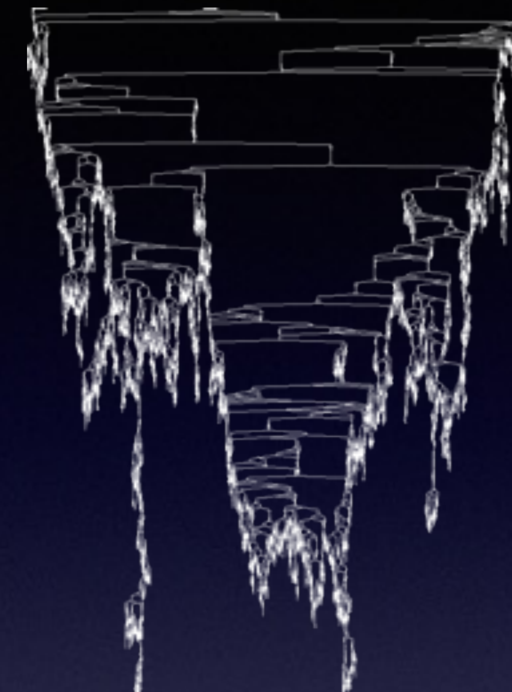
- Binary trees: $\mathcal{B}=\mathcal{Z}+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$



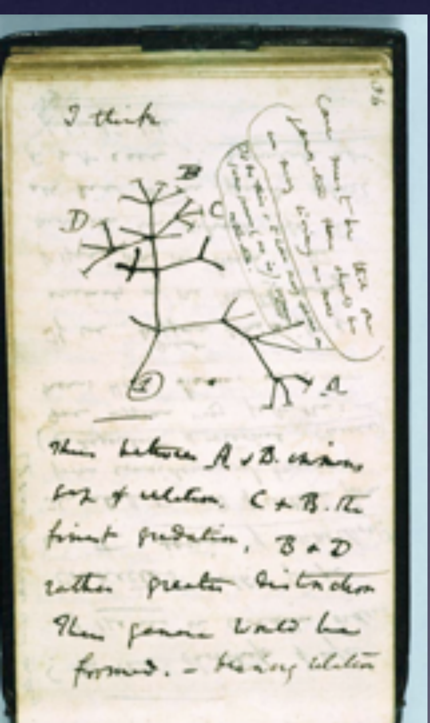
- Permutations: Perm=SET(CYC($\mathcal{Z}$));
- Trees: $\mathcal{T}=\mathcal{Z}\times\mathrm{SET}(\mathcal{T}(\mathcal{Z}))$;

Combinatorial specifications

***Language**: 1, $\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.*



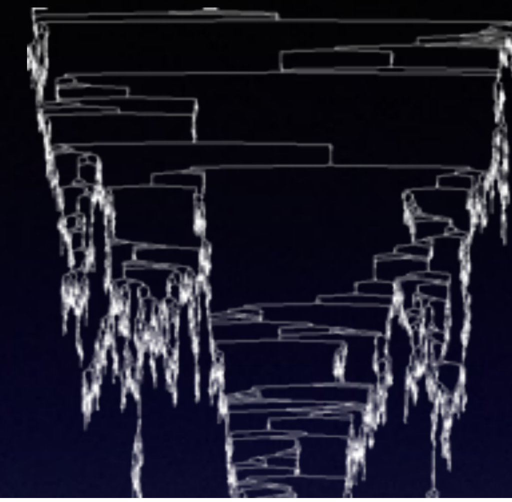
- Binary trees: $\mathcal{B}=\mathcal{Z}+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$



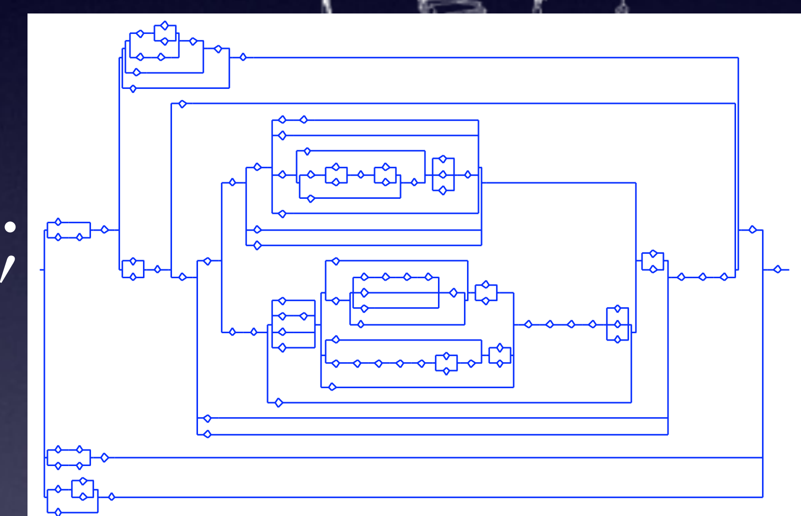
- Permutations: Perm=SET(CYC($\mathcal{Z}$));
- Trees: $\mathcal{T}=\mathcal{Z}\times\mathrm{SET}(\mathcal{T}(\mathcal{Z}))$;
- Functional graphs: $\mathcal{F}=\mathrm{SET}(\mathrm{CYC}(\mathcal{T}(\mathcal{Z})))$;

Combinatorial specifications

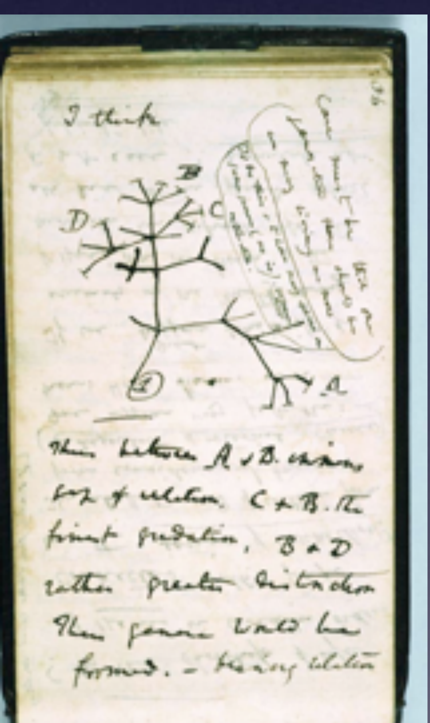
Language: *1,$\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.*



- Binary trees: $\mathcal{B}=\mathcal{Z}+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$



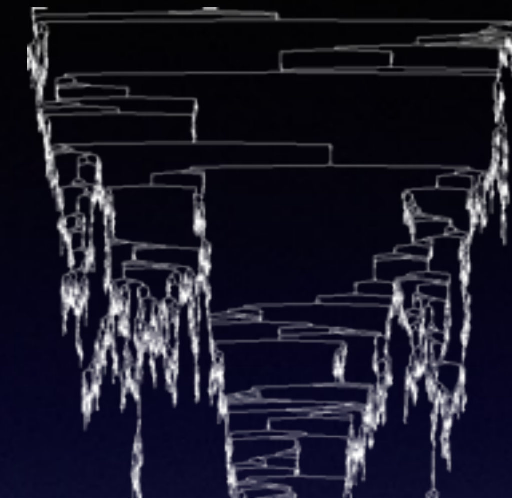
- Permutations: Perm=SET(CYC($\mathcal{Z}$));
- Trees: $\mathcal{T}=\mathcal{Z}\times\mathrm{SET}(\mathcal{T}(\mathcal{Z}))$;



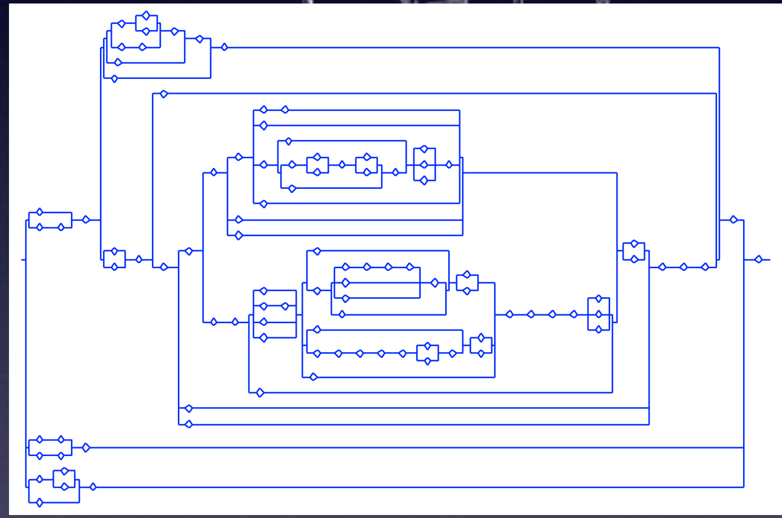
- Functional graphs: $\mathcal{F}=\mathrm{SET}(\mathrm{CYC}(\mathcal{T}(\mathcal{Z})))$;
- Series-parallel graphs:
  $\mathcal{G}=\mathcal{Z}+\mathcal{S}+\mathcal{P}$, $\mathcal{S}=\mathrm{SEQ}_{>0}(\mathcal{Z}+\mathcal{P})$,$\mathcal{P}=\mathrm{SET}_{>0}(\mathcal{Z}+\mathcal{S})$;

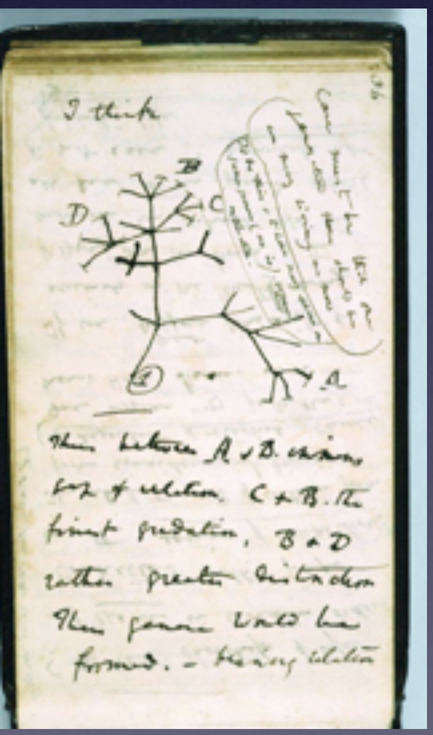
Combinatorial specifications

Language: *1, $\mathcal{Z}$, +, ×, Seq, Set, Cyc and recursion.*

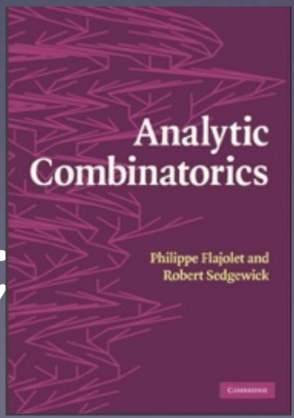


- Binary trees: $\mathcal{B}=\mathcal{Z}+\mathcal{Z}\times\mathcal{B}\times\mathcal{B}$



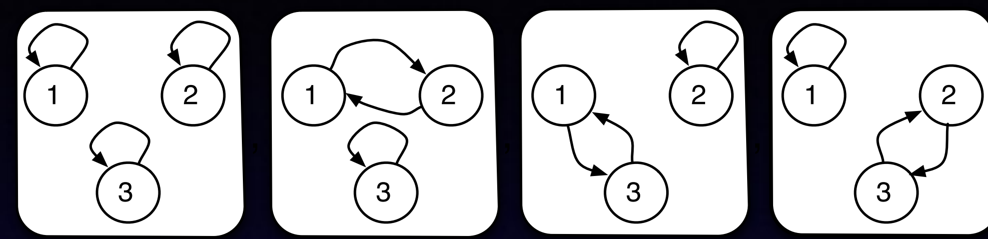


- Permutations: Perm=$\text{Set}(\text{Cyc}(\mathcal{Z}))$;
- Trees: $\mathcal{T}=\mathcal{Z}\times\text{Set}(\mathcal{T}(\mathcal{Z}))$;
- Functional graphs: $\mathcal{F}=\text{Set}(\text{Cyc}(\mathcal{T}(\mathcal{Z})))$;
- Series-parallel graphs: $\mathcal{G}=\mathcal{Z}+\mathcal{S}+\mathcal{P}$, $\mathcal{S}=\text{Seq}_{>0}(\mathcal{Z}+\mathcal{P})$, $\mathcal{P}=\text{Set}_{>0}(\mathcal{Z}+\mathcal{S})$;



- ...hundreds of examples in “the purple book”.

Two kinds of Generating Functions

$\mathrm{Inv}[\{1,2,3\}] = \{$  $\}$

4 involutions;

3 of them permuted by $\mathfrak{S}_3 \rightarrow$ 2 unlabelled structures.

Exponential generating function:

$$F(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!}, \qquad f_n = \text{nb. labelled structs of size } n. \quad \mathrm{Inv}_3(z) = \frac{2}{3} z^3$$

Ordinary generating function:

$$\tilde{F}(z) = \sum_{n=0}^{\infty} \tilde{f}_n z^n, \qquad \tilde{f}_n = \text{nb. unlabelled of size } n. \quad \widetilde{\mathrm{Inv}}_3(z) = 2z^3$$

A Dictionary for Generating Functions

Language: *1*, $\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.

Structure	EGF	OGF
$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$	$\tilde{A}(z)+\tilde{B}(z)$
$\mathcal{A}\times\mathcal{B}$	$A(z)\times B(z)$	$\tilde{A}(z)\times\tilde{B}(z)$
$\mathrm{Seq}(\mathcal{A})$	$\dfrac{1}{1-A(z)}$	$\dfrac{1}{1-\tilde{A}(z)}$
$\mathrm{Set}(\mathcal{A})$	$\exp(A(z))$	$\exp(\sum \tilde{A}(z^i)/i)$
$\mathrm{Cyc}(\mathcal{A})$	$\log\dfrac{1}{1-A(z)}$	$\sum\dfrac{\phi(i)}{i}\log\dfrac{1}{1-\tilde{A}(z^i)}$

A Dictionary for Generating Functions

***Language**: 1, $\mathcal{Z}$, +, ×, SEQ, SET, CYC and recursion.*

Structure	EGF	OGF
$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$	$\tilde{A}(z)+\tilde{B}(z)$
$\mathcal{A}\times\mathcal{B}$	$A(z)\times B(z)$	$\tilde{A}(z)\times\tilde{B}(z)$
$\mathrm{Seq}(\mathcal{A})$	$\dfrac{1}{1-A(z)}$	$\dfrac{1}{1-\tilde{A}(z)}$
$\mathrm{Set}(\mathcal{A})$	$\exp(A(z))$	$\exp(\sum \tilde{A}(z^i)/i)$
$\mathrm{Cyc}(\mathcal{A})$	$\log\dfrac{1}{1-A(z)}$	$\sum\dfrac{\phi(i)}{i}\log\dfrac{1}{1-\tilde{A}(z^i)}$

Regular and context-free languages.

