Function spaces vs. Scaling functions: Some issues in image classification

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Abstract: Criteria based on the computation of fractal dimensions have been used in order to perform image analysis and classification; we show that such criteria often amount to determine the regularity of the image in some classes of function spaces, and that looking for richer criteria naturally leads to the introduction of new classes of function spaces. We will investigate the properties of some of these classes, and show which type of additional information they yield for the initial image.

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1 Introduction

Since the 1970s, tools derived form fractal geometry have been used in order to derive parametres of fractal nature, which give a new type of information on the image studied, and can be used for classification. Fractal objects often present two related aspects: One is analytic, and consists in scaling-invariance properties, and the other is geometric and is expressed by the fractal dimension of the object. For instance, the one-dimensional Brownian motion is scaling invariant ($B(ax)$ has the same law as $a^{1/2}B(x)$), and its sample paths have fractal dimension $3/2$.

Let us start by considering the geometric aspect, which is supplied by fractal dimensions. The simplest notion of dimension which can be used is the *box dimension*:

Definition 1 *Let A be a bounded subset of $\mathbb{R}^d$; if $\epsilon > 0$, let $N_\epsilon(A)$ be the smallest number of balls of radius ϵ required to cover A.*

The upper box dimension of A is

$$\overline{dim}_B(A) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

The lower box dimension of A is

$$\underline{dim}_B(A) = \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}.$$

The following important feature makes this notion useful in practical applications: If both limits coincide, then the box dimension can be computed through a regression on a log-log plot ($\log N_\epsilon(A)$ vs. $\log \epsilon$):

$$dim_B(A) = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(A)}{-\log \epsilon}. \tag{1}$$

As such, this tool has a rather narrow field of applications in image processing; indeed, it applies only when a particular "fractal" set can be isolated in the image. However, the fact that the limit exists in (1) points towards another possible feature, which is much more common: It shows that some quantities display an approximate power-law behavior through the scales. In practice, this property holds for many other quantities than $N_\epsilon(A)$ and the associated power-law exponents can thus yield collections of important parameters, which can be used in image classification. Loosely speaking, the purpose of multifractal analysis is to:

- Introduce new quantities which present these power-law behaviors,
- study their mathematical properties, and in particular relate them with scales of function spaces,
- determine implications for "fractal features" which are present in the image.

2 Scaling functions and function spaces

Let us now briefly pass into review the different quantities which have been used up to now as possible candidates for scaling-invariance features. First, we mention that these quantities usually depend on (at least) one auxiliary parameter p, and therefore the exponents which are derived are not one real number, but a function of this parameter p, hence the term *scaling functions* used in order to charcaterize these collections of exponents. Note that the use of a whole function in order to perform classification yields a much more precise tool than the use of one single number.

Let us now be more specific; we start with what was historically the first example of a scaling function. It was introduced by N. Kolmogorov in the context of fully developed turbulence, with a motivation which was quite similar to ours: Take advantage of the (hypothetic) scaling invariance of fully developed turbulence in order to derive a collection of "universal" parameters which could be computed on experimental data, and use it in order to determine if a given model is correct or not.

Let $f : \mathbb{R}^d \to \mathbb{R}$. the scaling function of f is the function $\eta(p)$ which satisfies

$$\int |f(x+h) - f(x)|^p dx \quad \sim \quad |h|^{\eta(p)}. \tag{2}$$

This loose definition means precisely that

$$\eta(p) = \liminf_{|h| \to 0} \frac{\log \left(\int |f(x+h) - f(x)|^p dx \right)}{-\log |h|}. \tag{3}$$

As in the case of the box dimension, we have to draw a distinction between the mathematical definition, whose purpose is to make sense in a general setting, and the practical evaluation of $\eta(p)$ which requires that the liminf is a real limit, and, in practice, that one can make a precise regression on the scales available in the data.

We will now see the function space interpretation of this initial scaling function. This interpretation will serve several purposes. First, it allows to derive several mathematical properties of this scaling function, but its main advantage will be to point the way towards variants and extensions of this scaling function

- whose computation are numerically easier and more accurate,
- which yield new families of scaling parameters.

This second motivation had unexpected consequences: We will see for instance that such new scaling functions allow to show the presence of "oscillating singularite " in the data, which was an important open issue in several applications.

The most straightforward function space interpretation of the scaling function is obtained through the use of the spaces $\text{Lip}(s, L^p)$ defined as follows.

Definition 2 *Let $s \in (0,1)$, and $p \in [1, \infty]$; $f \in$ Lip$(s, L^p(\mathbb{R}^d))$ if $\int |f(x)|^p dx < \infty$ and $\exists C > 0$ such that $\forall h > 0$,*

$$\int |f(x+h) - f(x)|^p \leq C|h|^{sp}. \tag{4}$$

It follows from this definition that, if $\eta(p) < p$, then

$$\eta(p) = \sup\{s : f \in \mathrm{Lip}(s/p, L^p(\mathbb{R}^d))\}. \tag{5}$$

In other words, the scaling function allows to determine which spaces $\mathrm{Lip}(s, L^p)$ contain the signal for $s \in (0,1)$, and $p \in [1, \infty]$. this reformulation has several advantages:

- Using clasical embeddings between function spaces, one can derive reformulations of the scaling function, which can lead tio a better implementation,
- such a reformulation can naturally lead to extensions outside of the range $s \in (0,1)$, and therefore apply to signals which are either smoother or rougher than allowed by this range,
- extensions for $p < 1$ would lead to a scaling function defined for $p < 1$, and therefore which will be more powerful for classification).

The simpler setting for these extension is supplied by the Besov spaces; this setting will offer the additional advantage of yielding a wavelet reformulation of the scaling function. In order to define Besov spaces, we need to recall the definition of a wavelet basis.

Orthonormal wavelet bases are a privileged tool to study multifractal functions for several reasons. A first one, exposed in this section, is that classical function spaces (such as Besov or Sobolev spaces) can be characterized by conditions bearing on the wavelet coefficients, see Section 2.2. We will just recall some properties of orthonormal and biorthogonal wavelet bases that will be useful in the following. We refer the reader for instance to [15, 16, 34, 39] for detailed expositions of this subject.

2.1 Orthonormal and biorthogonal wavelet bases

Orthonormal wavelet bases are of the following form: There exists a function $\varphi(x)$ and $2^d - 1$ functions $\psi^{(i)}$ with the following properties: The functions $\varphi(x-k)$ ($k \in \mathbb{Z}^d$) and the $2^{dj/2}\psi^{(i)}(2^j x - k)$ ($k \in \mathbb{Z}^d$, $j \in \mathbb{Z}$) form an orthonormal basis of $L^2(\mathbb{R}^d)$. This basis is r-smooth if φ and the $\psi^{(i)}$ are C^r and if the $\partial^\alpha \varphi$, and the $\partial^\alpha \varphi \psi^{(i)}$, for $|\alpha| \le r$, have fast decay.

Therefore, $\forall f \in L^2$,

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_i c^i_{j,k} \psi^{(i)}(2^j x - k); \tag{6}$$

the $c^i_{j,k}$ are the wavelet coefficients of f

$$c^i_{j,k} = 2^{dj} \int_{\mathbb{R}^d} f(x) \psi^{(i)}(2^j x - k) dx, \tag{7}$$

and

$$C_k = \int_{\mathbb{R}^d} f(x) \varphi(x-k) dx. \tag{8}$$

Remarks: In (6), we do not choose the L^2 normalisation for the wavelets, but rather an L^∞ normalisation which is better fitted to the study of Hölder regularity. The L^1 normalisation of (7) follows accordingly.

Note that (7) and (8) make sense even if f does not belong to L^2; indeed, if one uses smooth enough wavelets, these formulas can be interpreted as a duality product betweeen smooth functions (the wavelets) and distributions. We will see the examples of Sobolev and Besov spaces.

We will also need decompositions on *biorthogonal wavelet bases*, which are a useful extension of orthonormal wavelet bases. A *Riesz basis* of an Hilbert space H is a collection of vectors (e_n) such that the finite linear expansions $\sum_{n=1}^N a_n e_n$ are dense in H and

$$\exists C, C' > 0: \quad \forall N, \quad \forall a_n, \quad C\sum_{n=1}^N |a_n|^2 \leq \left\| \sum_{n=1}^N a_n e_n \right\|_H^2 \leq C' \sum_{n=1}^N |a_n|^2.$$

Two collections of functions (e_n) and (f_n) form *biorthogonal bases* if each collection is a Riesz basis, and if $\langle e_n | f_m \rangle = \delta_{n,m}$. When such is the case, any element $f \in H$ can be written

$$f = \sum_{n=1}^{\infty} \langle f | f_n \rangle e_n. \tag{9}$$

Biorthogonal wavelet bases are couples of Riesz bases of L^2 which are, of the form: on one side,

$$\varphi(x-k), \ \ (k \in \mathbb{Z}^d) \quad \text{and} \quad 2^{dj/2}\psi^{(i)}(2^j x - k), \ \ (k \in \mathbb{Z}^d, \ j \in \mathbb{Z})$$

and, on the other side,

$$\tilde{\varphi}(x-k) \ \ (k \in \mathbb{Z}^d) \quad \text{and} \quad 2^{dj/2}\tilde{\psi}^{(i)}(2^j x - k), \ \ (k \in \mathbb{Z}^d, \ \ j \in \mathbb{Z}).$$

Therefore, $\forall f \in L^2$,

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_i c_{j,k}^i \psi^{(i)}(2^j x - k); \tag{10}$$

where

$$c_{j,k}^i = 2^{dj} \int_{\mathbb{R}^d} f(x)\tilde{\psi}^{(i)}(2^j x - k)dx, \quad \text{and} \ C_k = \int_{\mathbb{R}^d} f(x)\tilde{\varphi}(x-k)dx. \tag{11}$$

We will see that biorthogonal wavelet bases are particularly well adapted to the decomposition of the Fractional Brownian Motion; indeed, well chosen biorthogonal wavelet bases allow to decorrelate the wavelet coefficients of these processes (the wavelet coefficients become independent random variables), and therefore greatly simplifies their analysis.

We will use more compact notations for indexing wavelets. Instead of using the three indices (i, j, k), we will use dyadic cubes. Since i takes $2^d - 1$ values, we can assume that it takes values in $\{0,1\}^d - (0, \dots, 0)$; we introduce:

- $\lambda \ \ (= \lambda(i,j,k)) \ \ = \dfrac{k}{2^j} + \dfrac{i}{2^{j+1}} + \left[0, \dfrac{1}{2^{j+1}}\right)^d.$
- $c_\lambda = c_{j,k}^i$

- $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$.

The wavelet ψ_λ is essentially localized near the cube λ; more precisely, when the wavelets are compactly supported

$$\exists C > 0 \quad \text{such that} \quad \forall i, j, k, \qquad supp\,(\psi_\lambda) \subset C\ \lambda$$

(where $C\ \lambda$ denotes the cube of same center as λ and C times wider). Finally, Λ_j will denote the set of dyadic cubes λ which index a wavelet of scale j, i.e. wavelets of the form $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ (note that Λ_j is a subset of the dyadic cubes of side 2^{j+1}). We take for norm on $\mathbb{R}^d$

$$\text{if } \ x = (x_1, \dots, x_d), \qquad |x| = \sup_{i=1,\dots,d} |x_i|;$$

so that the diameter of a dyadic cube of side 2^{-j} is exactly 2^{-j}.

Among the families of wavelet bases that exist, two will be particularly useful for us:

- Lemarié-Meyer wavelets, such that φ and $\psi^{(i)}$ both belong to the Schwartz class;
- Daubechies wavelets, such that the functions φ and $\psi^{(i)}$ can be chosen arbitrarily smooth and with compact support.

If the wavelets are r-smooth, they have a corresponding number of vanishing moments, see [39]:

$$\text{If } |\alpha| < r, \quad \text{then} \quad \int_{\mathbb{R}^d} \psi^{(i)}(x) x^\alpha dx = 0.$$

Therefore, if the wavelets are in the Schwartz class, all their moments vanish.

In order to have a common notation for wavelets and functions φ, when $j = 0$, we note ψ_λ the function $\varphi(x - k)$ (where λ is, in this case, the unit cube shifted by k)).

Finally, Λ_j will denote the set of dyadic cubes λ which index a wavelet of scale j, i.e. wavelets of the form $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ (note that Λ_j is a subset of the dyadic cubes of side 2^{j+1}), and Λ will denote the union of the Λ_j for $j \geq 0$.

Remark: Indexing by dyadic cubes will prove useful for the following reason: The wavelet ψ_λ is essentially localized near the cube λ; more precisely, when the wavelets are compactly supported

$$\exists C > 0 \quad \text{such that} \quad \forall i, j, k, \qquad supp\,(\psi_\lambda) \subset C \cdot \lambda.$$

This property will play an important role in the definition of the wavelet leaders below where this indexing by dyadic cubes is necessary for a simple and natural definition.

2.2 Wavelets and function spaces

A remarkable property of wavelet bases is that they supply bases not only in the L^2 setting, but also for most function spaces that are used in analysis.

Proposition 1 *Let $s > 0$ and $p, q \in (0, \infty]$. Let ψ_λ be an r-smooth wavelet basis with $r > \sup(s, s + d(\frac{1}{p} - 1))$. A function f belongs to the Besov space $B_p^{s,q}(\mathbb{R}^d)$ if and only if $(c_k) \in l^p$ and*

$$\sum_{j\in\mathbb{Z}} \left(\sum_{\lambda\in\Lambda_j} \left[2^{(s-d/p)j} |c_\lambda| \right]^p \right)^{q/p} \leq C \tag{12}$$

(using the usual convention for l^∞ when p or q in infinite).

This definition is very close to saying that f and its fractional derivatives of order at most s belong to L^p (see [] for precise embeddings with Sobolev spaces). Let $s \in (0, 1)$, and $p \in [1, \infty]$; then the following embeddings hold

$$B_p^{s,1} \hookrightarrow \text{Lip}(s, L^p(\mathbb{R}^d)) \hookrightarrow B_p^{s,\infty}.$$

Furthermore, the following embeddngs between Besov spaces hold: If $s \geq 0$, $p > 0$ and $0 < q_1 < q_2$, then $\forall \epsilon > 0$,

$$B_p^{s+\epsilon,\infty} \hookrightarrow B_p^{s,q_1} \hookrightarrow B_p^{s,q_2} \hookrightarrow B_p^{s,\infty}.$$

Thus $B_p^{s,q}$ is "very close" to Lip(s, L^p). (Recall also that $B_\infty^{\alpha,\infty} = C^\alpha(\mathbb{R}^d)$.) In particular, these embeddings imply that Kolmogorov's scaling function satisfies

$$\eta_f(p) = \sup\{s : f \in B_p^{s/p,\infty}\}. \tag{13}$$

This property allows to redefine the scaling function by (13). This redefinition has two advantages: On one hand, it extends the scaling function to all values of $p > 0$, on the other hand, it suggests an alternative way to compute it, through wavelet coefficients. Indeed, it follows form (12) and (13) that the scaling function of f is

$$\forall p > 0, \qquad \zeta_f(p) = \liminf_{j\to+\infty} \frac{\log\left(2^{-dj} \sum_{\lambda\in\Lambda_j} |d_\lambda|^p\right)}{\log(2^{-j})}. \tag{14}$$

We can now take (14) for definition of the scaling function (usually referred to as the *wavelet scaling function*). Note that its interpretation in terms of function spaces implies that it is independent of the (smooth enough) wavelet basis. We also recall that a similar formula was previoulsy introduced by A. Arneodo , E. Bacry and J.-F. Muzy, using the continuous wavelet transform, see []. An additional advantage of using (14) as a definition is that the scaling function is well defined even if f is not a function (in the most general case, it can be a tempered distribution); note that this degree of generality may be necessary: indeed, in all generality, a picture is a discretization of the light intensity, which is a non-negative quantity. Therefore the most general mathematical modeling which takes into account only this a priori information amounts to make the assumption that f is a measure (indeed non-negative distributions necessarily are measures, by a famous theorem of L. Schwartz); we note that a posteriori estimation of the smoothness of images using the wavelet scaling function shows that, indeed, some types of natural images are not smoother than measures.

We will pay special attention to the case $p = +\infty$: f belongs to $B^s_\infty(\mathbb{R}^d)$ if and only if $(C_k) \in l^\infty$ and

$$\exists C,\ \forall \lambda, \qquad |c_\lambda| \ \leq C 2^{-sj}. \tag{15}$$

The spaces B^s_∞ coincide with the uniform Lipschitz spaces $C^s(\mathbb{R}^d)$; for instance, if $1 < s < 1$, an equivalent definition is given by: $f \in L^\infty$ and

$$\exists C,\ \ \forall x, y \qquad |f(x) - f(y)| \leq C|x - y|^s.$$

The *uniform Hölder exponent* of f is

$$H_f^{min} = \sup\{s :\ f \in C^s(\mathbb{R}^d)\}; \tag{16}$$

it yields an additional parameter for image processing and classification that will prove important in the following.

Note that the uniform Hölder exponent of f can be derived from the scaling function

$$H_f^{min} = \lim_{p \to +\infty} \eta_f'(p);$$

it can also be derived directly from the wavelet coefficients of f; indeed, it follows from (37) and the wavelet characterization of the Besov spaces B^s_∞ that, if

$$\omega_j = \sup_{\lambda \in \Lambda_j} |c_\lambda|,$$

then

$$H_f^{min} = \liminf_{j \to +\infty} \quad \frac{\log(\omega_j)}{\log(2^{-j})}. \tag{17}$$

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Donner le biais pour un brownien

Besides its use as a classification tool, let us mention another problem posed by function-space modeling when applied to real-life signals: Data are always available with a finite resolution; therefore, assumeing that images are function (or perhaps distributions) defined on $\mathbb{R}^2$ (or a subset of $\mathbb{R}^2$ such as a square or a rectangle) is an idealization which is convenient for mathematical modeling. However, real-life images are sampled and given by a finite array of numbers (usually of size 1024×1024). This practical remark has an important consequence: The problem of finding which function spaces contain a particular image is ill-posed. Indeed, given any "classical" space of functions defined on a square, and such an array of numbers, one can find a function in this space that will have the preassigned values at the corresponding points of the grid. In other words, paradoxically, essentially any "standard" function space could be used. Let us however show extreme consequences of this simple remark.

Recall that the Fourier transform of a function $f(x_1, x_2)$ is defined by

$$\hat{f}(\xi_1, \xi_2) = \int_{\mathbb{R}^2} f(x_1, x_2) e^{-i(x_1\xi_1 + x_2\xi_2)} dx_1 dx_2.$$

One can, for instance, assume that images are *band-limited* which means that their Fourier transforms vanish outside a ball centered at 0, and whose radius is proportional to the inverse of the sampling width (according to Shannon's theorem); note that this assumption is often done, in particular in deblurring and denoising algorithms. This assumption implies that the model used is composed of C^∞ functions; however it would lead to incompatibilities, for instance if we want to use a realistic model which includes discontinuites along edges (which, as we saw, is a natural requirement).

Another commonly met pitfall is that an image is given by grey-levels, and thus takes values in $[0, 1]$. Therefore, it may seem appropriate to use a modeling by bounded functions, and this is indeed a classical assumption. We will see that the wavelet techniques we introduced allow to discuss this assumption, and show that it is not satisfied for most images.

The resolution of the paradox we raised in this section requires the use of *multiscale techniques* such as the one supplied by wavelet analysis. Let us consider for instance the last example we mentioned: Starting with a discrete image, given by an array of 1024×1024 numbers all lying between 0 and 1, how can we decide that it can be modeled or not by a bounded function? It is clear that, if we consider the image at only one scale (the finest scale in order to lose no information), then the answer seems to be affirmative. However, as mentioned earlier, any other space would also do. One way to solve the difficulty is to consider the image at all the scales available (in theory, there are 10 of them, since $1024 = 2^{10}$) and inspect if certain quantities behave **through this range of scales** as it is the case for a bounded function. If not, we can give an unexpected negative answer to our problem, but this negative answer should however be understood as follows: *The image considered is a discretization at a given scale of a "hidden function" defined on a square (to which we have no access) and, if the scaling properties of this "hidden function" are, at all scales, the same ones as we observe in the range of scales available, then it is not bounded.*

The recipe in order to settle this point is the following: one uses (38) in order to determine numerically the value of H_f^{min},which is done by a regression on a log-log plot, and using Proposition **??**, it follows that, if $H_f^{min} < 0$, then the image is not bounded, and if $H_f^{min} > 0$, then the image is bounded. Of course, if the numerical value obtained for H_f^{min} is close to 0 (i.e. if 0 is contained in the confidence interval which can be obtained using statistical methods, see [44, 46]) then the issue remains unsettled.

Later refinements and extensions of the scaling functions were an indirect consequence of its interpretation in terms of fractal dimensions of Hölder singularities, proposed by G. Parisi and U. Frisch in their seminal paper []. In order to explain their argumentation, we first recall the definition associated with pointwise regulariy. The most widely used one is supplied by the *Hölder regularity*, which has been considered since the end of the 19th century.

Definition 3 *Let $f: \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function, $x_0 \in \mathbb{R}^d$ and let $\alpha \geq 0$; f belongs to $C^\alpha(x_0)$ if there exist $C > 0$, $R > 0$ and a polynomial P of degree less than α such that*

$$\text{if } \; |x - x_0| \leq R, \quad \text{then} \qquad |f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha.$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : \; f \quad \text{is} \quad C^\alpha(x_0)\}.$$

The isohölder sets are

$$E_H = \{x_0 : \quad h_f(x_0) = H\}.$$

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Note that Hölder exponents met in signal processing often lie between 0 and 1, in which case the Taylor polynomial $P(x - x_0)$ boils down to $f(x_0)$ and the definition of the Hölder exponent means that, heuristically,

$$|f(x) - f(x_0)| \sim |x - x_0|^{h_f(x_0)}.$$

The idea behind the derivation proposed in [] is that, if f is not smooth on a large set of locations, then, at a given scale h, the increments $f(x+h) - f(x)$ will bring a large contribution to (2), and therefore the kowledge of the scaling function should yield some information of the size of the sets where f has a given Hölder regularity. Using statistical physics formalism leads to a natural conjecture, usually referred to as the *multifractal formalism* concerning the size of the sets of singularities of f. In order to recall it, we start by recalling the notion of "size" which is adapted to this problem.

Definition 4 *Let $A \subset \mathbb{R}^d$. If $\epsilon > 0$ and $\delta \in [0, d]$, we denote*

$$M_\epsilon^\delta = \inf_R \left(\sum_i |A_i|^\delta \right),$$

where R is an ϵ-covering of A, i.e. a covering of A by bounded sets $\{A_i\}_{i \in \mathbb{N}}$ of diameters $|A_i| \leq \epsilon$. The infimum is therefore taken on all ϵ-coverings.

For any $\delta \in [0, d]$, the δ-dimensional Hausdorff measure of A is

$$mes_\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta.$$

There exists $\delta_0 \in [0, d]$ such that

$$\begin{array}{ll} \forall \delta < \delta_0, & mes_\delta(A) = +\infty \\ \forall \delta > \delta_0, & mes_\delta(A) = 0. \end{array}$$

This critical δ_0 is called the Hausdorff dimension of A, and is denoted by $dim(A)$.

If E is empty then, by convention, $\dim_H(E) = 0$.

If f is locally bounded, then the function $H \to dim(E_H)$ is called the *spectrum of singularities* of f.

Note that, in distinction with the box dimension, the Hausdorff dimension cannot be computed via a regression on a log-log plot. Therefore it can be estimated on experimental data only using indirect methods. We will see that the use of the multifractal formalism is one of them. This heuristic formula, proposed by Paris and Frisch, is

$$dim(E_H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p)) . \tag{18}$$

One mets several problems when one tries to apply this formula. On one hand, the scaling function is properly defined only for $p > 0$ (though one can check that it still makes sense for large classes of processes when $p > -1$, see [] for a discussion of this point); on the other, several natural processes, which are used in signal or image modeling, are counterexamples, see [] for a discussion of this point. Finally, the only result relating the spectrum of singularities and the scaling function, is very partial, and stated in Theorem 1 below, see [].

A *uniform Hölder function* is a function satisfying $H_f^{min} > 0$. In particular, it is continuous. One can prove the following relationship between the scaling function of a function and its pointwise Hölder singularities, see [24].

Theorem 1 *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a uniform Hölder function. Then*

$$dim(E_H) \leq \inf_{p > p_0} (d + Hp - \eta_f(p)) .$$

where p_0 *is such that* $\eta_f(p_0) = dp_0$.