Examples

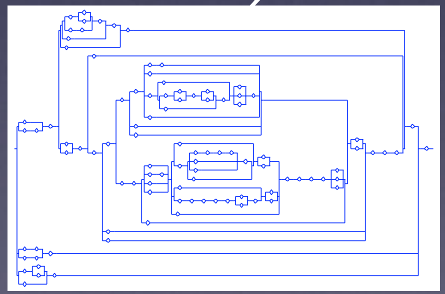
Binary trees: $\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$

$$\longrightarrow B(z) = z + zB(z)^2 = \tilde{B}(z)$$

Cayley trees: $\mathcal{T} = \mathcal{Z} \times \textsc{Set}(\mathcal{T})$

$$\longrightarrow T(z) = z \exp(T(z));$$

$$\longrightarrow \tilde{T}(z) = z \exp\left(\tilde{T}(z) + \frac{1}{2}\tilde{T}(z^2) + \frac{1}{3}\tilde{T}(z^3) + \cdots\right)$$



Series-parallel graphs:

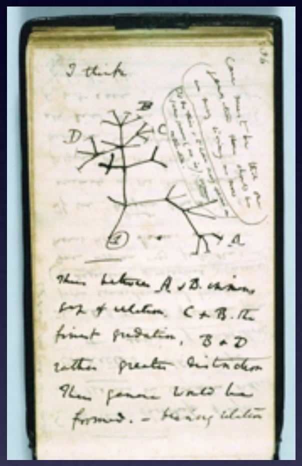
$$\mathcal{G} = \mathcal{Z} + \mathcal{S} + \mathcal{P}, \mathcal{S} = \textsc{Seq}_{>0}(\mathcal{Z} + \mathcal{P}), \mathcal{P} = \textsc{Set}_{>0}(\mathcal{Z} + \mathcal{S})$$

$$\longrightarrow \left\{ G(z) = z + S(z) + P(z), S(z) = \frac{1}{1 - z - P(z)} - 1, P(z) = e^{z+S(z)} - 1 \right\}$$

Examples

Binary trees: $\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$

$$\longrightarrow B(z) = z + zB(z)^2 = \tilde{B}(z)$$

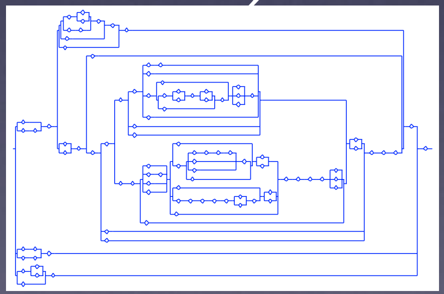


Cayley trees: $\mathcal{T} = \mathcal{Z} \times \textsc{Set}(\mathcal{T})$

$$\longrightarrow T(z) = z\exp(T(z));$$

$$\longrightarrow \tilde{T}(z) = z\exp\left(\tilde{T}(z) + \frac{1}{2}\tilde{T}(z^2) + \frac{1}{3}\tilde{T}(z^3) + \cdots\right)$$

Series-parallel graphs:



$$\mathcal{G} = \mathcal{Z} + \mathcal{S} + \mathcal{P}, \mathcal{S} = \textsc{Seq}_{>0}(\mathcal{Z} + \mathcal{P}), \mathcal{P} = \textsc{Set}_{>0}(\mathcal{Z} + \mathcal{S})$$

$$\longrightarrow \left\{ G(z) = z + S(z) + P(z), S(z) = \frac{1}{1 - z - P(z)} - 1, P(z) = e^{z+S(z)} - 1 \right\}$$

II. Mini-introduction to complex analysis

Basic Definitions and Properties

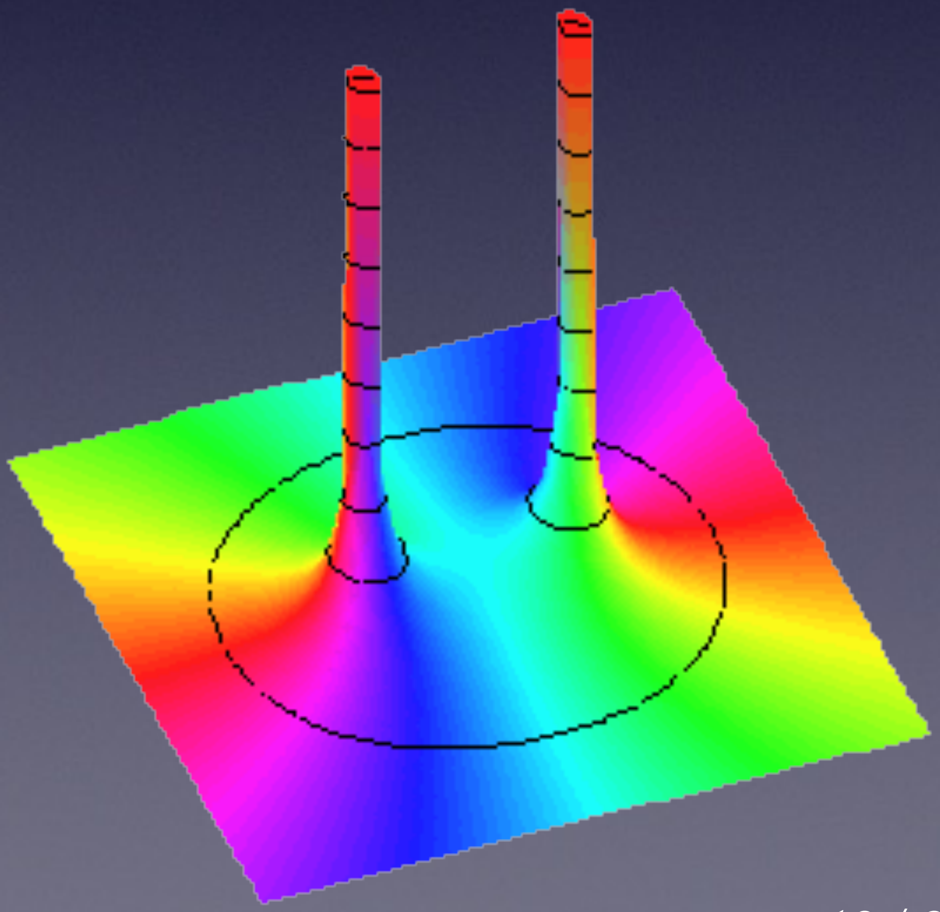
- **Def.** f: $D \subset \mathbb{C} \to \mathbb{C}$ is *analytic at a* if it is the sum of a power series in a disk around a.
- **Prop.** f,g analytic at a, then so are $f+g$, $f \times g$, f'.
- g analytic at a, f analytic at $g(a)$, then $f \circ g$ analytic at a.
- Same def and prop in *several variables*.

Examples

f	analytic at 0?	why
polynomial	Yes	
$\exp(x)$	Yes	$1 + x + x^2/2! + \cdots$
$\frac{1}{1-x}$	Yes	$1 + x + x^2 + \cdots \ (\lvert x\rvert < 1)$
$\log \frac{1}{1-x}$	Yes	$x + x^2/2 + x^3/3 \cdots \ (\lvert x\rvert < 1)$
$\frac{1-\sqrt{1-4x}}{2x}$	Yes	$1 + \cdots + \frac{1}{k+1}\binom{2k}{k}x^k + \cdots \ (\lvert x\rvert < 1/4);$
$\frac{1}{x}$	No	infinite at 0
$\log x$	No	derivative not analytic at 0
$\sqrt{x}$	No	derivative infinite at 0

Analytic Continuation and Singularities

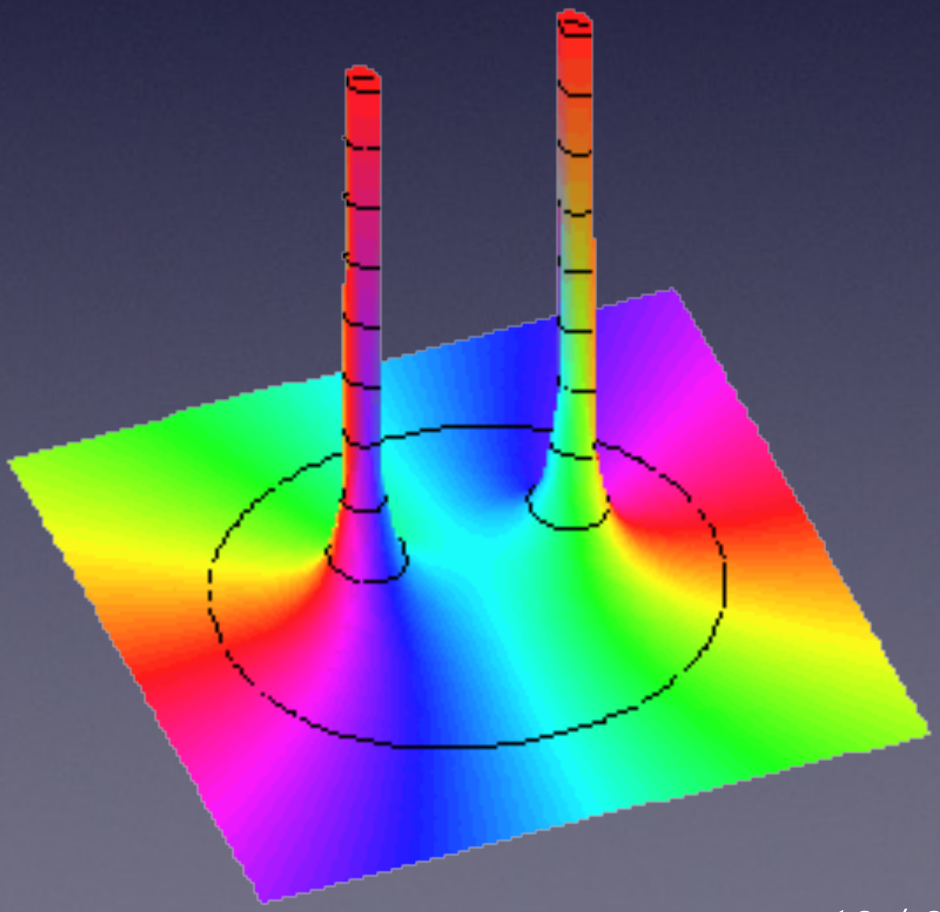
- **Def**. Analytic in a region (= connected, open, ≠ø): at each point.
- **Prop**. f analytic in $R \subset S$. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S.)



Analytic Continuation and Singularities

- **Def**. Analytic in a region (= connected, open, ≠ø): at each point.
- **Prop**. f analytic in $R \subset S$. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S.)

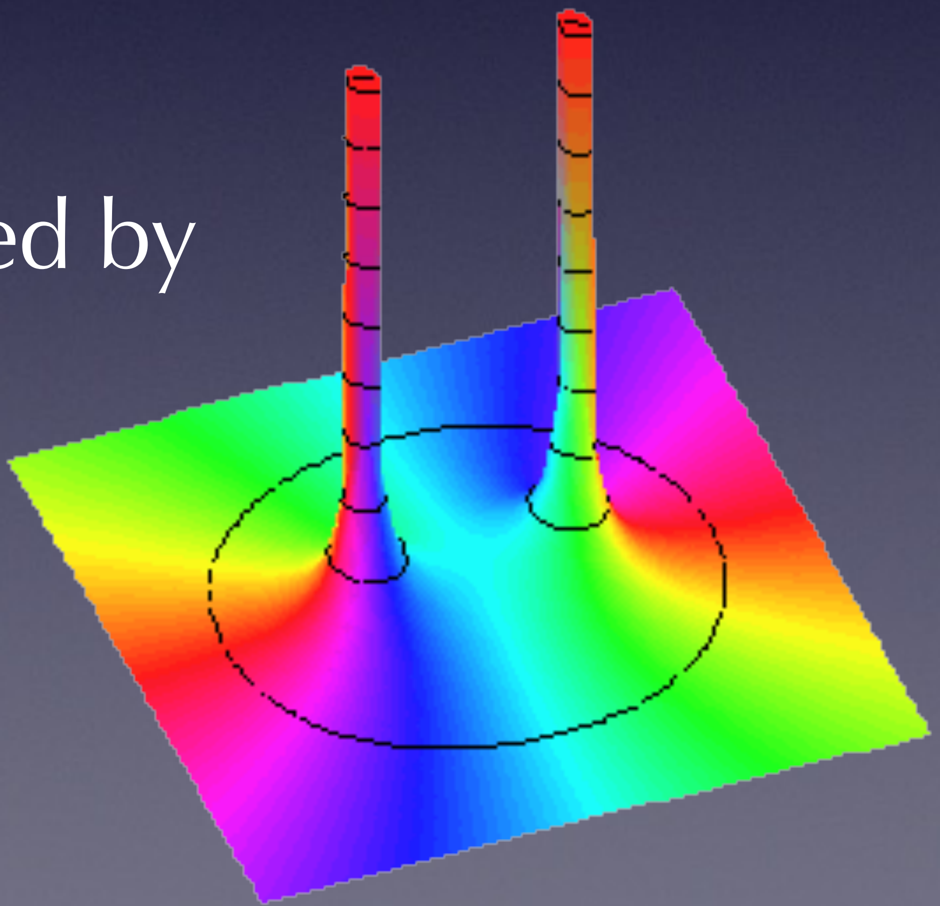
More Defs.



Analytic Continuation and Singularities

- **Def**. Analytic in a region (= connected, open, ≠ø): at each point.
- **Prop**. *f* analytic in $R \subset S$. There is at most one analytic function in S equal to *f* on R (the analytic continuation of *f* to S.)

More Defs.
Singularity: a point that cannot be reached by analytic continuation;



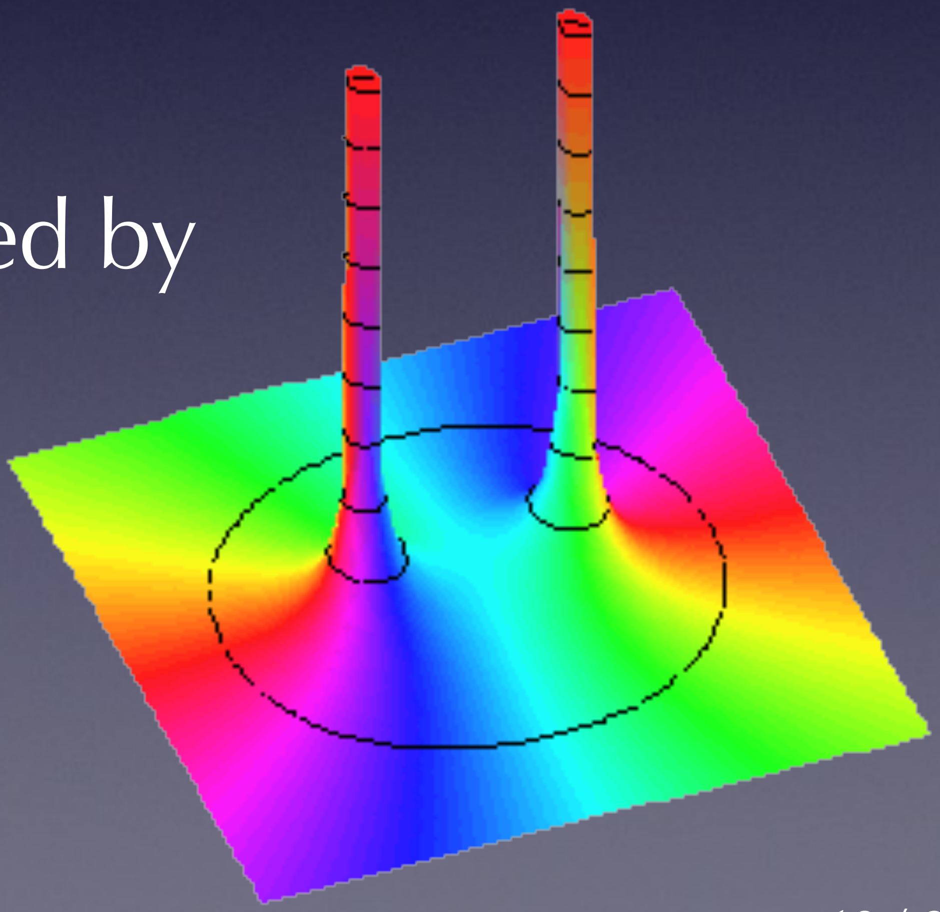
Analytic Continuation and Singularities

- **Def**. Analytic in a region (= connected, open, ≠ø): at each point.
- **Prop**. f analytic in $R \subset S$. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S.)