Therefore, a natural line of research was to look for an "improved" scaling function, i.e. one such that (18) would have a wider range of validity, and for which the upper bound supplied by Theorem 1 would be sharper. This lead to the construction of the *wavelet leader scaling function*, which we now recall. The "basic ingredients" in this formula are no more wavelet coefficients, but wavelet leaders, i.e. local suprema of wavelet coefficients. The reason is that pointwise smoothness can be expressed much more simply in terms of wavelet leaders than of wavelet coefficients.

Let λ be a dyadic cube; 3λ is the cube of same center and three times wider. If f is a bounded function, the *wavelet leaders* of f are the quantities

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$

Let $x_0 \in \mathbb{R}^d$; $\lambda_j(x_0)$ is the dyadic cube of width 2^{-j} which contains x_0; and

$$d_j(x_0) = d_{\lambda_j(x_0)} = \sup_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|.$$

It is important to require f to be bounded; otherwise, the wavelet leaders of f can be infinite. The reason for introducing wavelet leaders is that they give an information on the pointwise

Hölder regularity of the function. Indeed, one can show that (see [24] and references therein) if f is a uniform Hölder function, then

$$h_f(x_0) = \liminf_{j\to+\infty} \left(\frac{\log\left(d_j(x_0)\right)}{\log(2^{-j})} \right).$$

Therefore, it is clear that a scaling function constructed with the help of wavelet leaders will incorporate pointwise smoothness information. For any $p \in \mathbb{R}$, let

$$T_f(p,j) = 2^{-2j} \sum_{\lambda\in\Lambda_j} |d_\lambda|^p.$$

The *leader scaling function* is defined by

$$\forall p \in \mathbb{R}, \qquad \zeta_f(p) = \liminf_{j\to+\infty} \frac{\log(T_f(p,j))}{\log(2^{-j})}.$$

An important property of the leader scaling function is that it is "well defined" for $p < 0$, which is not the case for the wavelet scaling function. By "well defined", we mean that it has the following robustness properties if the wavelets belong to the Schwartz class (they still partly hold otherwise, see [**?**, 24]) :

- ζ_f is independent of the wavelet basis.
- ζ_f is invariant under the addition of a C^∞ perturbation.
- ζ_f is invariant under a C^∞ change of variable.

Note that the wavelet scaling function does not possess these properties when p is negative.

The leader scaling function can also be given a function-space interpretation for $p > 0$. Let $p \in (0,\infty)$; a function f belongs to the *Oscillation space* $\mathcal{O}_p^s(\mathbb{R}^d)$ if and only if $(C_k) \in l^p$ and

$$\exists C,\ \forall j, \qquad \sum_{\lambda\in\Lambda_j} \left[2^{(s-d/p)j} d_\lambda\right]^p \leq C.$$

Then

$$\zeta_f(p) = \sup\{s : f \in \mathcal{O}_p^{s/p}\}.$$

Properties of oscillation spaces are investigated in [**?**, 24].

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Montrer qu'on obtient des definitions equivalentes en prenant les leaders restreints

We denote by $\mathcal{L}(u)$ the transform of a concave function u, i.e.

$$\mathcal{L}u(H) = \inf_{p\in\mathbb{R}} \left(d + Hp - u(p)\right).$$

The *leader spectrum* of f is defined through a Legendre transform of the leader scaling function as follows

$$L_f(H) = (\mathcal{L}\zeta_f)(H).$$

Of course, the leader spectrum of f has the same robustness properties as the leader scaling function.

Theorem 2 *If f is uniform Hölder then,*

$$\forall H, \qquad dim(E_H) \leq L_f(H).$$

!!!

These basic questions did not receive a single clear-cut answer. Several notions of singularities can be used, leading to different choices of "global quantities", and leading to estimates on several notions of "size" of sets.

The formulas that relate the scaling functions with the dimensions of singularities are reffered to as *multifractal formalisms*, see Section 4; their range of validity is still far from being well understood, large classes of examples and counterexamples being known.

All notions of pointwise singulartity which have been considered in this context are variants on the notion of "regularity exponent", which, roughly speaking, associates the exponent γ to the singularity $|x-x_0|^\gamma$ at x_0 (if $\alpha \notin 2\mathbb{N}$), see [27, 40] for explicit general definitions of the notion of regularity exponent); such exponents include the (most widely used) Hölder exponent (see definition 7), the p-exponent (see definition 8), and the weak-scaling exponent (which we will not consider in this paper ???, see [40]. However, one can wish to have information on how the function considered oscillates near the singularity at x_0: Consider for instance the "chirps"

$$F_{\gamma,\beta}(x) = |x-x_0|^\gamma \sin\left(\frac{1}{|x-x_0|^\beta}\right), \tag{19}$$

for a given regularity γ, their oscillatory behavior in the neighbourhod of x_0 increases with β; in this example, β parametrizes a "degree of oscillation". We will use a finer description of singularities by introducing an additional "oscillation" parameter, that allows to draw distinctions between functions which share the same Hölder exponent. Measuring such an additional exponent rises additional difficulties, one of them being that several mathematical definitions have been proposed, yielding different types of information. Note that the case $\beta = 0$, i.e. $F_{\gamma,0}(x) = |x-x_0|^\gamma$ is usually reffered to as a "cusp".

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Mettre ici une illustration de cusp, et de chirps avec le meme alpha, mais $\beta = 0$, et deux β positifs franchement differents

One motivation for the detection of singularities such as (19) is that the existence of such behaviors has been conjectured in some physical data, such as fully developed turbulence, see []. Note also that the detection of isolated chirps in a noisy signal is a key issue in the study of gravitational waves, see []. We will see another motivation, which is an internal mathematical requirement in multifractal analysis: one is often obliged to compute a primitive of the signal (or more generally a fractional integral) before performing its multifractal analysis (see Section**??**). In that case, the singularity sets can be modified is a way which is difficult to predict if singularities such as (19) are present in the signal. Therefore understanding what multifractal analysis yields in this case requires the consideration of such behaviors.

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Developper ici la motivation turbulence? citer au moins Hunt, Vassilicos, Moffat, Lundgren, autres? est-ce que ces articles ont eu une suite dans la communaute turbulence?

In Section 3, we discuss the notion of pointwise exponents, starting with the Hölder exponent, and then showing how considering this exponent for simultaneously the function itself and its fractional integrals allows to put into light the oscillatory properties of the function considered.

We discuss the wavelet characterizations of these quantities, and consider the particular case of functions that do not display oscillating singularites (i.e. “cusp functions”).

Section 4 deals with the corresponding multifractal formalisms. We start by recalling the basic ideas underlying the construction of the multifractal formalism for the Hölder exponent, and based on wavelet leaders.

In Section 5 we show how they can be extended to fractional integrals of the function, and which information the multifractal analysis of these fractional integrals yield about the presence of oscillating singularities. We will focus on two clases of examples which yield oscillating singularities: Lacunary Wavelet esries and Random wavelet series.

In a second companion paper, a multifractal formalism fitted to the obtention of the oscillation exponent only will be constructed and its properties will be investigated. We will also extend it into a grandcanonical multifractal formalism, yielding the dimensions of the sets of points where the couple composed of the Hölder and the oscillation exponent take a given value.

Some results proved in this paper have been announced in [2, 28, 45].

3 Pointwise exponents

We will usually assume that the mathematical objects we deal with are functions defined on $\mathbb{R}^d$. Let us briefly discuss the functional setting that we will use: L^∞ or L^p spaces are not a convenient setting because the most standard models, such as Brownian motion, FBM or Lévy processes grow at infinity, and therefore do not belong to theses spaces. A right model should locally correspond to these spaces, but should allow some growth at infinity. On the other hand, a non controlled growth (as allowed by the spaces L^p_{loc}) should be prohibited, for a technical reason: Our analysis will heavily be based on wavelet techniques, and, if wavelets are only assumed to have fast decay, so that the computation of the wavelet coefficients might lead to a divergent integral. This remark shows that a polynomial growth is the maximum we can allow; and it is compatible with Brownian or FBM sample paths. Therefore we make the following asumption.

Definition 5 *A function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $L^1_{SG}(\mathbb{R}^d)$ (L^1 with slow growth) if $f \in L^1_{loc}$ and*

$$\exists C, A > 0 : \qquad \int_{B(0,R)} |f(x)|dx \leq C(1+R)^A.$$

Furthermore, the local regularity definition that we will use requires a preliminary precaution the corresponding exponent α may be modified if the values taken by f are modified on a set of vanishing measure. In order to turn this problem, we always assume in the following that functions considered are redefined into a new functions $\tilde{f}$ at every point according to the formula

$$\tilde{f}(x_0) = \liminf_{r\to 0} \frac{1}{Vol(B(x_0,r))} \int_{B(x_0,r)} f(x)dx, \tag{20}$$

where $B(x_0, r)$ denotes the open ball centered at x_0 and of radius r.

Because of Lebesgue's differentiation theorem, f is thus redefined at most on a set of vanishing measure. We will still denote by f this (possibly) new function $\tilde{f}$. Note that, if f is continuous, it is not altered by this redefinition. In all that follows, we implicitly assume that all functions we consider are canonically redefined using (20). The assumptions that we made have the following consequences:

- One can always compute the wavelet coefficients of a function $f \in L^1_{SG}(\mathbb{R}^d)$, and the corresponding wavelet series converges to f locally in L^1.
- If x_0 is a Lebesgue point of f the wavelet series converges at x_0 towards $f(x_0)$.
- If f is continuous in a neighbourhood of x_0, then convergence is uniform on compact sets

We will have to make additional assumptions, depending on the notion of pointwise regularity which is used: We will often make the assumption that they are locally bounded with polynomial increase. Dealing with the p-exponent allows to relax this hypothesis, see [27, 30]. However, we will see in Section ??? how to deal with measures, or even how to fit the most general setting of distributions.

3.1 Regularity exponents

The first pointwise exponents which have been considered fall in the class of *regularity exponents* (as opposed to oscillation exponents, with wich we will deal later). This notion has first been developed in its general form by Y. Meyer: The following definition for a pointwise regularity criterium is implicit in [40].

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S.J. : IL peut etre plus astucieux de se placer directement dans le cadre un peu plus restrictif de [27] Le probleme etant que ca necessite d'introduire tout de suite un cadre technique assez lourd: il faut la notion d'integration fractionaire pour la condition:

$$\textbf{if} \quad f \in H^\alpha(x_0), \quad \textbf{then} \quad \forall \beta > 0, \;\; f^{-\beta} \in H^{\alpha+\beta}(x_0).$$

Il faut aussi demander la γ-stabilite, en mentionnant que l'exposant de Holder ne la verifie pas.

Definition 6 *Let $\alpha_0 \in \mathbb{R}$. A collection of vector spaces of functions (or distributions) $H^\alpha(x_0)$ (indexed by $\alpha > \alpha_0$ and $x_0 \in \mathbb{R}$) defines a pointwise regularity exponent if:*

- *This sequence of spaces is decreasing when α increases.*
- $f(.) \in H^\alpha(x_0) \Rightarrow f(. - y_0) \in H^\alpha(x_0 + y_0)$.
- *If $f - g$ is C^∞ in a neighbourhood of x_0, then $f \in H^\alpha(x_0) \Leftrightarrow g \in H^\alpha(x_0)$.*
- *The function $|x - x_0|^\alpha$ locally belongs to $H^\alpha(x_0)$ and does not belong $H^\gamma(x_0)$ if $\gamma > \alpha$.*

Let us mention a few examples: The most widely used notion of pointwise regularity criterium is supplied by the *Hölder regularity* (for which $\alpha_0 = 0$), which has been widely used since the end of the 19th century.

Definition 7 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function, $x_0 \in \mathbb{R}^d$ and let $\alpha \geq 0$; f belongs to $C^\alpha(x_0)$ if there exist $C > 0$, $R > 0$ and a polynomial P of degree less than α such that*

$$\text{if } |x - x_0| \leq R, \quad \text{then} \qquad |f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha.$$

One difficulty for using this notion is that it requires f to be locally bounded. We will discuss in Section ??? how to detremine numerically if requirement is fulfilled; when it is not the case, one can use the following generalization introduced by Calderón and Zygmund in 1961, see [12].

Definition 8 *Let $p \in [1, +\infty)$, $\alpha > -d/p$ and let f be a function locally in $L^p(\mathbb{R}^d)$; f belongs to $T^p_\alpha(x_0)$ if if there exist $C > 0$ and a polynomial P of degree less than α such that*

$$\forall r \leq R, \quad \left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C r^\alpha. \tag{21}$$

Remarks: This notion requires the weaker assumption that $f \in L^p_{loc}(\mathbb{R}^d)$. We will discuss in Section ??? how to check this requirement. An advantage over the Hölder exponent is that it allows to consider negative Hölder exponents, which is a mandatory requirement in many applications, see [5, 30]: for instance, the function $f(x) = \frac{1}{|x-x_0|^\gamma}$ belongs locally to L^p if $\gamma < d/p$, in which case it has a p-exponent equal to $-\gamma$, as expected.

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Ê **SJ: donner aussi une ref a la suite du papier si on rediscute vraiment ce probleme, dans le cadre: motivation pour l'integration fractionnaire**

Ê

(we also refer to [1, 40] for a discussion of the weaker notion of *weak-scaling regularity*, which we won't deal with here).

Once a notion of pointwise regularity $H^\alpha(x_0)$ satisfying the properties listed in Definition 6 is picked, one defines the corresponding pointwise exponent as

$$r_f(x_0) = \sup\{\alpha : \quad f \in H^\alpha(x_0)\} \tag{22}$$

The *Hölder exponent*, which is denoted by $h^s_f(x_0)$, corresponds to the $C^\alpha(x_0)$ regularity and the *p-exponent*, which is denoted by $h^p_f(x_0)$) corresponds to the $T^p_\alpha(x_0)$ regularity.

It follows from the definition of r_f that

$$\forall \lambda, \mu \in \mathbb{R} \qquad r_{\lambda f + \mu g}(x_0) \geq \inf(r_f(x_0), r_g(x_0))$$

and, if $\lambda \neq 0$, $\mu \neq 0$ and $r_f(x_0) \neq r_g(x_0)$, then

$$r_{\lambda f + \mu g}(x_0) = \inf(r_f(x_0), r_g(x_0)). \tag{23}$$

3.2 Oscillation exponents

Let us now come back to our initial motivation of defining oscillation exponents: We wish to describe strong local oscillations which display the same qualitative feature as in (19). A first natural idea is to check if this can be done by comparing the values taken by different regularity exponents. and see if one can infer from this information the value of β. Let us check what this idea yields on the regularitry exponents that we already introduced; it is straightforward to check which spaces $C^\alpha(x_0)$ or $T^p_\alpha(x_0)$ contain the functions $F_{\gamma,\beta}$ defined in (19); an easy computation shows that $F_{\gamma,\beta} \in C^\alpha(x_0)$ and $F_{\gamma,\beta} \in T^p_\alpha(x_0)$ if and only if $\alpha \leq \gamma$. In other words, checking which $T^p_\alpha(x_0)$ spaces contain the $F_{\gamma,\beta}$ does not allow to draw a difference between them for a fixed γ, therefore, we have to follow a different path. An indications pointing to another idea is supplied by the following remark. Let $\gamma > 0$, be given and let us estimate the primitive of (19). Since

$$\int_0^x |t-x_0|^\gamma \sin\left(\frac{1}{|t-x_0|^\beta}\right) dt = \int_0^x \frac{|t-x_0|^{\gamma+\beta+1}}{\beta}\left(\cos\left(\frac{1}{|t-x_0|^\beta}\right)\right)' dt,$$

it follows that, in the neighbourhood of x_0, this primitive is the sum of

$$\frac{|x-x_0|^{\gamma+\beta+1}}{\beta}\cos\left(\frac{1}{|x-x_0|^\beta}\right)$$

and higher order terms; thus the Hölder exponent of $F_{\gamma,\beta}$ is γ, but the Hölder exponent of its primitive is $\gamma+\beta+1$; this simple comuttaion points towards the clue that the oscillation exponent can be recovered by comparing the regularity exponents of f and of its primitive.

Before proposing a precise mathematical procedure which allows to recover β, let us mention a natural requirements that a notion of "oscillation exponent" should satisfy in order to be of pratical use. The definition used should allow for possible superpositions and "mixtures"; indeed, in the spirit of multifractal analysis, we do not expect these local behaviors to appear only in an isolated, "perfect" form as in (19), but rather for a dense set of values of x_0, and with possible corruptions by noise. Therefore one should find a key feature of (19) that characterizes the exponent β, and use it as a general definition of oscillating singularity. We noticed that β should be recovered by comparing the regularity exponents of f and of its primitive. We will need a slight extension of this remark in order to obtain a definition of oscillation exponent which fulfills this requirement. For that purpose, we introduce the notion of *local fractional integral*.

Definition 9 *Let f be an L^2 function; the fractional integral of order s of f is the operator $(Id-\Delta)^{-s/2}$ defined as the convolution operator which amounts to multiply the Fourier transform of f with $(1+|\xi|^2)^{-s/2}$.*

Let ϕ be a C^∞ compactly supported function satisfying $\phi(x) = 1$ for x in a neighbourhood of x_0. If $f \in L^1_{SG}$, its local fractional integral of order s is

$$f^{(-s)} = (Id-\Delta)^{-s/2}(\phi f). \tag{24}$$

*The Hölder exponent of the local fractional integral of f of order s at x_0 is called the **fractional Hölder exponent of** f **at** x_0 and denoted by*

$$h^s_f(x_0) = h_{f^{(-s)}}(x_0).$$

Lemma 1 *The definition of $h^s_f(x_0)$ does not depend on the function ϕ which is chosen.*

Proof: Denote by $f_1^{(-s)}$ and $f_2^{(-s)}$ the local fractional integrals of f corresponding to two different functions ϕ_1 and ϕ_2; $f\phi_1 - f\phi_2$ vanishes in a neighborhood of x_0, and therefore is C^∞ in a neighborhood of x_0. The local regularity properties of the operator $(Id - \Delta)^{-s/2}$ imply that $(Id - \Delta)^{-s/2}(f\phi_1 - f\phi_2)$ also is C^∞ in a neighborhood of x_0; therefore $f_1^{(-s)}$ and $f_2^{(-s)}$ differ by a C^∞ function, and thus share the same Hölder exponent.

Properties of the fractional Hölder exponent have been investigated in [6, 33]. In particular one can show that it is a concave function.

Illustrations pedagogiques? par exemple: le calcul de h^s_f pour un cusp, un chirp, un chirp + FBM; dans ce cas, on doit voir la cassure: d'abord la pente $1 + \beta$, puis la pente 1, pour s assez grand.

It follows from the concavity of the fractional Hölder exponent that

$$\text{either Ê} \quad \forall s > 0, \;\; h^s_f(x_0) = +\infty \qquad \text{orÊ} \quad \forall s > 0, \;\; h^s_f(x_0) < +\infty;$$

an example of the first case is given by

$$|x - x_0|^\alpha \sin\left(\exp \frac{1}{|x - x_0|}\right).$$

When this first occurence happens, there is no ambiguity in order to define an oscillation exponent: All possible definitions agree on the value $+\infty$. Therefore, in the following, we suppose that the second case occurs.

The definition of the oscillation exponent that we will choose is motivated by the simple but important remark that follows.

Lemma 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, and denote by I^s_f an iterated primitive of order s of f. If s is an integer, then*

$$h^s_f(x_0) = h_{I^s_f}(x_0).$$

Proof: Since the result is clearly local, we can assume that f is supported in a neighbourhood of x_0, and therefore belongs to L^2, which allows to use the Fourier transform without any restriction, and also allows to assume that $\phi = 1$ in the definition of the local fractional integral. Up to a polynomial term, one derives I^s_f from f by multiplying $\hat{f}$ by $(i\xi)^{-s}$; this iterated primitive has the same Hölder exponent as the one obtained using instead $|\xi|^{-s}$ since the corresponding operators are either the same (up to a multiplicative constant), or deduce from each other by an Hilbert transform (the Fourier multiplier by $sign\ (\xi)$); and applying the Hilbert transform does not modify the pointwise Hölder exponent since $f^{(-s)}$ is uniform Hölder , see [21] (the notion of uniform Hölder function is recalled in Definition 13). The result follows by noticing that

$$\frac{1}{(1 + |\xi|^2)^{s/2}} - \frac{1}{|\xi|^s} \sim \frac{C}{|\xi|^{s+2}} \qquad \text{when} \qquad |\xi| \to +\infty,$$

and therefore the corresponding operator is a uniformly smoothing operator which, for any $\alpha \in \mathbb{R}$, maps $C^\alpha(\mathbb{R})$ to $C^{\alpha+s+2}(\mathbb{R})$. Therefore,

$$h_{f^{(-s)}-I_f^s}(x_0) \geq 2 + \sup(h_{f^{(-s)}}(x_0), h_{I_f^s}(x_0)),$$

and, using (23), we get $h_{f^{(-s)}}(x_0) = h_{I_f^s}(x_0)$.

It follows from this lemma that results in dimension 1 which are checked by hand through the computation of primitives can then be extended in any dimension and to the non-integer case by using fractional integrals; for instance, let us check that

$$\forall s \geq 0, \qquad h^s_{F_{\alpha,\beta}}(x_0) = \alpha + (1+\beta)s. \tag{25}$$

Indeed, a straightforward integration by parts shows that this result holds for iterated primitives, and thus, using Lemma 2, it also holds for fractional integrals of integer order; the result then follows immediately by the concavity of the fractional Hölder exponent. Therefore, a natural definition for the oscillation exponent of an arbitrary function at x_0 is to use the slope of the fractional Hölder exponent, i.e. of the function

$$s \to h^s_f(x_0). \tag{26}$$

In the case of $F_{\alpha,\beta}$, this function is linear, and the definition is unambiguous; however, it is not always the case, and one can show (see [6, 33]) that, in general, functions defined by (26) only satisfy the following properties, which are characteristic of the functions $h(s)$ which are fractional Hölder exponent, i.e. for which there exists a function f satisfying $h^s_f(x_0) = h(s)$.

Proposition 2 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function, and let $x_0 \in \mathbb{R}^d$. The function $s \to h^s_f(x_0)$ is concave; therefore it has everywhere a left and a right derivative. These derivatives satisfy*

$$\forall s \geq 0, \qquad \frac{\partial(h^s_f(x_0))}{\partial s} \geq 1.$$

Furthermore, these properties characterize fractional Hölder exponents.

It follows from this characterization that $h^s_f(x_0)$ is, in general, not a linear function of s, and therefore many choices are possible for its slope. In practice, only two choices have been used up to now, leading to two different exponents:

- The chirp exponent (choice of the slope "at infinity")

$$\gamma_f(x_0) = \lim_{s\to+\infty} \frac{\partial}{\partial s}(h^s_f(x_0)) - 1, \tag{27}$$

- the oscillation exponent (choice of the slope "at the origin")

$$\beta_f(x_0) = \lim_{s\to 0} \frac{\partial}{\partial s}(h^s_f(x_0)) - 1, \tag{28}$$

(see [31] for properties associated with chirp exponents, and [6] for properties associated with oscillation exponents). The drawback of using (27) is that this notion is very unstable: If g is an arbitrary smooth (but not C^∞) function, one can have

$$\gamma_f(x_0) = \Gamma > 0 \quad \text{but} \quad \gamma_{f+g}(x_0) = 0.$$

A simple example of this phenomenon is given by the functions: $f(x) = x\sin(1/x)$ and $g = |x|^a$ for an $a \notin 2\mathbb{N}$ and arbitrarily large. Therefore, the chirp exponent does not satisfy the requirement stated at the beginning of section 3.2. These instabilities cannot occur when using the oscillation exponent as shown by the following result, which states that, as soon as one imposes this stabilitry requirement, then the choice of $\beta_f(x_0)$ is canonical.

Proposition 3 *Let f be a locally bounded function satisfying: $\forall s > 0$, $h^s_f(x_0) < +\infty$. Then $\beta_f(x_0)$ is the only quantity which satisfies the following properties:*

- *It is deduced from the function $s \to h^s_f(x_0)$.*
- *If $h_g(x_0) > h_f(x_0)$, then $\beta_{f+g}(x_0) = \beta_f(x_0)$ (the oscillation exponent of f is not altered under the addition of a function which is smoother than f).*
- *It yields the exponent β for the functions $F_{\alpha,\beta}$.*

This propositions justifies the fact that, from now on, we will use the exponent $\beta_f(x_0)$ in order to measure oscillations.

Proof: Let us first check that the oscillation exponent satisfies these properties. We only have to check the second one; indeed, the first one follows from the definition, and the last one has already been proved. Because of the finiteness assumption for h^s_f, and its concavity, it is continuous; therefore, it remains strictly smaller than $h_g(x_0)$ in a small neighbourhood of x_0, and thus

$$\exists \eta > 0 : \quad \forall s < \eta, \qquad h^s_f(x_0) < h^s_g(x_0),$$

so that $h^s_{f+g}(x_0) = h^s_f(x_0)$.