More Defs.

Singularity: a point that cannot be reached by analytic continuation;

Pole: isolated singularity a and $(z-a)^m f$ analytic for some $m \in \mathbb{N}$;



Analytic Continuation and Singularities

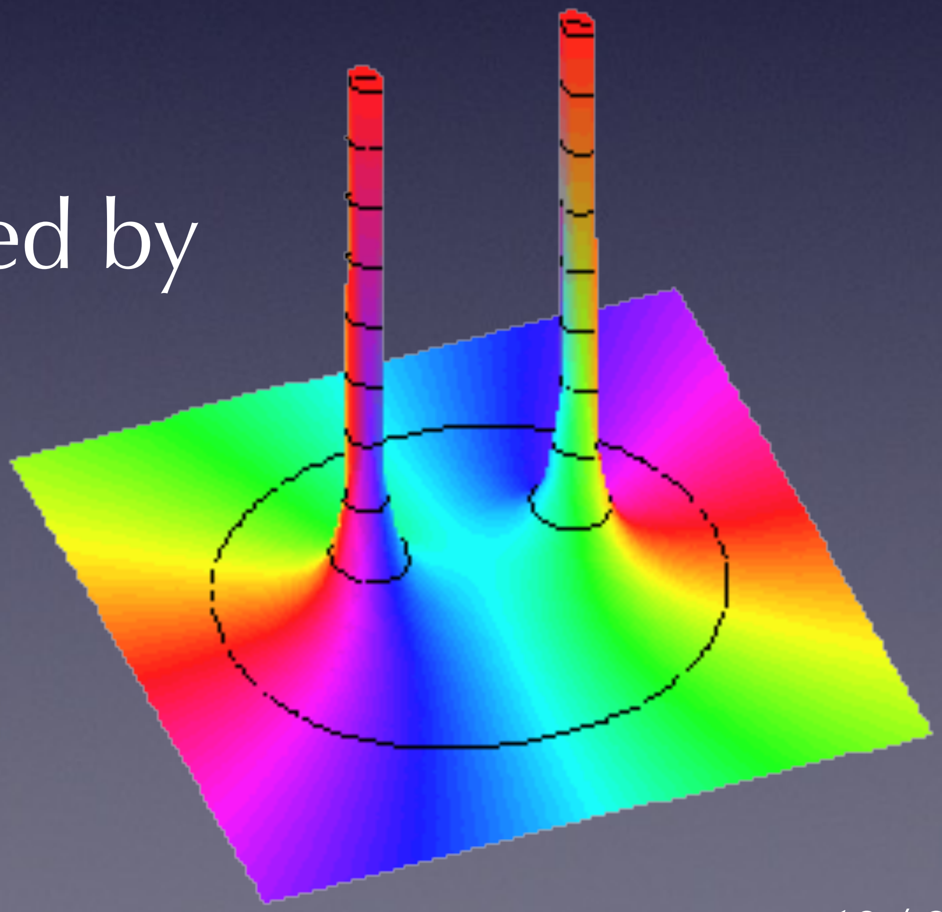
- **Def**. Analytic in a region (= connected, open, ≠ø): at each point.
- **Prop**. f analytic in $R \subset S$. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S.)

More Defs.

Singularity: a point that cannot be reached by analytic continuation;

Pole: isolated singularity a and $(z-a)^m f$ analytic for some $m \in \mathbb{N}$;

Residue at a pole a: coeff of $(z-a)^{-1}$;

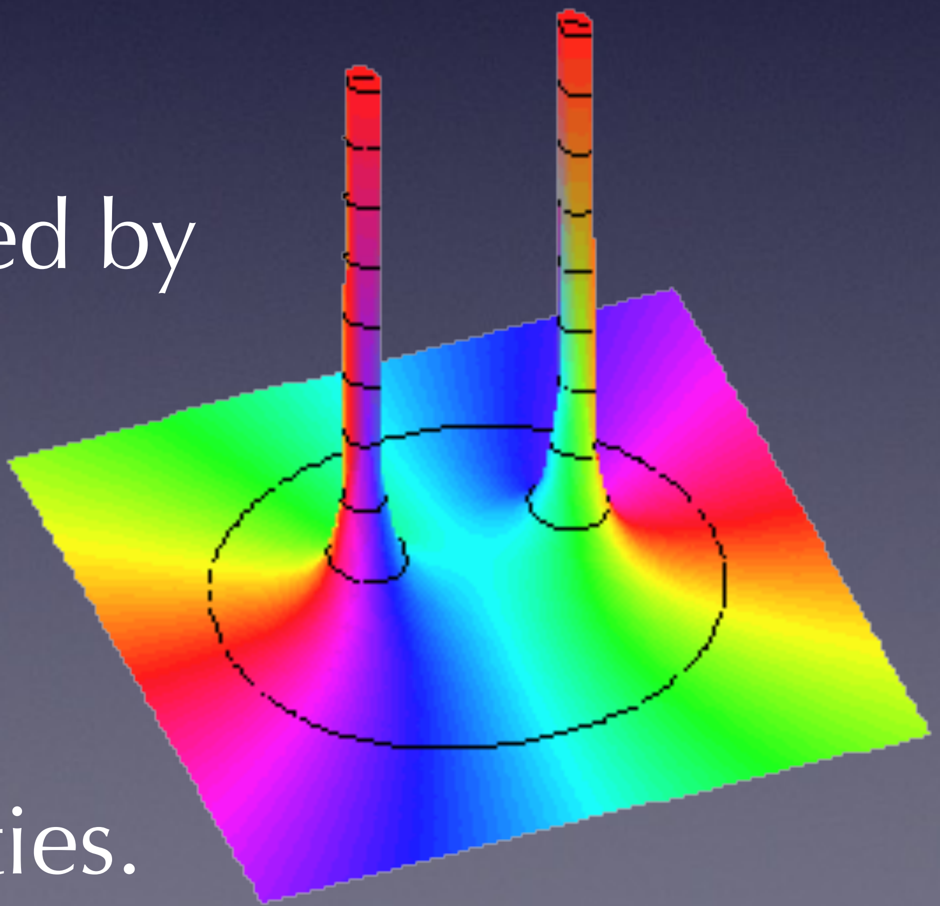


Analytic Continuation and Singularities

- **Def**. Analytic in a region (= connected, open, ≠ø): at each point.
- **Prop**. f analytic in $R \subset S$. There is at most one analytic function in S equal to f on R (the analytic continuation of f to S.)

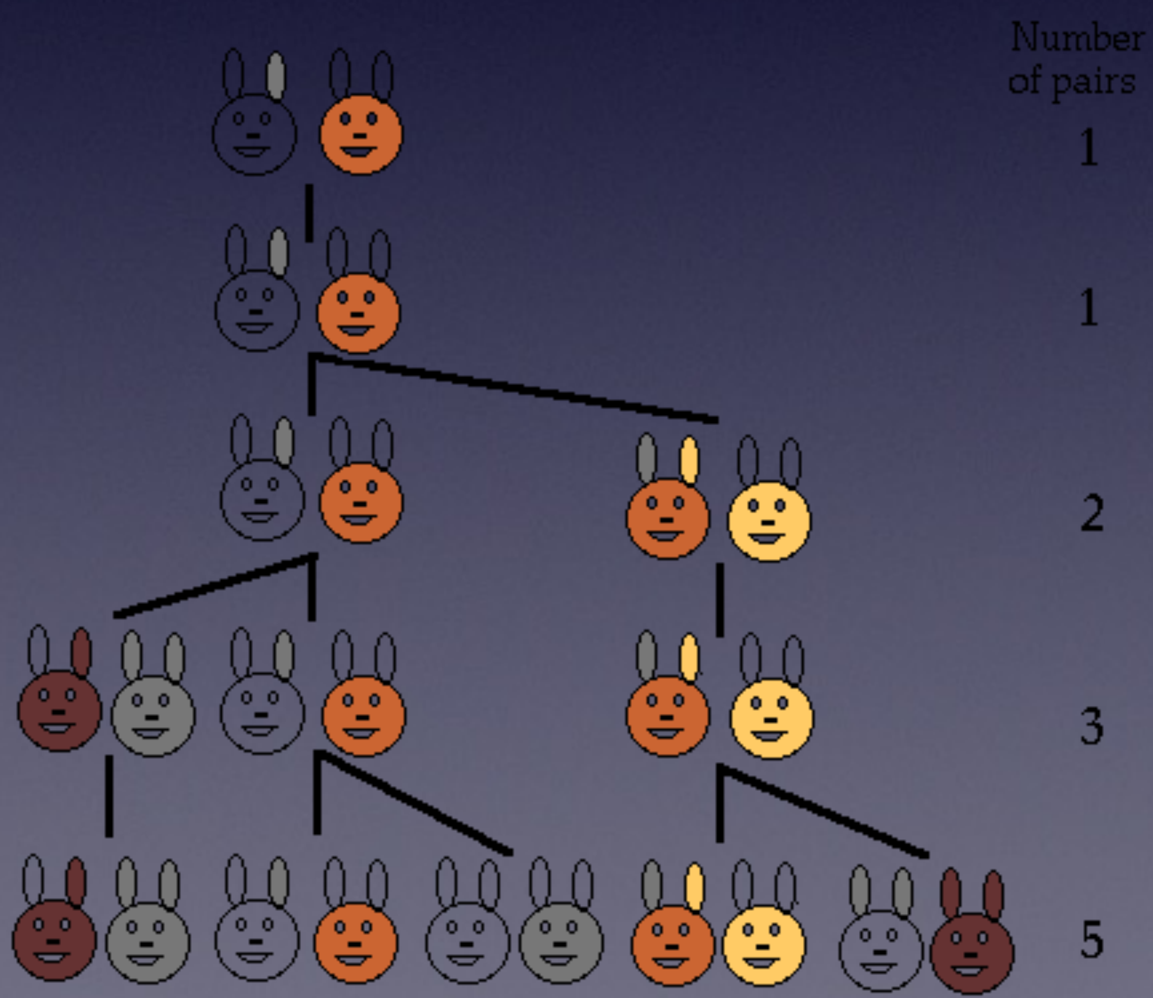
More Defs.

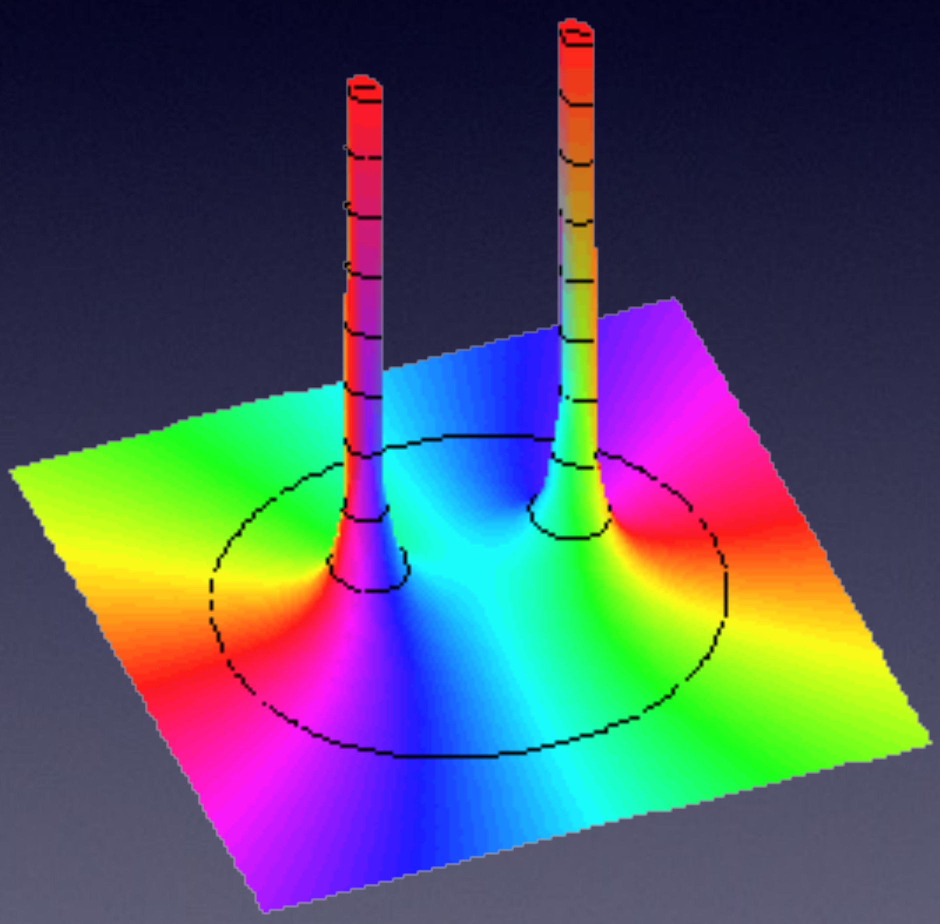
Singularity: a point that cannot be reached by analytic continuation;
Pole: isolated singularity a and $(z-a)^m f$ analytic for some $m \in \mathbb{N}$;
Residue at a pole a: coeff of $(z-a)^{-1}$;
f meromorphic in R: only polar singularities.