Let us now prove the converse result. The stability requirement implies that the quantity considered cannot be function of $h^s_f(x_0)$ for an $s > 0$; therefore it is a "germ property" at $s = 0$, and therefore, a function of the value at 0 of the function $s \to h^s_f(x_0)$ and its derivatives. Since $h^s_f(x_0)$ can be an arbitrary concave function, higher order derivatives do not exist in general; therefore, only $h_f(x_0)$ and the first derivative can be involved. Finally, the fact that the exponent takes the value β for the functions $F_{\alpha,\beta}$ implies that it is given precisely by (28).

Remarks: In practice, one cannot directly measure the oscillation exponent since it involves the estimation of how the Hölder exponent evolves under a fractional integration of "infinitesimal" order, and one rather estimates the evolution under a fractional integration of given fixed order s, thus obtaining the *s-oscillation exponent*:

$$\beta_f(s, x_0) = \frac{h^s_f(x_0) - h_f(x_0)}{s} - 1. \tag{29}$$

Note that, because of the concavity of $h^s_f(x_0)$,

$$\forall s > 0, \qquad \gamma_f(x_0) \le \beta_f(s, x_0) \le \beta_f(x_0)$$

and

$$\beta_f(s, x_0) = 0 \iff \beta_f(x_0) = 0. \tag{30}$$

Definition 10 *A locally bounded function f has a cusp singularity at x_0 if $\beta_f(x_0) = 0$.*

Note that (30) is of practical importance for the following reason: If one is interested in a qualitative information such as the existence or absence of oscillating singularities (and not the precise values taken by β_f), then, (30) shows that, in practice, it is equivalent to work with the s-oscillation exponent and the oscillation exponent in order to obtain the required information.

Definition 11 *A locally bounded function f has a cusp singularity at x_0 if $\beta_f(x_0) = 0$.*

We defined the notion of oscillation exponent through a construction derived from the Hölder exponent (by comparing how it evolves through fractional integration); however, one can notice that the only specific property of the fractional Hölder exponent that we used is its concavity, which holds for regularity exponents, in a very general setting, as shown by the following proposition.

Proposition 4 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function or a distribution, and let $x_0 \in \mathbb{R}^d$. Let H^α be a scale of regularity exponents satisfying the hypotheses of Definition* **??**. *The function $s \to H^\alpha_{f^{-s}}(x_0)$ is concave; therefore it is differentiable, except possibly at a countable number of points; it has everywhere a left and a right derivative; these derivatives satisfy*

$$\forall s \ge 0, \frac{\partial (H^\alpha_{f^{-s}}(x_0))}{\partial s} \ge 1.$$

Proof: Assez facile si on se place dans le cadre des "gentle spaces" de [27]. peut-etre vrai dans le cadre le plus general? dans un premier temps je mettrai ici la demo "gentle spaces"

It follows from Proposition 4 that one can define alternative oscillation exponents in the same way, but using the p-exponent instead of the Hölder exponent; more generally, one can use the abstract setting of pointwise smoothness based on *gentle spaces* which is studied in [27]. In any of these setting, we see that the oscillation exponent is not a "primary notion"; it has to be based on a prior notion of pointwise regularity exponent, and is obtained by inspecting how this regularity exponent behaves under fractional integration: The oscillation exponent is *subordinated* to the Hölder exponent. This will explain why, in practice, it is difficult to obtain a direct access to the oscillation exponent; and it will have important consequences for the corresponding multifractal formalisms.

Let us now recall the characterization of oscillating singualties, that was discoverd by J.-M. Aubry, cf [8].

Definition 12 *A function $g \in L^\infty(\mathbb{R}^d)$ is called indeÞnitely oscillating if and only if $\exists \omega \in \mathcal{S}(\mathbb{R}^d)$ such that such $\forall N \in \mathbb{N}$, it can be written*

$$g = \omega + \sum_{|\alpha|=N+1} \partial^\alpha g_\alpha$$

with each $g_\alpha \in L^\infty(\mathbb{R}^d)$.

This is a natural generalization (in particular to higher dimen- sions) of the oscillating behavior of the sine function. Then, if $h < h_f(x_0)$ and f can be written, is the neughbourhood of x_0 as

$$f(x) = |x - x_0|^h g\left(\frac{x - x_0}{|x - x_0|^{\beta+1}}\right) + r(x),$$

where r is smoother that the first term at x_0 (see [8]) for a more precise statement).

3.3 Wavelets and pointwise regularity

We will now recall or derive the wavelet characterizations of the quantities we introduced. This is a mandatory requirement for constructing multifractal formalisms adapted to the determination of oscillation exponents. Indeed, it has already been shown that, even for the Hölder exponent, wavelet techniques are necessary in order to recover the whole spectrum of singularities, see [1, 5, 24]. Therefore it will be all the more true for quantities such as oscillation exponents, which are based on Hölder exponents.

The functional setting we pick in this section is the following: We assume that the functions we consider are locally bounded, and satisfy the following assumption.

Definition 13 *A function $f : \mathbb{R}^d \to \mathbb{R}$ is a uniform Hölder function if f is locally bounded and satisfies*

$$\exists a,\ C,\ \epsilon > 0 \quad \textit{such that} \qquad |f(x) - f(y)| \leq C|x - y|^\epsilon (1 + |x| + |y|)^a.$$

This setting allows to deal with stochastic processes used in modeling, such as Brownian motion, or Fractional Brownian motions for instance, but typically, not with processes with discontinuities, such as Lévy processes. This setting is required for the wavelet characterization of the Hölder exponent, see ??? We will see below how to check this condition in practice, and we will see in Section ??? how to deal with more general settings which are pertinent for applications, such a measures, or even distributions.

We will deduce Hölder and oscillation exponents from discrete quantities, which will either be wavelet coefficients, or quantities based on wavelet coefficients: The *wavelet leaders*. In order to be fully efficient, these notions have to be defined in the rather general setting supplied by *biorthogonal wavelet bases*, which are a useful extension of orthonormal wavelet bases. We first recall the notion of *biorthogonality*.

A *Riesz basis* of L^2 is a collection of vectors (e_n) such that the finite linear expansions $\sum_{n=1}^N a_n e_n$ are dense in L^2 and

$$\exists C, C' > 0 : \quad \forall N, \quad \forall a_n, \quad C \sum_{n=1}^N |a_n|^2 \leq \left\| \sum_{n=1}^N a_n e_n \right\|_H^2 \leq C' \sum_{n=1}^N |a_n|^2.$$

Two collections of functions (e_n) and (f_n) form *biorthogonal bases* if each collection is a Riesz basis, and if $\langle e_n | f_m \rangle = \delta_{n,m}$. When such is the case, any element $f \in L^2$ can be written

$$f = \sum_{n=1}^{\infty} \langle f | f_n \rangle e_n. \tag{31}$$

Biorthogonal wavelet bases are couples of Riesz bases of L^2 which are of the following form: On one side,

$$\varphi(x-k), \ \ (k \in \mathbb{Z}^d) \quad \text{and} \quad 2^{dj/2} \psi^{(i)}(2^j x - k), \ \ (k \in \mathbb{Z}^d, \ j \in \mathbb{Z}),$$

and, on the other side,

$$\tilde{\varphi}(x-k) \ \ (k \in \mathbb{Z}^d) \quad \text{and} \quad 2^{dj/2} \tilde{\psi}^{(i)}(2^j x - k), \ \ (k \in \mathbb{Z}^d, \ \ j \in \mathbb{Z}).$$

Therefore,

$$\forall f \in L^2, \qquad f(x) = \sum_{k \in \mathbb{Z}^d} C_k \varphi(x-k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_i c^i_{j,k} \psi^{(i)}(2^j x - k); \tag{32}$$

where

$$c^i_{j,k} = 2^{dj} \int_{\mathbb{R}^d} f(x) \tilde{\psi}^{(i)}(2^j x - k) dx, \quad \text{and} \quad C_k = \int_{\mathbb{R}^d} f(x) \tilde{\varphi}(x-k) dx. \tag{33}$$

Remarks:

We will see that biorthogonal wavelet bases are particularly well adapted to the decomposition of the Fractional Brownian Motion; indeed, well chosen biorthogonal wavelet bases allow to decorrelate the wavelet coefficients of these processes (the wavelet coefficients become independent random variables), and therefore greatly simplifies their analysis. They are also well adapted to the analysis of some operators, such as fractional derivations or integrations, since, up to a normalization factor, these operators map an orthonormal wavelet basis on one element of a couple of Riesz bases. Since these operators play a key role in the definition of oscillation exponents, this explains why the boiorthognal setting will be mandatory in the following.

In (32), the L^2 normalisation for the wavelets is not used, but rather an L^∞ normalisation which is better fitted to the study of Hölder regularity. The L^1 normalisation of (33) follows accordingly.

Note that (33) makes sense even if f does not belong to L^2; indeed, if one uses smooth enough wavelets, these formulas can be interpreted as a duality product betweeen smooth functions (the wavelets) and distributions, and convergence usually takes place in the corresponding spaces (see [39] for precise results). In the setting we picked, one can easlily show that, though f does not necessarily belong to L^2, (32) still makes sense, and the convergence holds locally in L^2.

We will use more compact notations for indexing wavelets: Instead of using the three indices (i, j, k), we will use an indexing by *dyadic cubes* which is obtained as follows. Recall that, if $j \in \mathbb{Z}$, a dyadic cube of scale j is of the form

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right), \tag{34}$$

where $k = (k_1, \dots k_d) \in \mathbb{Z}^d$.

Each point $x_0 \in \mathbb{R}^d$ is contained in a unique dyadic cube of scale j, denoted by $\lambda_j(x_0)$. We denote by $C \cdot \lambda_j(x_0)$ the cube of same center and orientation as $\lambda_j(x_0)$ and C times wider; i.e., if $\lambda_j(x_0)$ is given by (34), then

$$3\lambda_j(x_0) = \left[\frac{k_1-1}{2^j}, \frac{k_1+2}{2^j}\right) \times \cdots \times \left[\frac{k_d-1}{2^j}, \frac{k_d+2}{2^j}\right).$$

Since i takes $2^d - 1$ values, we can assume that it takes values in $\{0,1\}^d - (0, \dots, 0)$, which allows to use the following compact notations:

- $\lambda \;\; (= \lambda(i,j,k)) \;\; = \dfrac{k}{2^j} + \dfrac{i}{2^{j+1}} + \left[0, \dfrac{1}{2^{j+1}}\right)^d.$
- $c_\lambda = c^i_{j,k}$
- $\psi_\lambda(x) = \psi^{(i)}(2^j x - k).$

In order to have a common notation for wavelets and functions φ, when $j = 0$, we note ψ_λ the function $\varphi(x - k)$ (where λ is, in this case, the unit cube shifted by k)).

Finally, Λ_j will denote the set of dyadic cubes λ which index a wavelet of scale j, i.e. wavelets of the form $\psi_\lambda(x) = \psi^{(i)}(2^j x - k)$ (note that Λ_j is a subset of the dyadic cubes of side 2^{j+1}), and Λ will denote the union of the Λ_j for $j \geq 0$.

Remark: Indexing by dyadic cubes will prove useful for the following reason: The wavelet ψ_λ is essentially localized near the cube λ; more precisely, when the wavelets are compactly supported

$$\exists C > 0 \quad \text{such that} \quad \forall i, j, k, \qquad supp\,(\psi_\lambda) \subset C \cdot \lambda.$$

This property will play an important role in the definition of the wavelet leaders below where this indexing by dyadic cubes is necessary for a simple and natural definition.

Pointwise Hölder regularity is characterized in terms of the following quantities.

Definition 14 *Let f be a uniform Hölder function. The wavelet leaders of f are*

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|. \tag{35}$$

We note $d_j(x_0) = d_{\lambda_j(x_0)}$.

The following result of [24] yields a characterization of pointwise regularity: Let f be a uniform Hölder function; then

$$\forall x_0 \in \mathbb{R}^d, \qquad h_f(x_0) = \liminf_{j \to +\infty} \left(\frac{\log\left(d_j(x_0)\right)}{\log(2^{-j})} \right). \tag{36}$$

In this proposition and in all that follows, it is implicitely understood that one uses a "smooth enough" wavelet basis. In the present case, it means that the wavelets are r-smooth for an $r > h$.