Example

Fibonacci numbers: $\dfrac{1}{1-z-z^2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots$





$$F_n = c\phi^n + c'\phi^{-n}$$

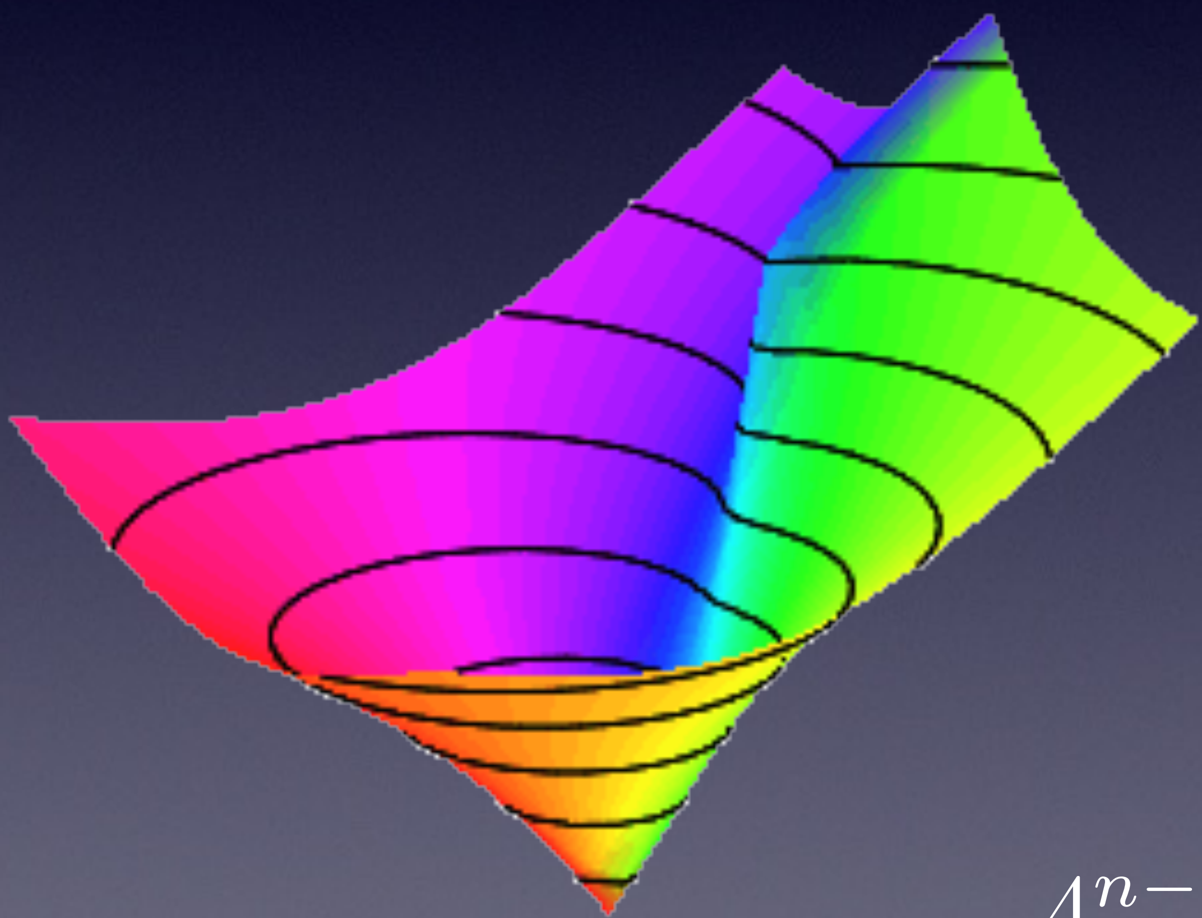
Examples

Binary trees: $$\frac{1-\sqrt{1-4z}}{2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots$$


Binary trees:

Examples

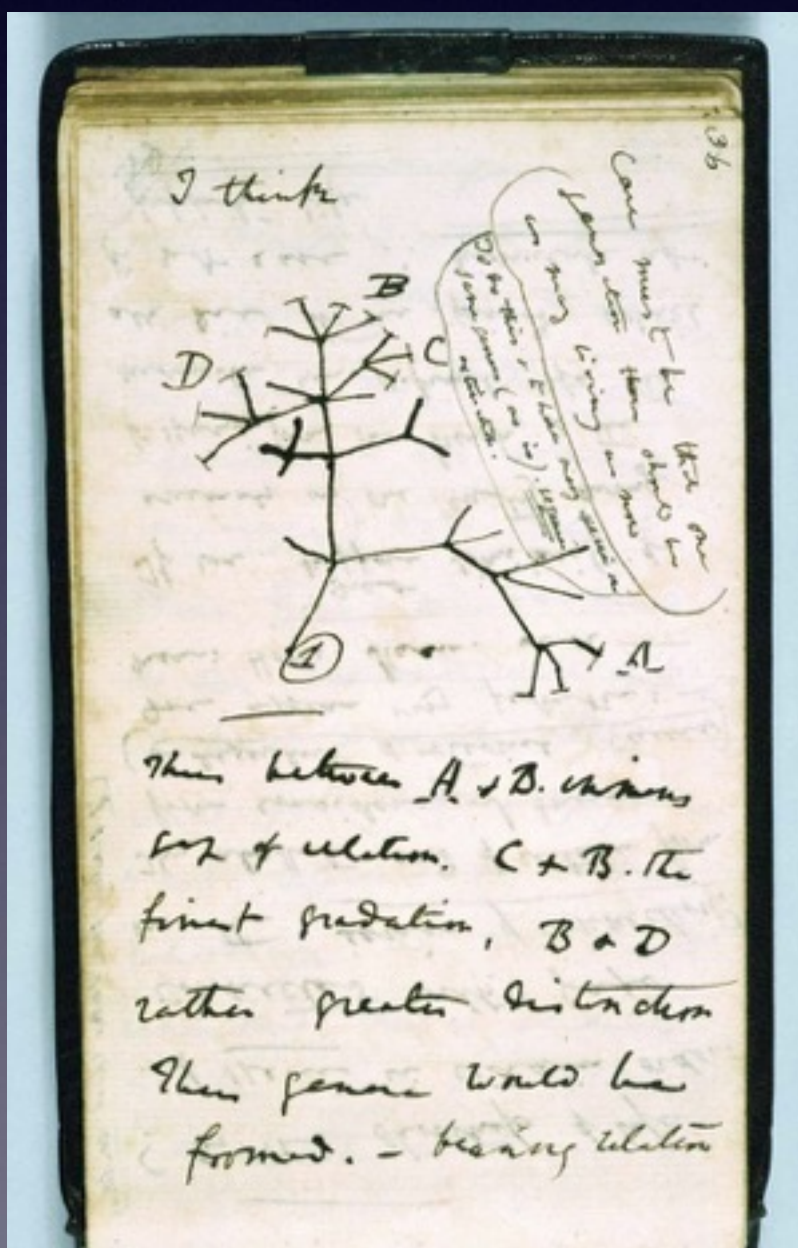
Binary trees: $$\frac{1-\sqrt{1-4z}}{2} = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots$$



$$B_n \sim \frac{4^{n-1} n^{-3/2}}{\sqrt{\pi}}$$

Examples

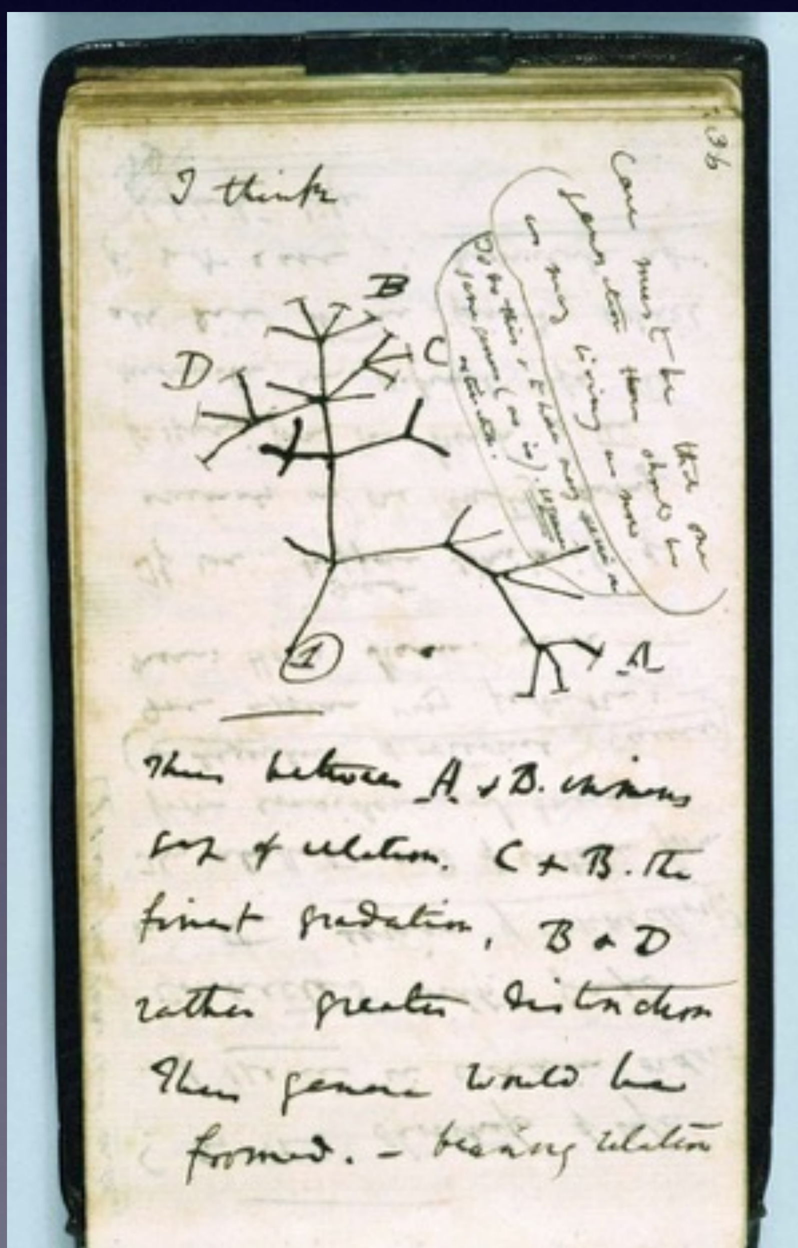
Cayley trees: $T(z) = z\exp(T(z)) = z + 2\frac{z^2}{2!} + 9\frac{z^3}{3!} + 64\frac{z^4}{4!} + \cdots$

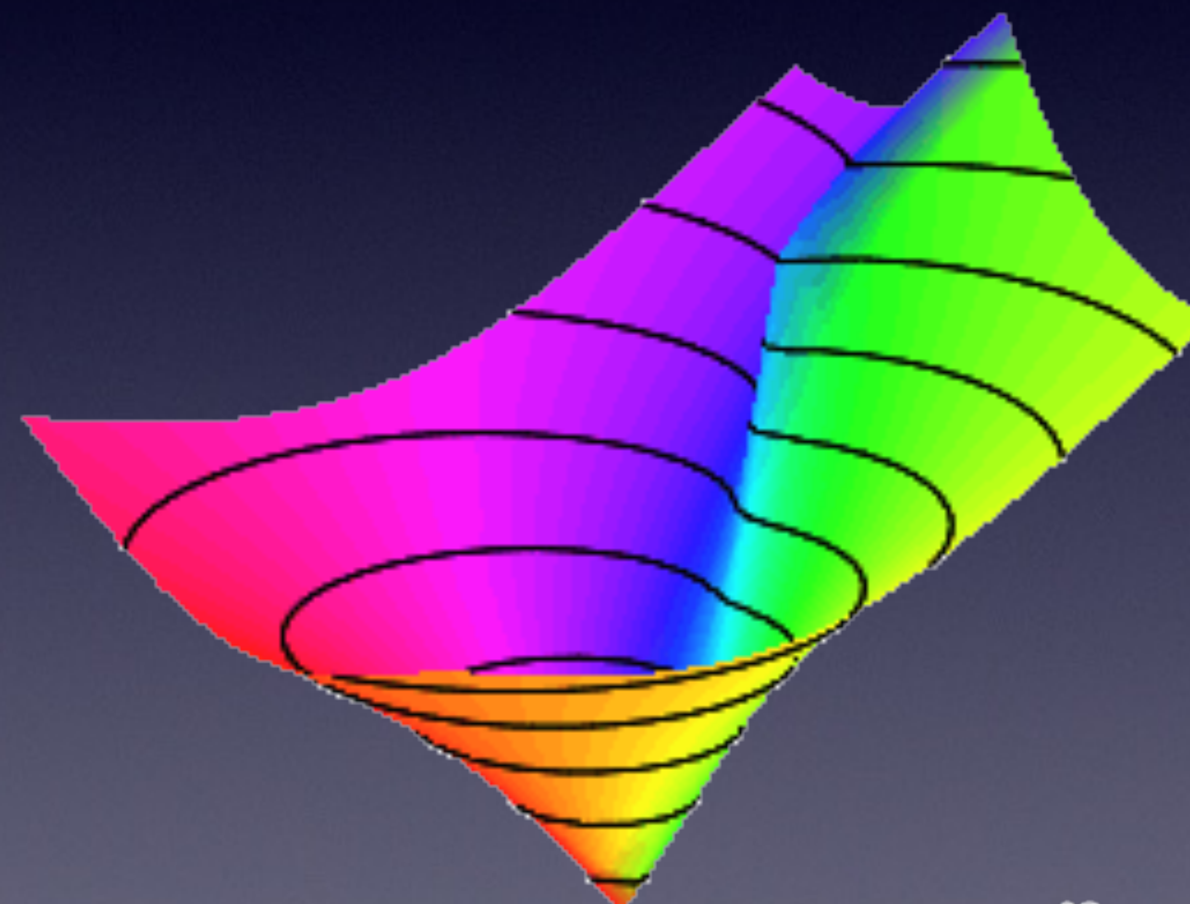


$$T_n \sim \frac{e^n n^{-3/2}}{\sqrt{2\pi}}$$

Examples

Cayley trees: $T(z) = z\exp(T(z)) = z + 2\frac{z^2}{2!} + 9\frac{z^3}{3!} + 64\frac{z^4}{4!} + \cdots$



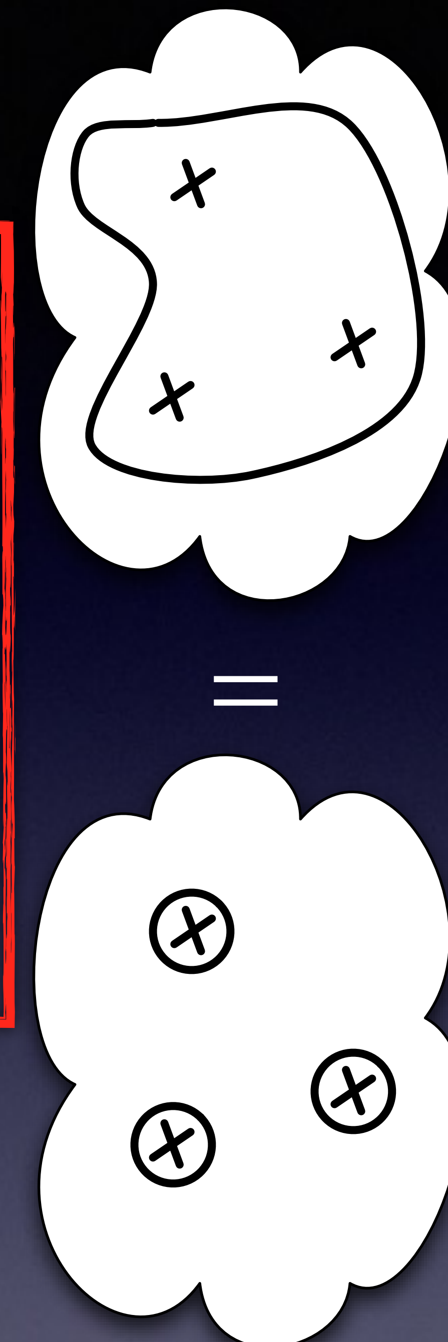


$$T_n \sim \frac{e^n n^{-3/2}}{\sqrt{2\pi}}$$

Integration

Prop. f meromorphic in a region R, γ a closed path in $\mathbb{C}$ encircling the poles $a_1,\ldots,a_m$ of f once in the positive sense. Then

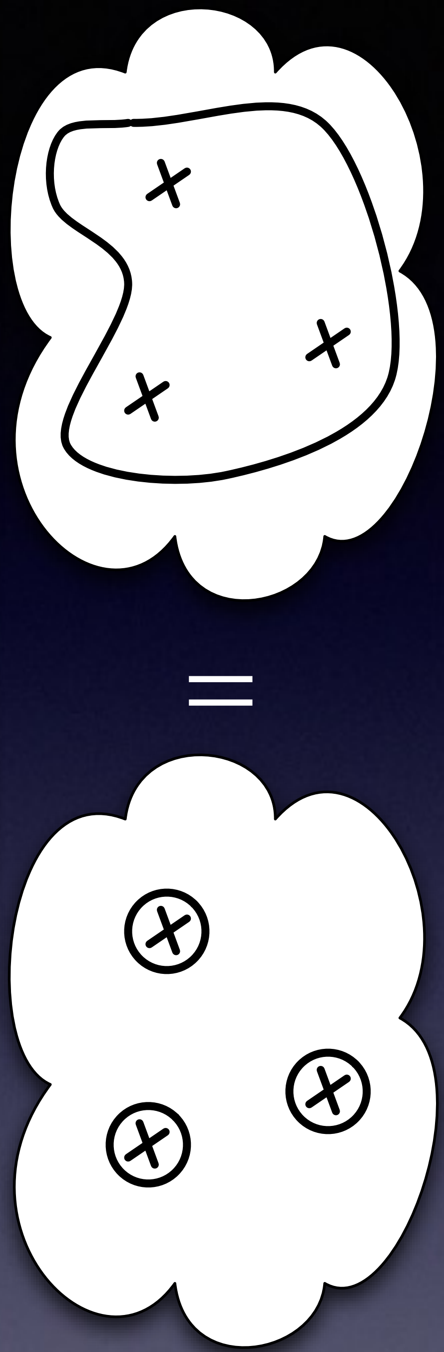
$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \mathrm{Res}(f; a_j).$$



Integration

Prop. f meromorphic in a region R, γ a closed path in $\mathbb{C}$ encircling the poles $a_1,\ldots,a_m$ of f once in the positive sense. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \mathrm{Res}(f; a_j).$$



Cor. If $f = a_0 + a_1 z + \cdots$ is analytic in R∋0, then

$$\boxed{a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz}$$

for any closed γ encircling 0 once in the positive sense.