For a real-life signal f, performing the multifractal analysis of f wil be based on the computation of wavelet leaders; therefore this procedure, however suffers an important restriction: It requires the function analyzed to be locally bounded. Note that this restriction is not attached to the particular use of of wavelet leaders: Indeed, the definition of Hölder regularity already requires the considered function to be locally bounded. Therefore a natural preliminary question is to wonder if signals or images which are commonly considered can indeed be modelled by locally bounded functions. This question has to be reformulated for the following reason: In practice real life signals are stored at a finite resolution, and therefore always appear as discrete, hence bounded objects. The proper way to deal with this apparent contradiction is to attach to the signal multiresolution quantities, and see how they behave through the scales which are available on the data. The procedure proposed in [2, 45, 46] is based on the computation of the following exponent.

Definition 15 *Let f be a tempered distribution defined on $\mathbb{R}^d$. The uniform Hölder exponent of f is*

$$H_f^{min} = \sup\{s : \ f \in C^s_{loc}(\mathbb{R}^d)\}. \tag{37}$$

Note that the definition of this exponent does not require any a priori assumption on f, and it can be derived directly from the wavelet coefficients of f; indeed, it follows from (37) and the wavelet characterization of the Hölder spaces that, if

$$\omega_j^R = \sup_{\lambda \in \Lambda_j \cap B(0,R)} |c_\lambda|,$$

then

$$H_f^{min} = \lim_{R \to \infty} \ \liminf_{j \to +\infty} \ \frac{\log\left(\omega_j^R\right)}{\log(2^{-j})}. \tag{38}$$

This formula shows that, in practice, H_f^{min} ca be computed through a simple regression on a log-log plot. The determination of H_f^{min} allows to settle if f is bounded or not. Indeed, it follows from (37) that

- if $H_f^{min} > 0$, then f is a uniform Hölder function,

- if $H_f^{min} < 0$, then $f \notin L^\infty_{loc}$.

Let us consider now two simple models in order to show possible numerical problems that can appear in the practical estimation H_f^{min}.

The first model is supplied by Lacunary Wavelet Series (LWS) $X_{\alpha,\gamma}$ of type (α, γ); they are defined on $[0, 1]^d$ (for $\alpha > 0$ and $\gamma < d$) as follows: A biorthogonal wavelet basis in the Schwartz class is used for the construction. One draws at random (uniformly) $2^{\gamma j}$ locations λ among the 2^{dj} dyadic cubes of width 2^{-j} included in $[0, 1]^d$, and the corresponding wavelet coefficients are set to the value $2^{-\alpha j}$ the others are set to the value 0. In order to define the LWS on $\mathbb{R}^d$, one repeats this construction on all cubes of width 1. In this case it is straightforward that $H_f^{min} = \alpha$ and, at each scale $\omega_j^R = 2^{-\alpha j}$ so that (38) yields, at each scale, an unbiased estimator of H_f^{min}.

The second example falls in the general model of *Random Wavelet Series*, that we will consider in details in Section ???. Let us describe the particular case that we consider now.

Definition 16 *Let ψ_λ be a biorthogonal wavelet basis in the Schwartz class. A Uniform Random Wavelet Series (URWS) of type β is a random field of the form*

$$X = \sum C_{j,k} 2^{-\alpha j} X_{j,k} \psi_{j,k},$$

where the $X_{j,k}$ are IID with common law X, which is a non-vanishing random variable which satisfies the tail estimate

$$\mathbb{P}(|X| \geq A) \sim C \exp(-B|A|^\beta),$$

and C, B and β are positive constants.

Note that this model includes the FBM (up to a C^∞ additive term). It follows from general results on RWS (see []) that this model yields a random field, with a constant Hölder exponent which is equal to α. Since the $X_{j,k}$ are independent, one obtains that

$$\mathbb{P}\left(\sup_k |X_{j,k}| \leq A\right) \sim 1 - C \exp(j \log 2 - BA^\alpha);$$

therefore $sup_k |X_{j,k}|$ is asymptotically equivalent to $Cj^{1/\alpha}$. It follows that

$$\omega_j = \alpha - \frac{\log j}{\alpha j \log 2}(1 + o(1));$$

therefore the estimator of H_f^{min} supplied by $\frac{\log(\omega_j)}{\log(2^{-j})}$ is biased by a term equivalent to $(\log j)/j$.

.

SJ: On pourrait faire le cas faiblement correle, qui donnerait aussi un estimateur biaise, et fournirait le resulatt pour un FBM dans une base d'onedelttes non necessairement adapte: poiur ca, ou sous-echnatillonne les guassiennes pour avoir de la "presuq'-independace " asymtotique, et on se ramene au cas precedent. Mais ca complque pas mal, pour une resulatt qui ne reste de toute facon qu'un exemple

A large proportion of signals and images have been found to have a negative exponent H_{min}, see []; hence they do not satisfy the regularity properties that are required in order to define the Hölder exponent. At this point, several options are possible. The first one, which we will investigate in Section ??? consists in performing a fractional intgration of large enough order on the data. Another one consists in using the p-exponent instead of the Hölder exponent; indeed, the p-exponent allows to deal with singularities of order as low as $-d/p$, see [26, 27, 30]. Note however that, in this case, one still has to check that f belongs to L^p_{loc}. Let us show how this can be performed numerically, since this verification is a consequence of the computation of the *wavelet scaling function*, which is an important quantity in multifractal analysis.

Definition 17 *Let f be a tempered distribution. Let*

$$S_f^R(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j \cap B(0,R)} |d_\lambda|^p.$$

The wavelet scaling function of f is

$$\forall p > 0, \qquad \zeta_f(p) = \lim_{R \to +\infty} \liminf_{j \to +\infty} \frac{\log(S_f^R(p,j))}{\log(2^{-j})}.$$

It follows from the wavelet characterization of Besov spaces that

$$\forall p > 0, \qquad \zeta_f(p) = \sup\{s' : f \in B^{s'/p,\infty}_{p,loc}\},$$

which shows that the wavelet scaling function is independent of the (smooth enough) wavelet basis which is chosen.

The embeddings between Besov and L^p spaces that if $f \in B^{s,\infty}_p$ for an $s > 0$, then $f \in L^p$ and conversely, if $f \in L^p$, then $\forall s < 0$, $f \in B^{s,\infty}_p$. The following simple practical criterium can therfore be used: A multifractal analysis based on the p-exponent can be performed if $\zeta_f(p) > 0$ and it cannot if $\zeta_f(p) < 0$.

.

SJ: Je me demande si on ne devrait pas donner cet exposant, (par exemple pour $p = 2$?) pour quelques signaux ou images, histoire de donner une premiere indication de faisabilité de la methode? (on les a deja, il n'y a pas de nouveaux calculs a faire).
Ê

Je pense qu'il y a aussi un biais dans l'estimation de la fonction d'echelle ondelettes. c'est facile a estimer dans la cas $p = 2$ et le "toy model" de ??? car, la quantite a estimer est une somme de carres de gaussiennes, donc un chi2 a 2^j degres de liberte, sur lequel on a des estimations explicites

Remarks: Pointwise regularity exponents usually have a wavelet characterization in terms of liminfs of quantities defined on the dyadic cubes. it is the case of the p-exponent, see [26] and of the more general exponents considered in [27]. It is important to note that a characterization given through a liminf is a key property in order to construct a multifractal formalism and show the corresponding upper bounds for spectra, as shown in [1] where such formalisms are constructed in the most general setting of exponents which are precisely defined by such liminfs. Note that alternative quantities, based on the WTMM (Wavelet Transform Maxima Method) have been advocated; however, though these quantities have a form similar to (35) (they are based on local suprema of the continuous wavelet transform), no result similar as (36) holds for the WTMM, see figure ???, and, therefore, the uper bounds stated in Theorem ?? are not expected to hold for the WTMM.

.

S.J.: on pourrait mettre les figures ou on montre les regressions log log qui donnent l'exposant de Hölder pour les cusps et chirps, comparant la MMTO et les leaders? il me semble qu'on les avait faits pour une conf, mais pas dans un article?

One difficulty that we will meet in the following is precisely that the oscillation exponent, because it is a subordinated exponent, cannot be characterized in such a way, i.e. as a liminf of quantities defined on the dyadic cubes.

3.4 Wavelets and fractional integration

In order to estimate oscillation exponents, one has to compare the pointwise Höder regularity of f and of fractional integrals of f. Therefore, we are led to the problem of computing fractional integrals of different orders. Another motivation for computing fractional integrals of signals is that very often, signals or images do not satisfy the uniform regularity assumption of Definition 13, as shown in [] where the uniform exponent of several types of signals and images has been estimated using the wavelet method, and shown to be often negative.

In specific applications, the problem of a negative uniform Hölder exponent has sometimes been solved by following an ad-hoc procedure which amounts to performing a fractional integration. It is the case for instance for the multifractal analysis of turbulence data using the WTMM (Wavelet transform Maxima Method, see []) where the normalization used in the continuous wavelet transform amounts to perform a fractional integration of order 1/2 (???). However, it is important to understand fully the mathematical implications of such a procedure for the following reasons:

- It allows to understand the hypotheses which are implicitely made of the signal, and therefore it opens the way to their correct validation (for instance by computing the uniform Hölder exponent).
- It opens the way to more flexibility by allowing to "tune" the amount of fractional integration, depending on the particular nature of the signal
- It allows a better understanding of he quantities effectively computed, which are related with the Hölder regularity of the function, but rather of a fractional integral, which allows to investigate how these quantities are related.
- The uniform Hölder exponent is, by itself, an important parameter which proved useful inclassification.

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SJ: Dans quels articles a-t-on trouve des H_{min} negatifs? J'en trouve juste un dans [2]; si on n'en a pas publie plus, il faudrait peut-etre mettre ici qqs exemples?

If H_f^{min} is negative, the characterization supplied by (36) does not hold and, since this characterization is the building block of the multifractal formalism, a multifractal analysis cannot be directly performed. A way to turn this problem is to "preprocess" such signals by first performing a fractional integration, which increases their uniform regularity, and, if it is of large enough order, transforms the signal or the image into a uniform Hölder function, see [2].

However, numerically, a fractional integration is difficult to realize; in practice, it is equivalent to perform a *pseudo-fractional integration* defined as follows.

Definition 18 *Let $s > 0$, let ψ_λ be an r-smooth wavelet basis with $r > s+1$ and let f be a function, or a distribution, with wavelet coefficients c_λ. The pseudo-fractional integral (in the basis ψ_λ) of f of order s, denoted by $I^s(f)$, is the function whose wavelet coefficients (on the same wavelet basis) are*

$$c_\lambda^s = 2^{-sj} c_\lambda.$$

This operation is numerically straightforward, and the following result shows that it retains the same properties as the fractional integral.

Proposition 5 *Let f be a uniform Hölder function. Then,*

$$\forall s > 0, \;\; \forall x_0, \qquad h_{I^s(f)}(x_0) = h_{f^{(-s)}}(x_0).$$

The proof of this result requires to introduce several tools; the first one is the algebras $\mathcal{M}^\gamma$, which are defined as follows.

Definition 19 *An infinite matrix $A(\lambda, \lambda')$ indexed by the dyadic cubes of $\mathbb{R}^d$ belongs to $\mathcal{M}^\gamma$ if*

$$|A(\lambda, \lambda')| \leq \frac{C \; 2^{-(\frac{d}{2}+\gamma)(j-j')}}{(1+(j-j')^2)(1+2^{\inf(j,j')} dist(\lambda, \lambda'))^{d+\gamma}}.$$

Matrices of operators which map an r-smooth wavelet basis onto another one belong to these algebras, as soon as $\gamma > r$, and more generally matrices (on wavelet bases) of pseudodifferential operators of order 0, such as the Hilbert transform in dimension 1, or the Riesz transforms in higher dimensions, belong to these algebras, see [39]. We denote by $\mathcal{O}p(\mathcal{M}^\gamma)$ the space of operators whose matrix on a r-smooth wavelet basis (for $r > \gamma$) belongs to $\mathcal{M}^\gamma$. Note that this space does not depend on the (smooth enough) wavelet basis which is chosen.

The second tool that we will need is the notion of *vaguelette* system.