III. From Generating Functions to Asymptotic Behaviour

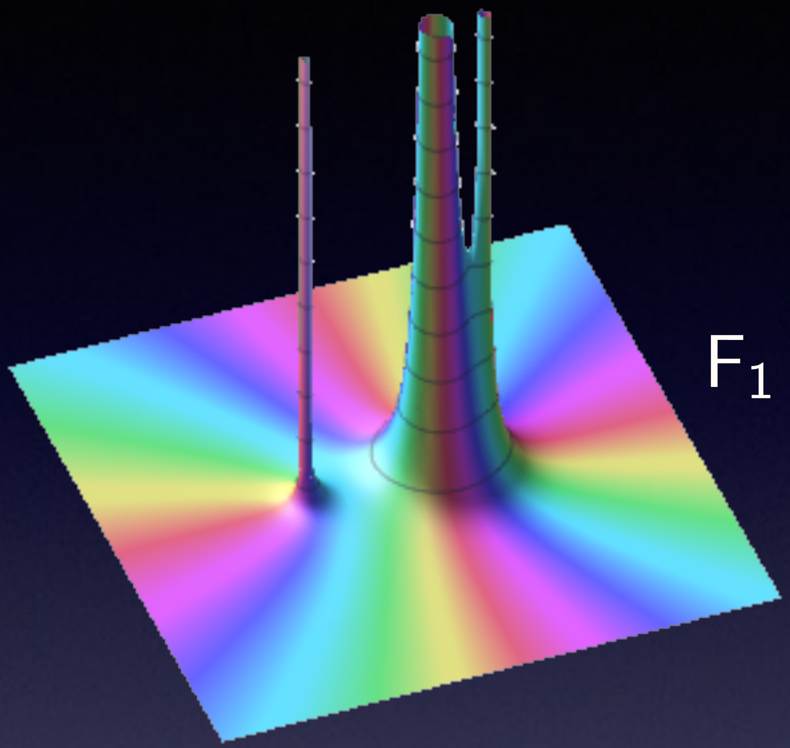
Philosophy:

Smallest singularity $\longleftrightarrow$ *exponential behaviour*
local behaviour $\longleftrightarrow$ *subexponential terms*

3 families cover most applications

Coefficients of Rational Functions

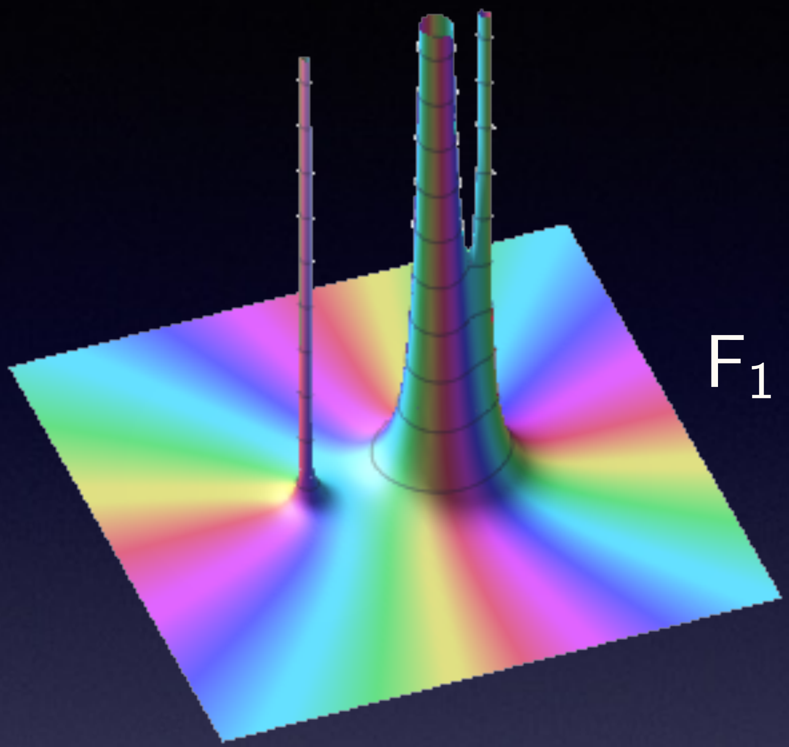
$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$



$$F_1 = 1 = \frac{1}{2\pi i}\oint \frac{1}{1-z-z^2}\frac{dz}{z^2}$$

Coefficients of Rational Functions

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

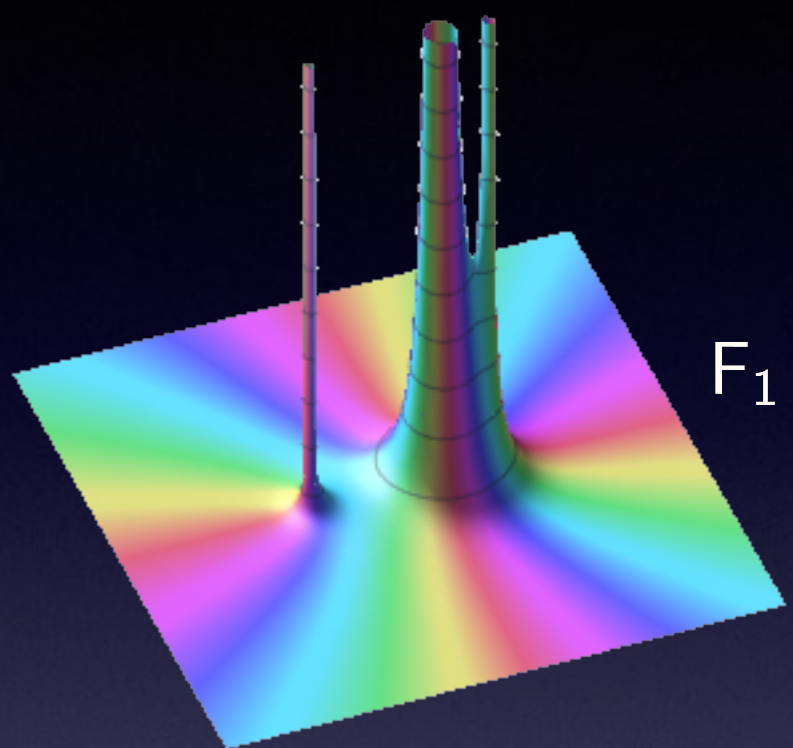


$$F_1 = 1 = \frac{1}{2\pi i}\oint \frac{1}{1-z-z^2}\frac{dz}{z^2}$$

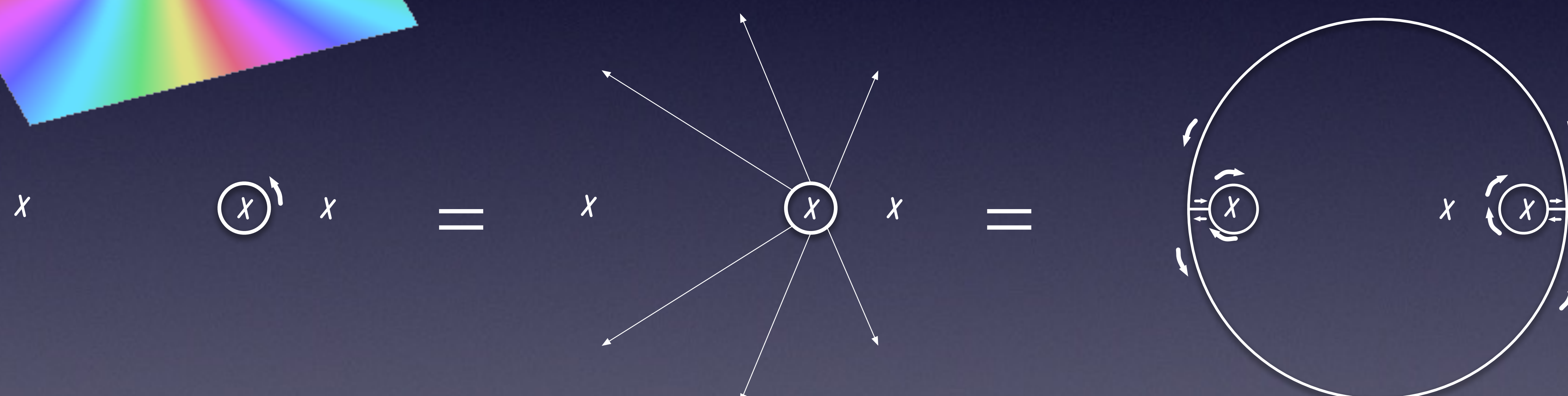
As n increases, the smallest singularities dominate.

Coefficients of Rational Functions

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$



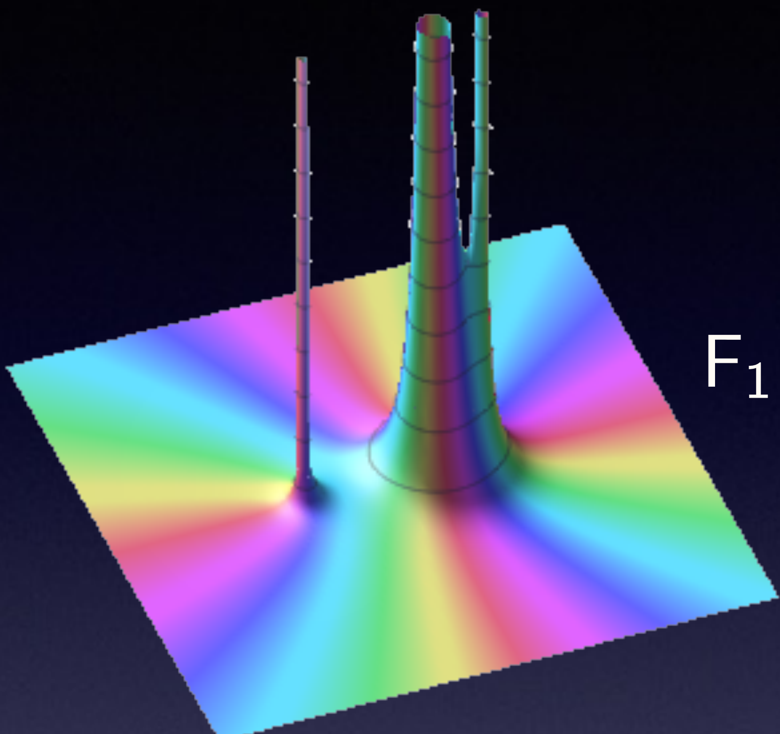
$$F_1 = 1 = \frac{1}{2\pi i}\oint \frac{1}{1-z-z^2}\frac{dz}{z^2}$$



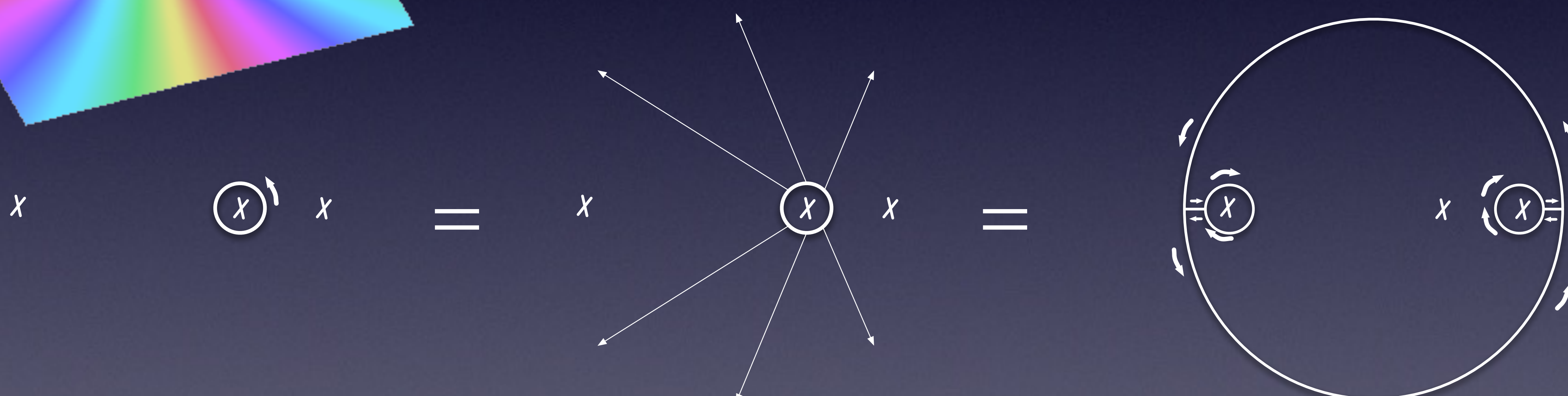
As n increases, the smallest singularities dominate.

Coefficients of Rational Functions

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$



$$F_1 = 1 = \frac{1}{2\pi i}\oint \frac{1}{1-z-z^2}\frac{dz}{z^2}$$



$$F_n = \frac{\phi^{-n-1}}{1+2\phi} + \frac{\overline{\phi}^{-n-1}}{1+2\overline{\phi}}$$

As n increases, the smallest singularities dominate.

Conway's sequence

Conway's sequence

1,11,21,1211,111221,...

Conway's sequence

1,11,21,1211,111221,...

Generating function for lengths:
$f(z)=P(z)/Q(z)$

Conway's sequence

1,11,21,1211,111221,...

Generating function for lengths:
$f(z)=P(z)/Q(z)$
with $\deg Q=72$.

Conway's sequence

1,11,21,1211,111221,...

Generating function for lengths:
$f(z)=P(z)/Q(z)$
with deg $Q=72$.

Conway's sequence

1,11,21,1211,111221,...