Definition 20 *A set of functions (θ_λ) indexed by the dyadic cubes of scale $j \geq 0$, forms a vaguelette system of order s if*

- *for any $j \geq 1$ the vaguelettes θ_λ of scale j have vanishing moment up to order $s+1$, i.e. if, for any multiindex α satisfying $|\alpha| \leq s+1$, then*

$$\int \theta_\lambda(x) x^\alpha dx = 0,$$

- *the θ_λ statisfy the following uniform decay estimates: For any multiindex α satisfying $|\alpha| \leq s+1$, then*

$$\forall N \in \mathbb{N}, \qquad \left| \frac{\partial^\alpha \theta_\lambda}{\partial x^\alpha} \right| \leq \frac{C_N 2^{(\alpha+d/2)j}}{(1+|2^j x - k|)^N}.$$

Biorthogonal vaguelette bases are couples of Riesz bases ψ^1_λ and ψ^2_λ which are both vaguelette systems and form biorthogonal bases. Therefore, $\forall f \in L^2$,

$$f(x) = \sum_\lambda c^1_\lambda \psi^2_\lambda, \tag{39}$$

where

$$c^1_\lambda = \int_{\mathbb{R}^d} f(x) \psi^1_\lambda dx. \tag{40}$$

Note that, as in the wavelet case, if f is uniform Hölder, then convergence also takes place locally uniformly.
Ê

The notions we introduced are related by the following key property, proved in [39]:

Proposition 6 *Let $\mathcal{M}$ be an operator which maps an r-smooth wavelet basis to a vaguelette system of order r; then, for any for $\gamma < r$, $\mathcal{M}$ belongs to $\mathcal{O}p(\mathcal{M}^\gamma)$.*

SJ: Attention l'utilisation d'une réciproque exigerait une legere modification de la notion de vaguelette (moments asymptotiquement nuls)

Proof of Proposition 5 : The proof is performed using the wavelet techniques developed in [39], such as the function spaces characterizations; therefore, we won't give a complete detailed proof, but only mention the main lines.

The first point consists in noticing that the systems

$$\psi^1_\lambda = 2^{sj}(Id - \Delta)^{-s/2}\psi_\lambda \quad \text{and} \quad \psi^1_\lambda = 2^{-sj}(Id - \Delta)^{s/2}\psi_\lambda$$

are biorthogonal vaguelette systems. This property is straightforward to check on the Fourier transform ψ^1_λ and ψ^2_λ, which, in this case, are completely explicit.

Note that

$$I^s(f) = \sum 2^{-sj} c_\lambda \psi_\lambda \quad \text{and} \quad f^{(-s)} = \sum 2^{-sj} c_\lambda \psi^1_\lambda.$$

If f is locally bounded, then $I^s(f)$ belongs locally to C^s (because of the wavelet characterization of C^s);

Assume now that $I^s(f)$ belongs to $C^\alpha(x_0)$; the operator that maps ψ_λ to ψ^1_λ belongs to $\mathcal{O}p(\mathcal{M}^\gamma)$, and therefore preserves the pointwise wavelet regularity criterium, as proved in [21]. Therefore, it is satisfied by $f^{(-s)}$ and, since $f^{(-s)}$ is uniform Hölder, the converse part of the wavelet pointwise regularity criterium implies that it belongs to $C^\beta(x_0)$ for any $\beta < \alpha$, see [21, 24]. The proof of the converse part is similar, using the biorthogonality of the ψ^1_λ and ψ^2_λ.

As regards signals and images which have a negative exponent H_{min}, another possibility for applying them multifractal analysis based on the Hölder exponent is to perform on the data a fractional integration of sufficently large order; indeed, the uniform regularity exponent H^{min}_f is always shifted exactly by s:

$$\forall f, \qquad H^{min}_{I^s(f)} = H^{min}_f + s.$$

This simple property shows a possible strategy in order to perform the multifractal analysis of a signal which is not bounded: First determine its exponent H^{min}_f; then, if $H^{min}_f < 0$, perform a fractional integration of order $s > -H^{min}_f$; it follows that the uniform regularity exponent of $I^s(f)$ is positive, and therefore it is a bounded function. However, this strategy rises an important problem: There is no canonical choice for the order of fractional integration, and the quantities considered (either the spectrum of singularities, or the scaling function) may depend on this order. This is one of the motivations for understanding how multifractal properties are modified under fractional integration.

3.5 Wavelet characterization of the oscillation exponent

The definition of the oscillation exponent requires to compare the Hölder exponent of f and of its fractional integrals. Therefore, it is is clear that information on the value of $\beta_f(x_0)$ will be deduced from wavelet characterization of the Hölder exponent. This can be performed directly on wavelet coefficients, or using wavelet leaders; hence two possible characterizations. The first characterization given by Proposition 7 is just a rewriting, in the orthonormal wavelet setting, of Corollary 1 of of [6] (deducing the orthonormal case from the continuous case is straightforward). In the following, the point x_0 is fixed, and we suppose that f is a uniform Hölder function. We introduce some notations. The first one is a weak form of the $\mathcal{O}$ notation of Landau: If F and G are two functions which tend to 0, $F = \overline{\mathcal{O}}(G)$ if

$$\liminf \frac{\log |F|}{\log |G|} \geq 1,$$

and the second one expresses the fact that two functions are of the same order of magnitude, disregarding "logarithmic corrections"

$$F \sim G \quad \text{if} \quad \lim \frac{\log |F|}{\log |G|} = 1.$$

Proposition 7 *Let f be a uniform Hölder function. The oscillating singularity exponents of f at x_0 are (h, β) if and only if its wavelet coefficients satisfy the following conditions:*

a) $c_\lambda = \overline{\mathcal{O}}\left(2^{-hj} + |\lambda - x_0|^h\right)$ *when* $j \to +\infty$ *and* $\lambda \to x_0$ *(here* $|\lambda - x_0|$ *denotes the distance between* λ *and* x_0*);*

b) there exists a sequence $\lambda_n \to x_0$ *of width* 2^{-j_n} *such that*

$$j_n \to +\infty, \;\; (2^{-j_n} + |\lambda_n - x_0|)^{1+\beta} \sim 2^{-j_n}; \quad \textit{and} \quad |c_{\lambda_n}| \sim 2^{-hj_n} + |\lambda_n - x_0|^h \quad (41)$$

c) β *is the smallest number such that (41) holds.*

We will call a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that (41) holds a *minimizing sequence* for f at x_0. These exponents can be given the following interpretation: Let us call a *minimizing sequence* for the Hölder exponent at x_0 a sequence λ_n for which the liminf is attained in

$$\frac{\log |C_{\lambda_n}|}{\log(2^{-j_n} + |\lambda_n - x_0|)}.$$

The exponent β is attained as a liminf of the sequence

$$\frac{\log(2^{-j_n})}{\log(2^{-j_n} + |\lambda_n - x_0|)},$$

where this liminf is taken on all subsequences which are minimizing for the Hölder exponent. So we see that the minimizing sequences for β are obtained as subsequences of the minimizing sequences for h. In this sense, we say that β is *subordinated* to h.

Remarks:

It is enlightening to split this condition into two subcases, depending whether $\beta = 0$ or $\beta > 0$:

- If $\beta = 0$, then Conditions b) and c) boil down to:

$$|\lambda_n - x_0| \sim 2^{-j_n} \quad \text{and} \quad |c_{\lambda_n}| \sim 2^{-hj_n};$$

  this condition makes precise the heuristic following which, for cusp singularities, the large wavelet coefficients stand in the "cone of influence" above x_0.

- If $\beta > 0$, then Condition b) can be rewritten:

$$|\lambda_n - x_0| \sim 2^{-j_n/(1+\beta)}; \quad \text{and} \quad |c_{\lambda_n}| \sim |\lambda_n - x_0|^h \sim 2^{-j_n h/(1+\beta)} \qquad (42)$$

In order to rewrite Proposition 7 as conditions bearing on wavelet leaders, we need to define the leaders associated with a pseudo-fractional integral.

Definition 21 *The s-leaders* d^s_λ *are the wavelet leaders associated with the pseudo-fractional integral of* f *of order* s

$$d^s_\lambda = \sup_{\lambda' \subset 3\lambda} \left(2^{-sj'} |c_{\lambda'}|\right). \qquad (43)$$

We will denote

$$d^s_j(x_0) = d^s_{\lambda_j(x_0)}.$$

It follows from (28) and Proposition 5 that

$$h_f^s(x_0) = \liminf_{j\to+\infty} \left(\frac{\log\left(d_j^s(x_0)\right)}{\log(2^{-j})} \right), \tag{44}$$

therefore, the oscillation exponent can be derived from the d^s_λ according to (28).

Proposition 8 *Let f be a uniform Hölder function. The oscillating singularity exponents of f at x_0 are (h, β) if and only if its wavelet leaders satisfy the following conditions:*

1. $d_j(x_0) = \overline{\mathcal{O}}\left(2^{-hj}\right)$,
2. *there exists a sequence $j_n \to \infty$ such that*

$$d_{j_n}(x_0) \sim 2^{-hj_n}, \tag{45}$$

3. *there exists a sequence $\lambda'_n \subset 3\lambda_{j_n}(x_0)$ such that:*

$$\begin{cases} j'_n = (1+\beta)j_n + o(j_n) \\ |c_{\lambda'_n}| \sim d_{j_n}(x_0), \end{cases}$$

4. *β is the smallest number such that 3) holds.*

Remarks: The last condition means that the supremum in the definition of the wavelet leader $d_{j_n}(x_0)$ is "almost" attained at a scale close to $(1+\beta)j$.

This proposition gives a practical interpretation of the notion of subordination we already introduced: The Hölder exponent is given by a liminf, i.e. a limit taken on particular susequences of cubes $\lambda_j(x_0)$; and the oscillation exponent is determined by inspecting at which scales the suprema are "almost" attained in the corresponding wavelet leaders.
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Proof of Proposition 8: The first two conditions are equivalent to the fact that h is the Hölder exponent, as proved in [24].

Let us check that 3) implies that $\beta_f(x_0) \leq \beta$. Indeed, after a pseudo-integration of order s, the sequence (λ_n, λ'_n) yields a wavelet coefficient $\tilde{c}_{\lambda'_n}$ of size

$$|\tilde{c}_{\lambda'_n}| = 2^{-sj'_n} |c_{\lambda'_n}| \sim 2^{-sj'_n} 2^{-hj_n}$$

which, using, (**??**) is larger than a quantity equivalent to $2^{-(h+(1+\beta)s)j_n}$. Therefore the wavelet leader $d^s_{j_n}(x_0)$ satisfies

$$2^{-(h+(1+\beta)s)j_n} = \overline{\mathcal{O}}\left(d^s_{j_n}(x_0)\right);$$

since this is true $\forall s > 0$, it follows that $\beta_f(x_0) \leq \beta$.

Let us now check that Conditions 1) to 4) imply that $\beta_f(x_0) \geq \beta$. Indeed, if 4) holds, $\forall \beta' < \beta$, $\exists h' > h$ such that, for j' large enough, and $\forall \lambda'$ of scale j', we have:

$$\forall j \quad \text{such that } \lambda' \subset \lambda_j(x_0) \quad \text{and} \quad j' \leq (1+\beta')j; \quad \text{then} \quad |c_{\lambda'}| \leq 2^{-h'j}.$$

We now split the wavelet coefficients $c_{\lambda'}$ into two collections: The first one is composed of those that satisfy this condition; but their contribution is $C^{h'}(x_0)$ and therefore, they have no influence on the oscillation exponent. As regards the others, they satisfy

$$\forall j \quad \text{such that} \;\; \lambda' \subset \lambda_j(x_0), \qquad j' \geq (1+\beta')j;$$

therefore, after a pseudo-fractional integration of order s,

$$|\tilde{c}_{\lambda'}| = 2^{-sj'}|c_{\lambda'}| = \overline{\mathcal{O}}\left(2^{-sj'}2^{-hj}\right) = \overline{\mathcal{O}}\left(2^{-(h+(1+\beta')s)j}\right)$$

so that $\beta_f(x_0) \geq \beta'$. Since this is true $\forall \beta' < \beta$, the result follows.

Let us now prove that a), b) and c) imply 3) and 4). We denote by λ_n the cube of the form $\lambda_n = \lambda_{j_n}(x_0)$ where

$$j_n = \left[-\log_2\left(2^{-j'_n} + |\lambda'_n - x_0|\right)\right],$$

where $[a]$ denotes the largest integer smaller than a. Then, clearly, $\lambda'_n \subset 3\lambda_n$

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SJ: a finir

SJ: llustrations possibles: le calcul par integration pseudo-fractionnaire et leaders du $h^s_f(x_0)$ d'un cusp, cusp + chirp, chirp, chirp + FBM plus régulier, éventuellement en 0 et en un point non dyadique.