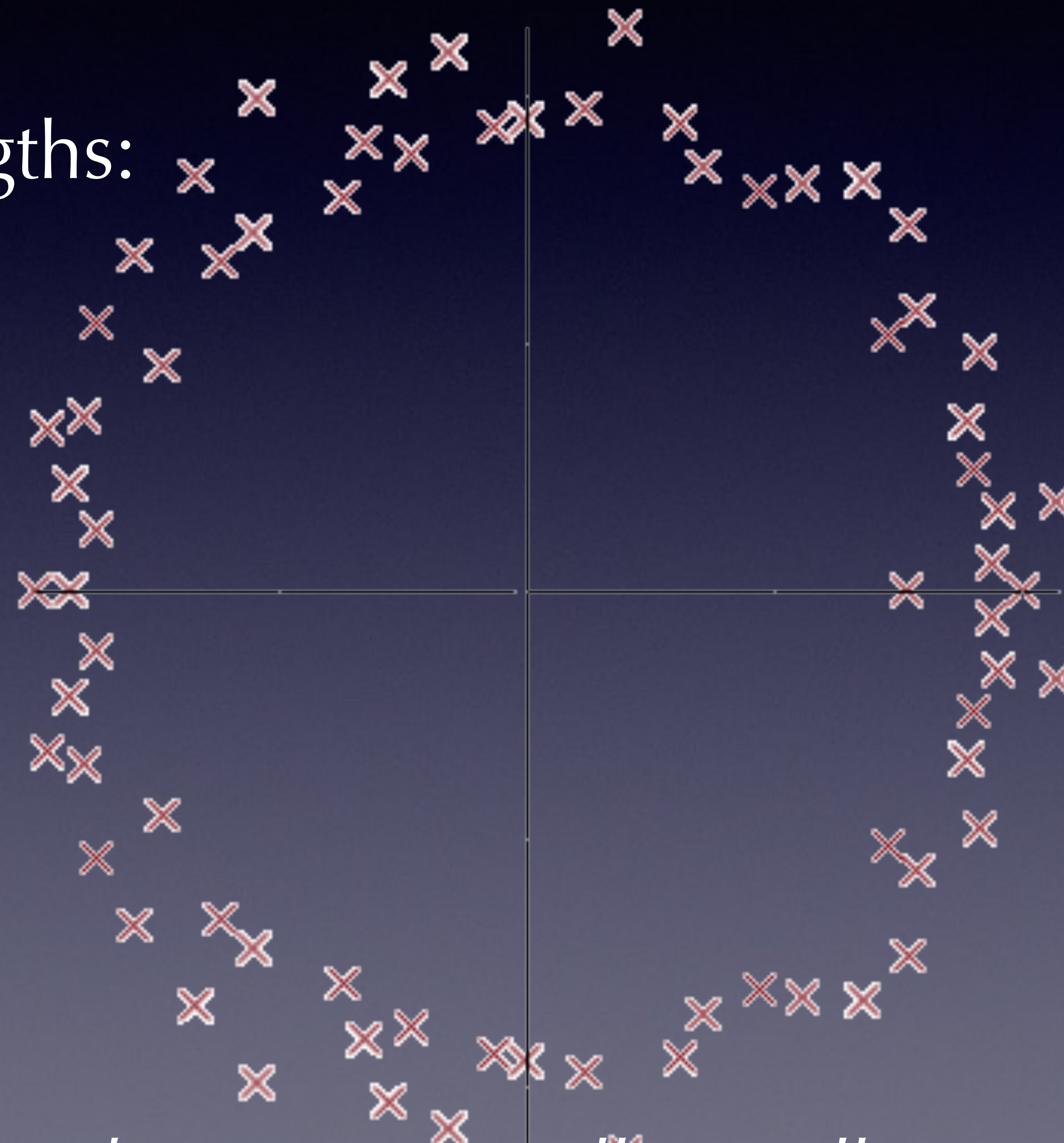
Generating function for lengths:
$f(z)=P(z)/Q(z)$
with deg Q=72.

Smallest singularity:
$\delta(f) \simeq 0.7671198507$

$\rho = 1/\delta(f) \simeq 1.30357727$

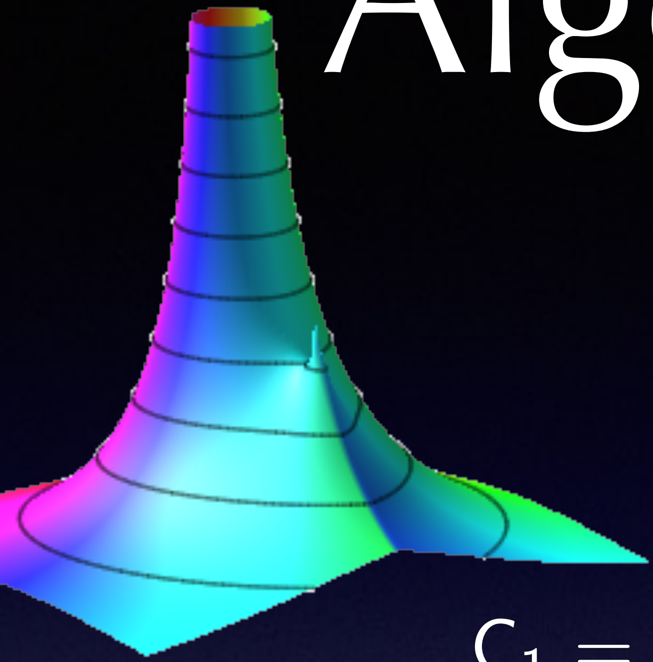
$$\boxed{\ell_n \simeq 2.04216\,\rho^n}$$

$\rho\ \mathrm{Res}(f,\delta(f))$



remainder exponentially small

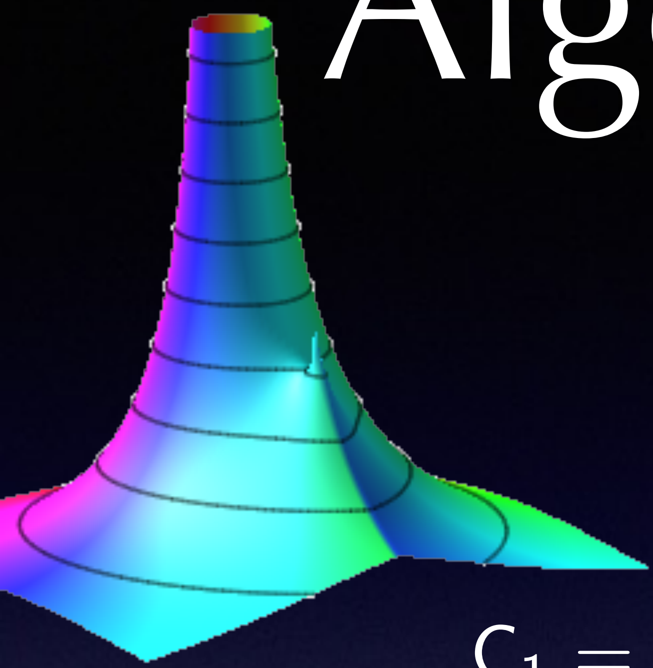
Algebraic Singularities



$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz$$

$$C_1 = 1 = \frac{1}{2\pi i} \oint \frac{1 - \sqrt{1-4z}}{2} \frac{dz}{z^2}$$

Algebraic Singularities



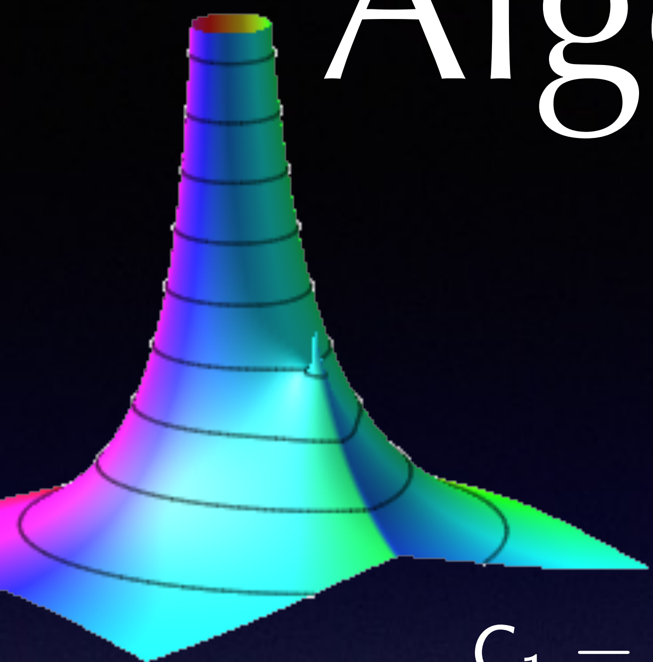
$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$C_1 = 1 = \frac{1}{2\pi i}\oint \frac{1-\sqrt{1-4z}}{2}\frac{dz}{z^2}$$

Hankel:

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int_{(0)}^{+\infty}(-t)^{-a}e^{-t}\,dt$$

Algebraic Singularities

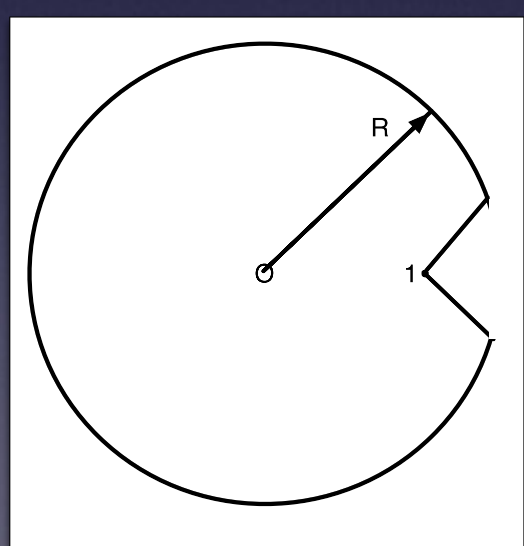


$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$C_1 = 1 = \frac{1}{2\pi i}\oint \frac{1-\sqrt{1-4z}}{2}\frac{dz}{z^2}$$

Hankel:

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int_{(0)}^{+\infty} (-t)^{-a} e^{-t}\,dt$$



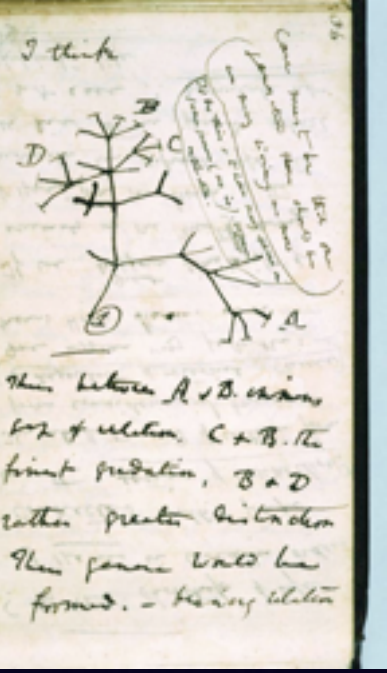
$\Delta(\varphi,R)$

Thm. [Flajolet-Odlyzko]
If f is analytic in $\Delta(\varphi,R)$ and $f=O((1-z)^a)$ when $z\to 1$, then $[z^n]f=O(n^{-a-1})$ when $n\to\infty$.

Method: expand, translate termwise, truncate.

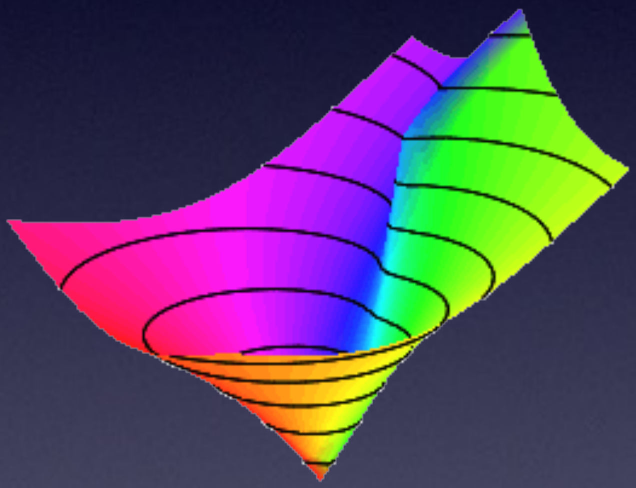
Example

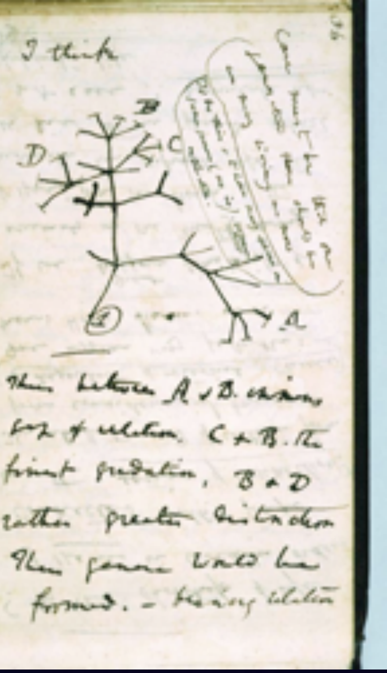
Cayley trees $\quad \mathsf{y} - \mathsf{z}\exp(\mathsf{y}) = 0$



Example

Cayley trees $\quad y - z\exp(y) = 0$

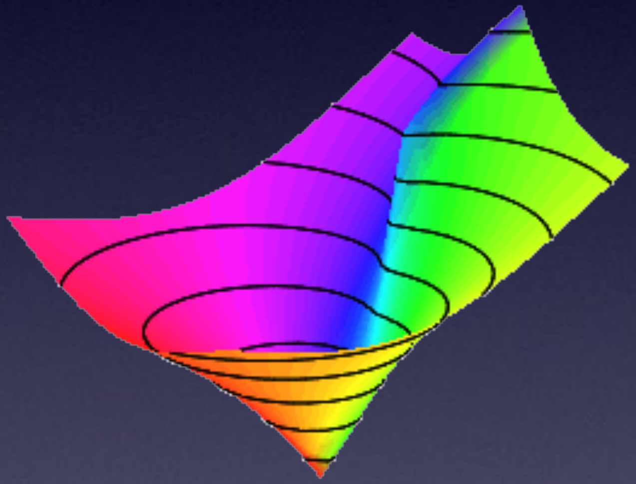


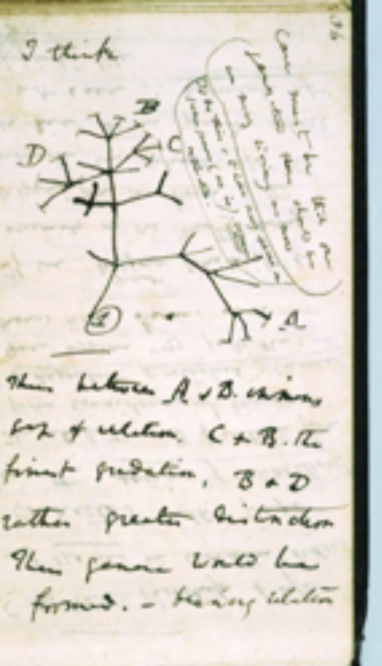


Example

Cayley trees $\quad y - z\exp(y) = 0$

Obstruction to Implicit Function Theorem: $\quad 1 - z\exp(y) = 0$





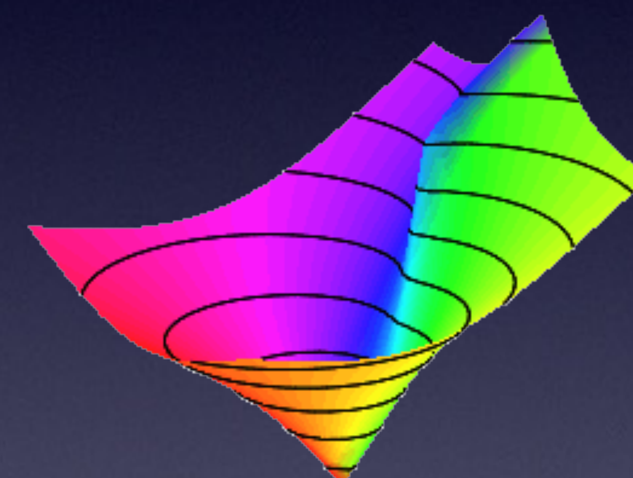
Example

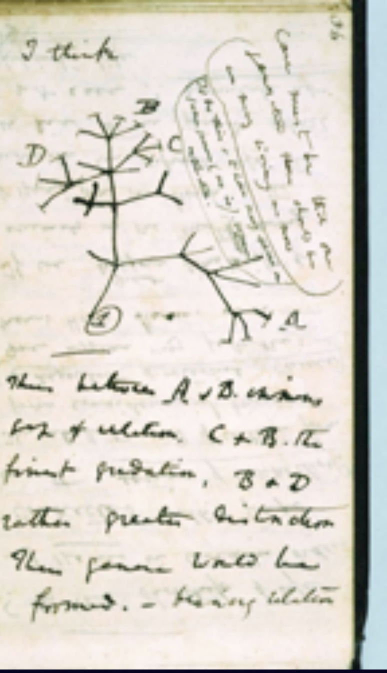
Cayley trees $\quad y - z\exp(y) = 0$

Obstruction to Implicit Function Theorem: $\quad 1 - z\exp(y) = 0$

Consequences:

Singularity at y=1, z=exp(-1);

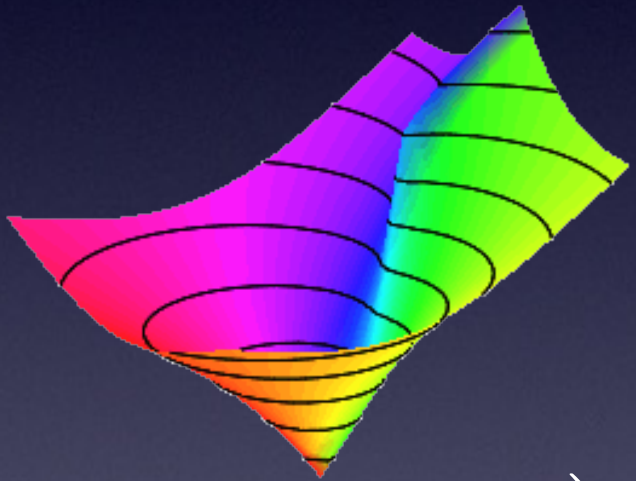




Example

Cayley trees $\quad y - z\exp(y) = 0$

Obstruction to Implicit Function Theorem: $1 - z\exp(y) = 0$

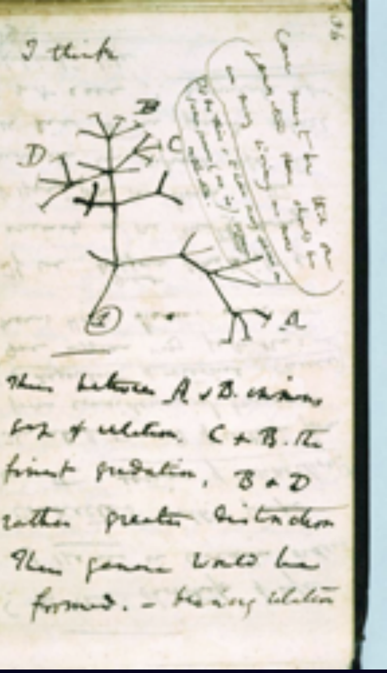


Consequences:

Singularity at y=1, z=exp(-1);
Local behaviour by inverting

$$z = y\exp(-y) = e^{-1}\left(1 - \frac{1}{2}(y-1)^2 + O((y-1)^3)\right)$$

$$\Rightarrow y = 1 - \sqrt{2}\sqrt{1-ez} + \frac{2}{3}(1-ez) + O((1-ez)^{3/2})$$

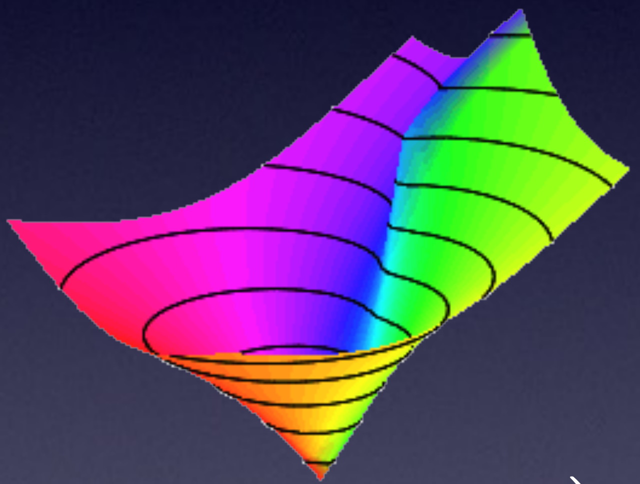


Example

Cayley trees $\quad y - z\exp(y) = 0$

Obstruction to Implicit Function Theorem: $\quad 1 - z\exp(y) = 0$

Consequences:



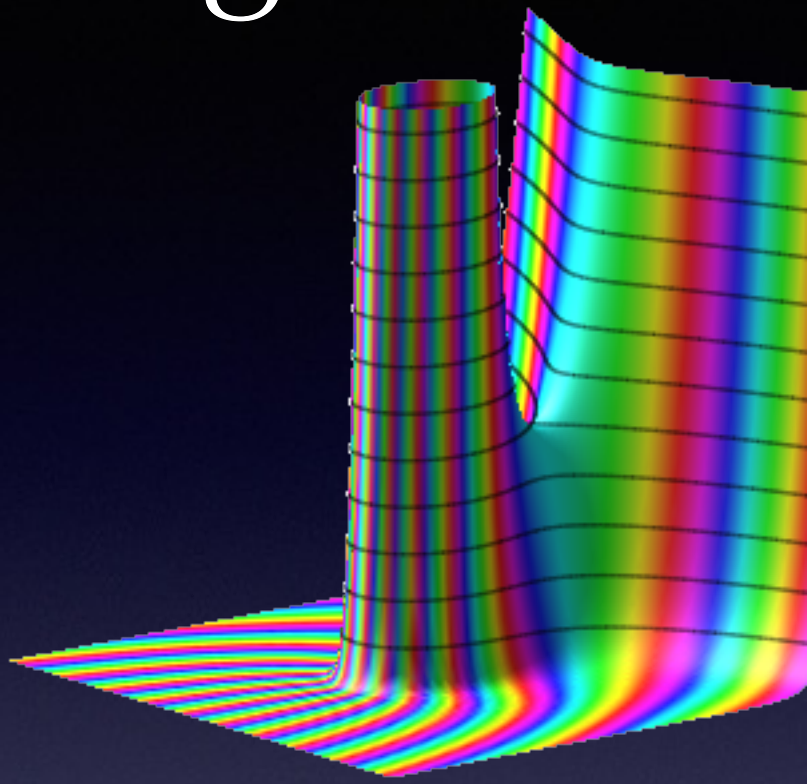
Singularity at y=1, z=exp(-1);
Local behaviour by inverting

$$z = y\exp(-y) = e^{-1}\left(1 - \frac{1}{2}(y-1)^2 + O((y-1)^3)\right)$$

$$\Rightarrow y = 1 - \sqrt{2}\sqrt{1-ez} + \frac{2}{3}(1-ez) + O((1-ez)^{3/2})$$

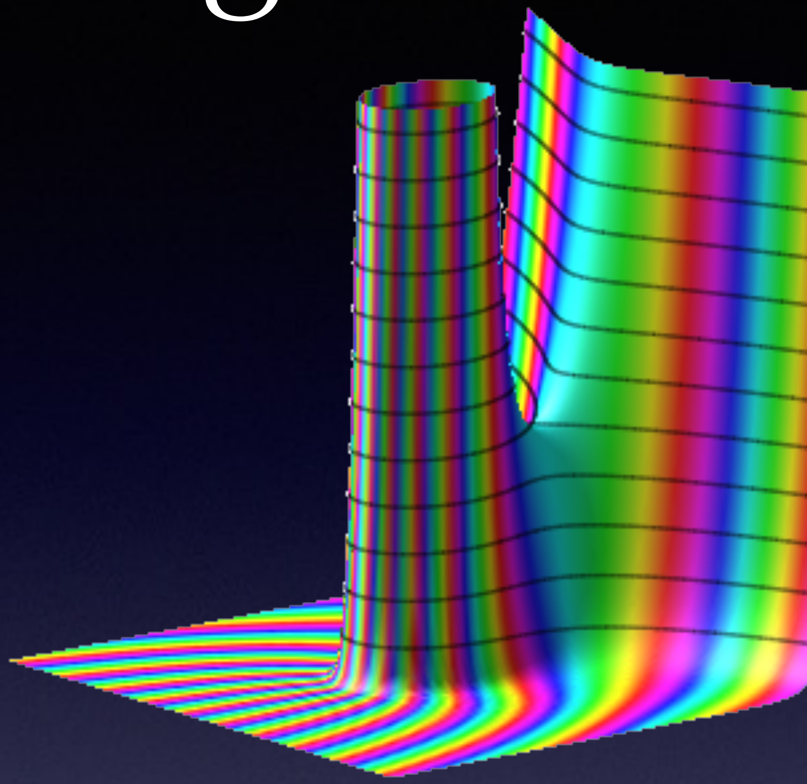
Coefficients: $\displaystyle \frac{T_n}{n!} = \frac{e^n}{\sqrt{2\pi}n^{-3/2}}(1 + O(1/n)).$

Functions with fast singular growth



$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz$$

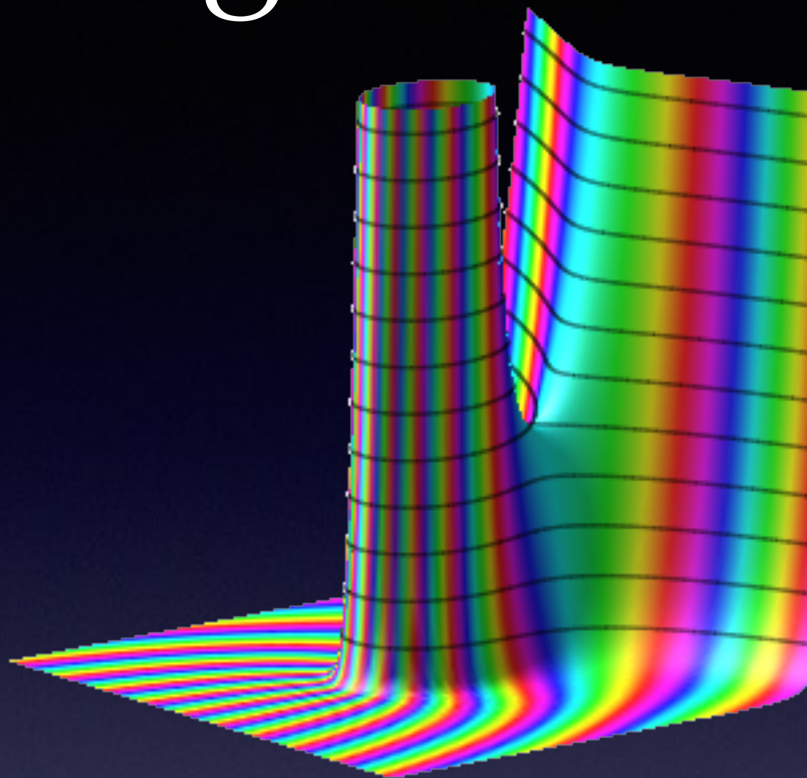
Functions with fast singular growth



$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

Saddle-point equation: $R_n \frac{f'}{f}(R_n) = n + 1$

Functions with fast singular growth

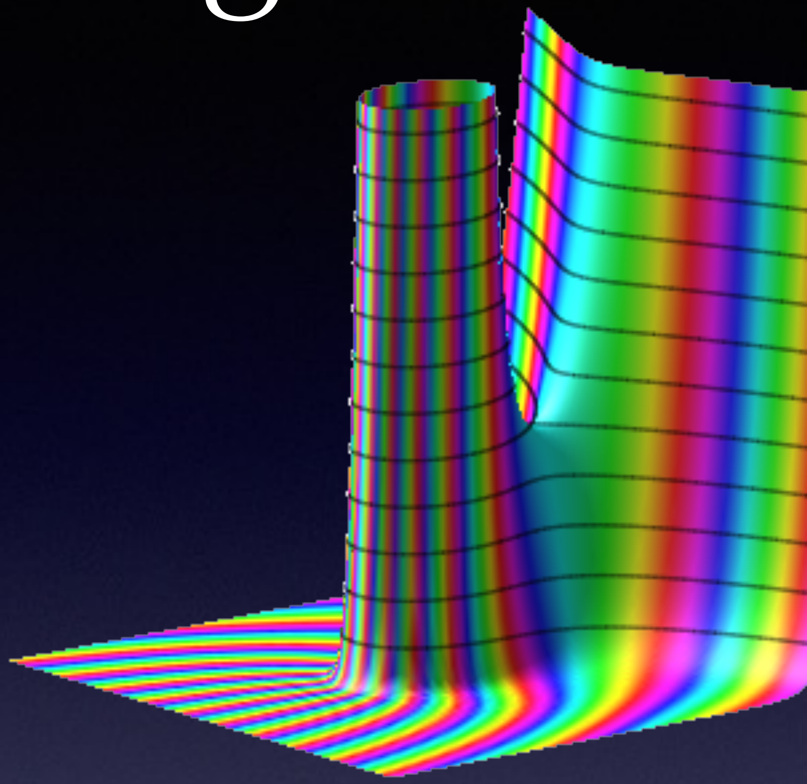


$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz$$

Saddle-point equation: $R_n \frac{f'}{f}(R_n) = n + 1$

Local behaviour: $\approx \frac{f(R_n)}{R_n^{n+1}} \exp(c_n \frac{(z - R_n)^2}{2})$

Functions with fast singular growth



$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz$$

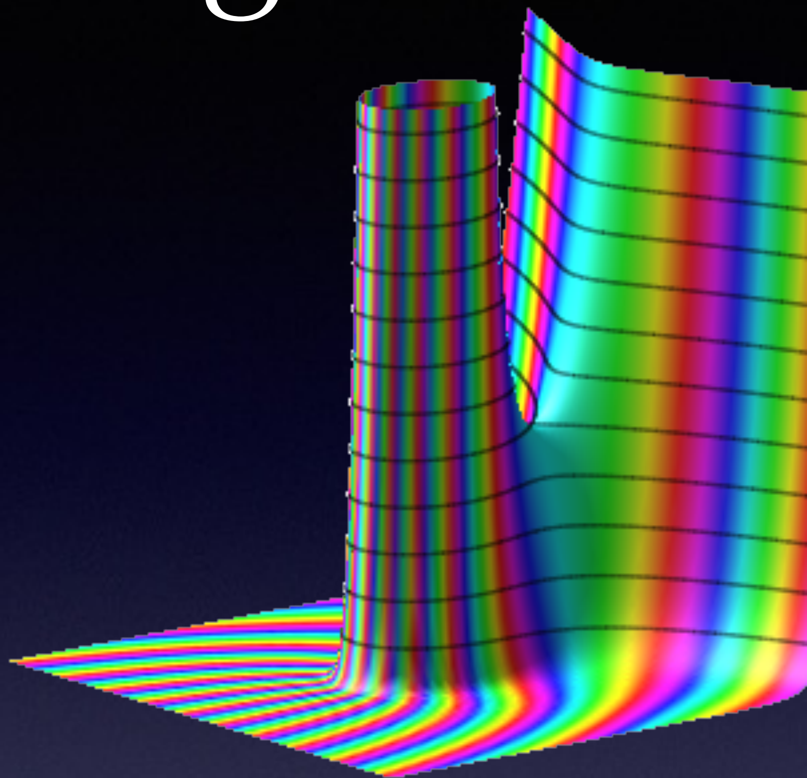
Saddle-point equation: $R_n \frac{f'}{f}(R_n) = n + 1$

Local behaviour: $\approx \frac{f(R_n)}{R_n^{n+1}} \exp(c_n \frac{(z - R_n)^2}{2})$

Approximate by a Gaussian integral:

$$a_n \sim \frac{1}{2} \frac{f(R_n)}{R_n^{n+1} \sqrt{2\pi c_n}}$$

Functions with fast singular growth



$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz$$

Saddle-point equation: $R_n \frac{f'}{f}(R_n) = n + 1$

Local behaviour: $\approx \frac{f(R_n)}{R_n^{n+1}} \exp(c_n \frac{(z - R_n)^2}{2})$

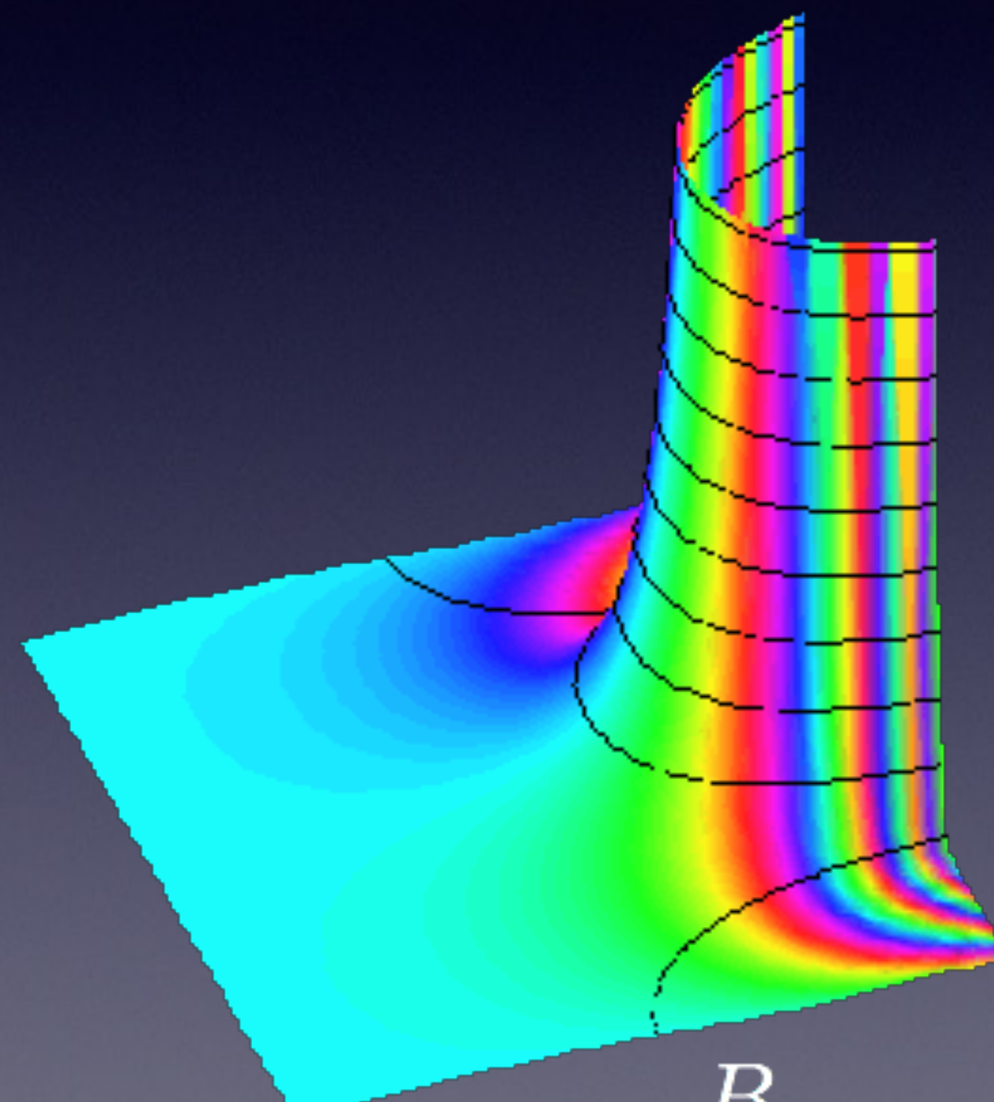
Approximate by a Gaussian integral:

$$a_n \sim \frac{1}{2} \frac{f(R_n)}{R_n^{n+1} \sqrt{2\pi c_n}}$$

Exercise: Stirling's formula (f=exp).

Other Example

Set partitions: $\exp(e^z - 1) = 1 + 1\frac{z}{1!} + 2\frac{z^2}{2!} + 5\frac{z^3}{3!} + 15\frac{z^4}{4!} + \cdots$



$$\log \frac{B_n}{n!} \sim -n \log \log n$$

These 3 cases cover most applications

Demo?

Concluding Remarks

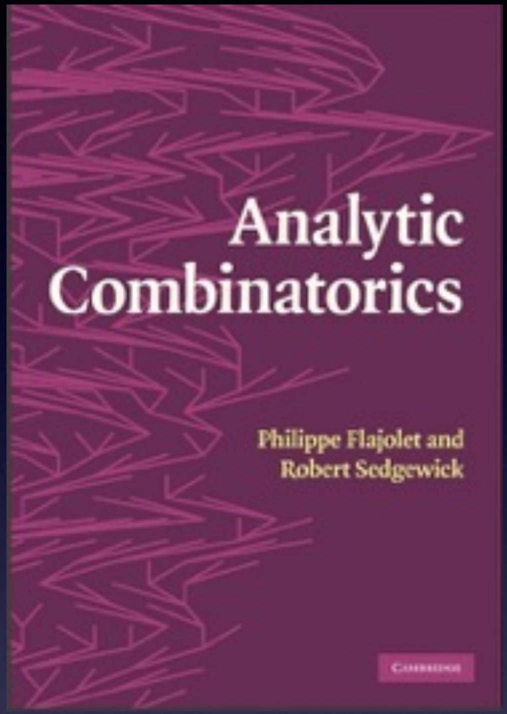
More is possible:

Full asymptotic expansions;
Limiting distributions;
Fast enumeration;
Random generation.

Concluding Remarks

More is possible:

Full asymptotic expansions;
Limiting distributions;
Fast enumeration;
Random generation.



Concluding Remarks

More is possible:

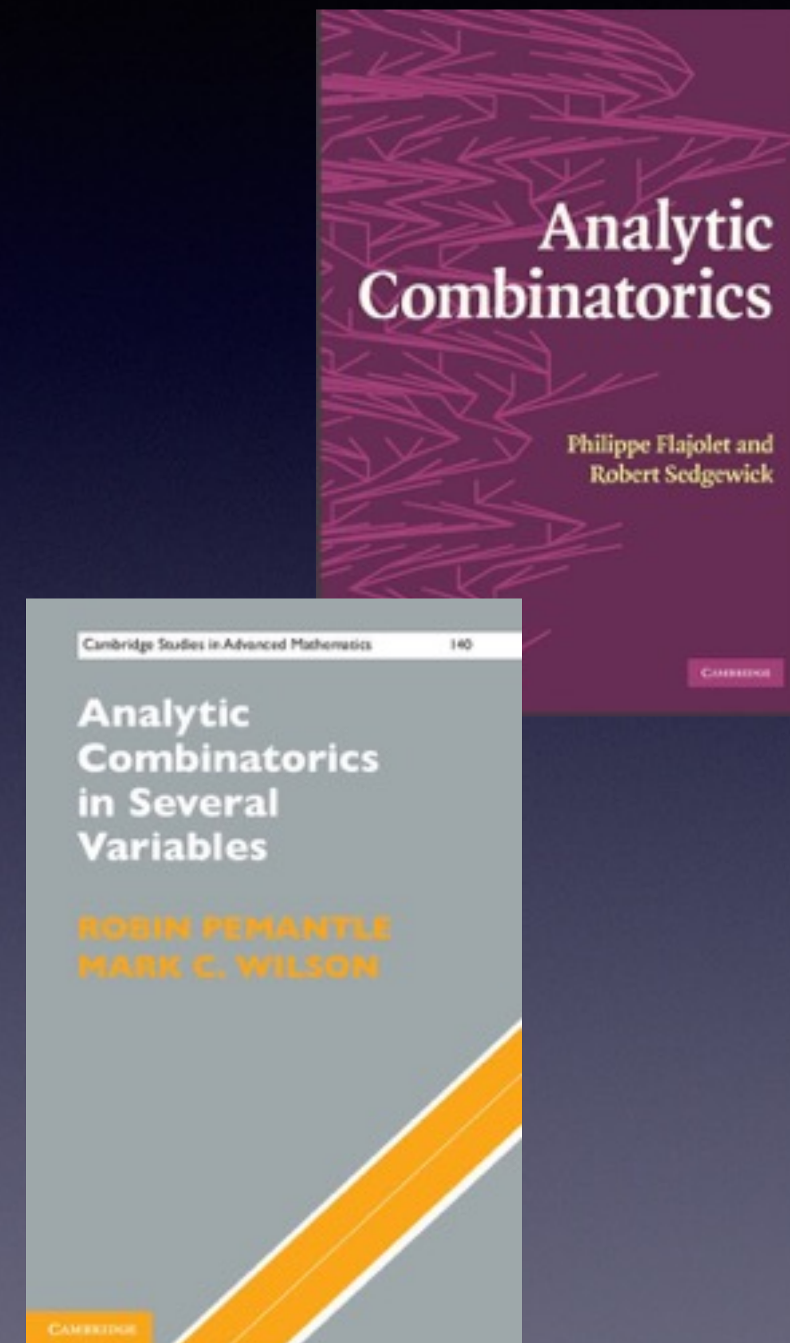
> *Full asymptotic expansions;*
> *Limiting distributions;*
> *Fast enumeration;*
> *Random generation.*

Concluding Remarks

More is possible:

Full asymptotic expansions;
Limiting distributions;
Fast enumeration;
Random generation.

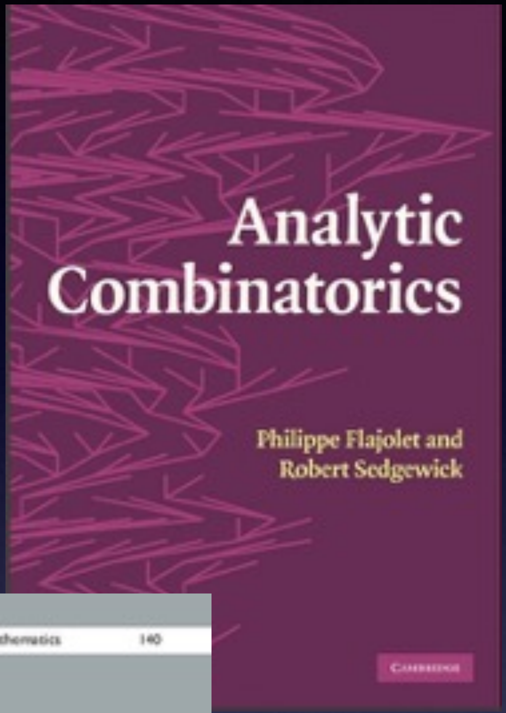
Next step: automate the multivariate case

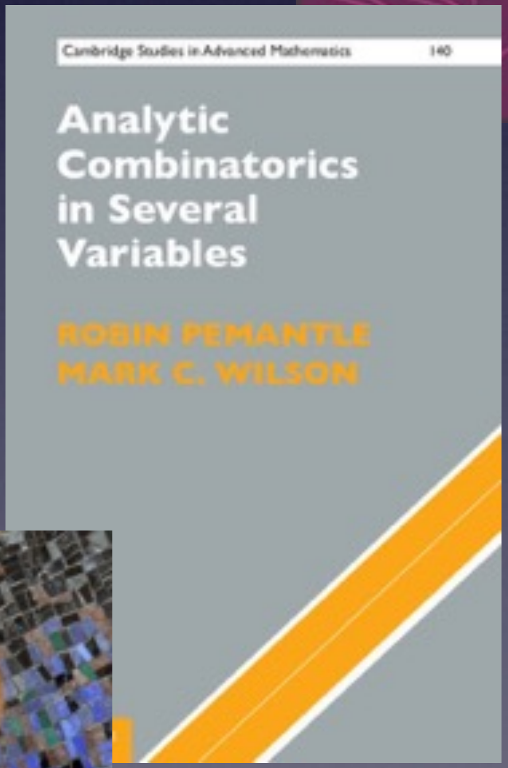


Concluding Remarks

More is possible:

Full asymptotic expansions;
Limiting distributions;
Fast enumeration;
Random generation.



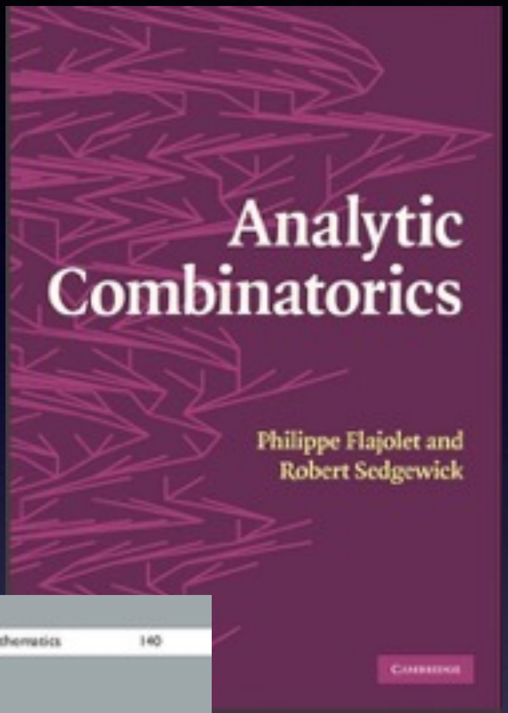


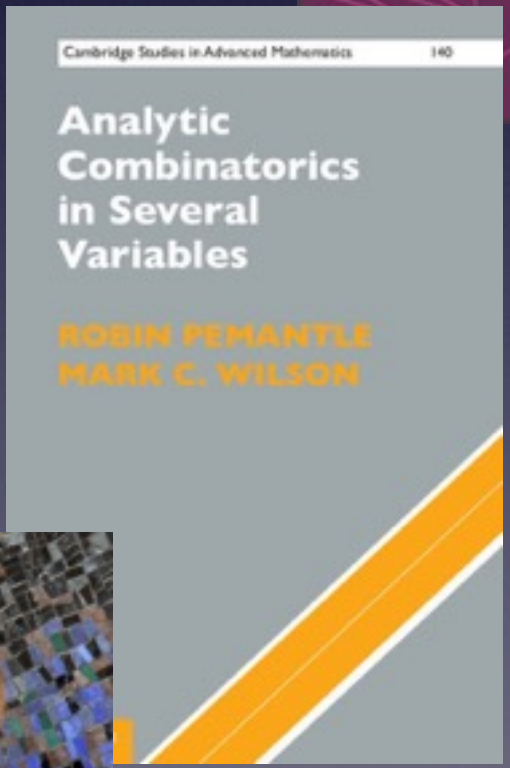
Next step: automate the multivariate case

Concluding Remarks

More is possible:

Full asymptotic expansions;
Limiting distributions;
Fast enumeration;
Random generation.





Questions?
Office #317

Next step: automate the multivariate case