3.6 Uniform cusp functions

One purpose of this paper is to supply new tools that will allow to determine if a function has oscillating singularities, i.e. if there exist points x_0 where $\beta_f(x_0) \neq 0$, and to obtain information on the size of these sets of points. It is therefore important to start with an analysis of the functions where no such points exist, i.e. for which $\beta_f(x_0) = 0$ everywhere.

We start by a characterization of a cusp at x_0; it directly follows from the leader characterization of oscillating singularities by taking $\beta = 0$ in Proposition 8.

Corollary 1 *Let f be a uniform Hölder function. f has a cusp at x_0 if and only if its wavelet coefficients satisfy the following condition*

$$\exists j_n \to +\infty, \; \exists \lambda' \subset \lambda_{j_n}(x_0) \text{ such that } \left\{ \begin{array}{l} j' = j_n + o(j_n) \\ |c_{\lambda'}| \geq 2^{-h_f(x_0)(j_n + o(j_n))} \end{array} \right. \tag{46}$$

It is natural to consider a "uniform" condition corresponding to (46).

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SJ: Expliquer pourquoi ca correspond a une version uniforme

Definition 22 *Let f be a uniform Hölder function; f is a uniform cusp function if there exist arbitrary smoooth biorthogonal wavelet bases such that the wavelet coefficients of f in these bases satisfy the following condition:*

There exist two functions $g(j)$ and $h(j)$ such that $g(j) = o(j)$, $h(j) = (j)$ and

$$\forall\lambda, \ \ \exists\lambda' \subset 3\lambda : \ \begin{cases} j' \leq j + g(j) \\ \\ |c_{\lambda'}| \geq d_\lambda 2^{-h(j)} \end{cases} \tag{47}$$

Remarks:

- It follows immediately from Proposition 1 that a uniform cusp function only displays cusps.
- An important feature of this definition is that, in order to be checked, is does not require the knowledge of $h_f(x_0)$. This is in sharp disctinction with Corollary 1, and, therefore, though it is a stronger requirement than Corollary 1, it will sometimes be easier to check, as shown by some of the examples that we will work out below.

We will see that the requirement of a uniform cusp function has many implications in multifractal analysis. A natural question is to wonder if this requirement is strong or if many models actually satisfy it. We start by mentioning a few examples, among standard models in turbulence or signal processing.

Our first example is the particular model of URWS already considered in Definition 16, and which includes the FBM as particular cases.

Proposition 9 *With probability 1, the sample paths of URWS are uniform cusp functions.*

Proof: We keep the same notations as in Definition 16. First, note that the tail estimates for the $X_{j,k}$ imply (by the Borel Cantelli lemma) that, for j large enough,

$$|X_{j,k}| \leq C_3 j^{1/\beta}.$$

It follows that the coefficients $2^{-\alpha j'}|X_{j',k'}|$ for $j' \geq j + (\log j)^2$ are bounded by

$$C_3(j + (\log j)^2)^{1/\beta} 2^{-\alpha(j+(\log j)^2)} := A_j.$$

Let us estimate the probability that, for a given dyadic cube λ, the supremum of the coefficients $2^{-\alpha j'}|X_{j',k'}|$ for $j' \leq j + (\log j)^2$, is bounded by A_j. This implies that for each couple (j', k') such that $\lambda' \subset \lambda$ and $j' \leq j$, the corresponding RV $X_{j',k'}$ satisfies

$$|X_{j',k'}| \leq C_3 2^{-\alpha(\log j)^2}(j)^{1/\beta}.$$

By the Borel Cantelli, lemma, this event does not occur for j large enough. But, if it is the case, il implies precisely that the supremum in the wavelet leader is attend for $j' \leq j + (\log j)^2$, so that the URWS is a uniform cusp function.

We now consider the setting supplied by *wavelet cascades*. We start by recalling their construction: the coefficients at the scale $j + 0$ are set to 1. One picks independent copies W_λ. of a RV W. The wavelet coefficients of the cascade, are defined for $j \geq 0$ by

$$C_\lambda = \epsilon_\lambda \prod_{\lambda \subset \lambda'} X_{\lambda'}$$

where the ϵ_λ are random signs. Convergence and uniform Hölder regularity of the RWC are proved in [] under the assumption that $X > 0$, $\log(X)$ has a negative expectation and under tail estimates ???

Proposition 10 *If X is lognormal, then the corresponding wavelet cascades are uniform cusp functions.*

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S.J. Le cas lognormal est direct car on a les lois explicites. Il est assez facile de montrer que c'est aussi vrai dans le cas sous-normal. C'est moins evident dans les cas "queues lourdes" Des estimations du type 'Berry-Essen" doivent permettre de traiter des cas faiblement correles. IL faudrait voir si les queues lourdes peuvent creer des singularites oscillantes. je suis un peu embete par l'article ABM qui ne donne pas le spectre de Holder, mais eulement de Legendre, et considere le FM avec des esperances et non trajectoire par trajectoire. Avant d'en faire plus, il faudrait savoir ce qui a ete fait de plus...

Another example of uniform cusp functions is supplied by a model of wavelet series associated with measures, introduced by Arneodo et al. in [] and by Barral and Seuret in [].

Definition 23 *Assume that μ is a probability measure defined on $\mathbb{R}^d$ and let ψ_λ be a wavelet basis in the Schwartz class. Let $\gamma \geq 0$ and $\delta > 0$ be two parameters. The wavelet series of parameters (γ, δ) associated with μ is defined by its wavelet coefficients*

$$c_\lambda = \mu(\lambda)^\gamma 2^{-\delta j}.$$

Note that, in this model, the measure μ can be either deterministic or random. Clearly, $|c_\lambda| \leq 2^{-\delta j}$ so that the function constructed is uniform Hölder. Moreover, if $\lambda' \subset \lambda$, then $\mu(\lambda') \leq \mu(\lambda)$ $c_{\lambda'} \leq c_\lambda$ so that we have indeed obtained a uniform cusp function.

Typical counterexamples of functions which are not uniform cusps are supplied by random wavelet series, as shown implicitely in [] where the scales at which the maxima in the wavelet leader definition are attained are actually estimated. We will come back to these processes in Section ???.

SJ Dans le cas RWS, separer deux cas: un ou les gros coefficients sont tres nombreux, et on est dans la cas cusp uniforme; c'est par exemple la cas du FBM; et le cas ou les gros coeff sont rares, et on a des singularites oscillantes.

4 Multifractal formalisms

The initial purpose of multifractal analysis is to relate information on the size of the sets with a given Hölder with global quantities, which are effectively computable on real data. These quantities, called *scaling functions*, should be effectively computable on experimental data;

Let $f(x)$ be a function: $\mathbb{R} \to \mathbb{R}$. The *structure function* of f is

$$S(p,l) = l \sum |f((n+1)l) - f(nl)|^p$$

If

$$S(p,l) \sim |l|^{\zeta_f(p)} \quad \text{when} \quad h \to 0, \tag{48}$$

then $\zeta_f(p)$ is the *scaling function* associated with f; (48) means that $\zeta_f(p)$ can indeed be computed by performing regressions on log-log plots, according to the formula

$$\zeta_f(p) = \liminf_{l \to 0} \frac{\log S(p,l)}{\log |l|}.$$

The first example for the use of such a quantity was provided by A. Kolomogorov, in 1941, in his seminal paper on turbulence [], who showed the relevance of $\zeta(2)$. Later, together with ????, he showed the relevance of $\zeta(3)$ in turbulence (indeed the exponent $p = 3$ plays a particular role, due to the form of the quadratic nonlinearty term in the Navier-Stokes equations. As finer models depending on more parameters were introduced, it became clear that the classification tools should also be based on more than one or two parameters but on the whole collection supplied by all values taken by the function $\zeta_f(p)$.

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The seminal ideas of multifractal analysis were introduced by G. Parisi and U. Frisch: They proposed a new interpretation for the scaling function of Kolmogorov in terms of the Hölder exponents of f, see [41]. Let us recall the definitions related with sizes of Hölder singularities.

Definition 24 *Let f be a locally bounded function. The isohölder sets of f are*

$$E_H = \{x_0 : \quad h_f(x_0) = H\}.$$

G. Parisi and U. Frisch suggested that the study of the scaling function can be related with the fractal dimension of the sets E_H. Two alternative definitions of fractal dimension have been introduced and are used in multifractal analysis.

Note however that all mathematical results concerning the validity of the multifractal formalism, point to two (slight) variants of size: The Hausdorff and Packing dimensions; one additional problem being that, in practice, one does not have a direct way to compute these dimensions on practical data (indeed, unlike the box dimension, they cannot be computed by regressions on log-log plots). In order to define the Hausdorff dimensions, we need to recall the notion of δ-dimensional Hausdorff measure.

Definition 25 *Let $A \subset \mathbb{R}^d$. If $\epsilon > 0$ and $\delta \in [0, d]$, we denote*

$$M_\epsilon^\delta = \inf_R \left(\sum_i |A_i|^\delta \right),$$

where R is an ϵ-covering of A, i.e. a covering of A by bounded sets $\{A_i\}_{i\in\mathbb{N}}$ of diameters $|A_i| \le \epsilon$. The infimum is therefore taken on all ϵ-coverings.

For any $\delta \in [0, d]$, the δ-dimensional Hausdorff measure of A is

$$mes_\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta.$$

There exists $\delta_0 \in [0, d]$ such that

$$\begin{array}{ll} \forall \delta < \delta_0, & mes_\delta(A) = +\infty \\ \forall \delta > \delta_0, & mes_\delta(A) = 0. \end{array}$$

This critical δ_0 is called the Hausdorff dimension of A, and is denoted by $dim(A)$.

The other notion of dimension we will use is the *packing dimension* which was introduced by C. Tricot, see [42, 43] (see also Chap. 5 of [38]): The *lower packing dimenson* is

$$Dim(A) = \inf \left\{ \sup_{i\in\mathbb{N}} \left(\underline{dim}_B A_i : A \subset \bigcup_{i=1}^{\infty} A_i \right) \right\} \tag{49}$$

(the infimum is taken over all possible partitions of A into a countable collection A_i). We will use this alternative notion in order to bound the dimensions of some sets of singularities. The dimensions we introduced can be compared as follows, see [38, 42, 43],

$$\forall A \subset \mathbb{R}^d, \qquad dim(A) \le Dim(A) \le \underline{dim}_B(A) \le \overline{dim}_B(A). \tag{50}$$

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SJ: Ne garder la definition de packing que si c'est vraiment utile a la fin pour les majorations de spectre

Definition 26 *Let* $dim(A)$ *denote the Hausdorff dimension of the set* A*. The function*

$$d_f(H) = dim(E_H)$$

is called the spectrum of singularities *of* f*. (One agrees that* $dim(\emptyset) = -\infty$*, i.e., if* H *is not an Hölder exponent present met the function* f*, then* $d_f(H) = -\infty$*.)*

The support of the spectrum is

$$supp(d_f) = \{H : E_H \neq \emptyset\} = \{H : d_f(H) \geq 0\}.$$

Note that a notion of spectrum can be attached to any local exponent. If one considers the p-exponent instead of the Hölder exponent, one obtains the p-spectrum considered in [26, 30]. General definitions concerning pointwise regularity exponents have been considered in [27, 29]. Similarly, one can define spectra associated with the oscillation exponent.

Two options are possible: Either one is only interested in the presence of oscillating singularities, in which case one associates a spectrum to the oscillation exponent only, or one is interested in recovering the full information concerning the dimensions of the sets of points where, simultaneously, the Hölder exponent takes a given value and the oscillation exponent takes another given value; then one associates a spectrum to the couple (H, β). Hence the following definition.

Definition 27 *Let* f *be a a uniform Hölder function, and let*

$$F_\beta = \{x_0 : \quad \beta_f(x_0) = \beta\}.$$

The β*-spectrum of* f *is*

$$\mathcal{D}(\beta) = dim(F_\beta).$$

Let

$$G_{H,\beta} = \{x_0 : \quad h_f(x_0) = H \ \text{ and } \ \beta_f(x_0) = \beta\}.$$

The grandcanonical spectrum $\mathcal{D}_f(H, \beta)$ *is*

$$\mathcal{D}_f(H, \beta) = dim(G_{H,\beta}).$$

In the second part of this paper, we will investigate multifractal formalisms constructed in order to yield these spectrums; In the present paper, we will only be interested in deriving some estimates on these spectrums from the integrated spectrums.

U. Frisch and G. Parisi proposed an interpretation of the initial scaling function which yields a relationship between the scaling function and the spectrum of singularities $d_f(H)$. This formula is backed by a heuristic argument derived from statistical physics, but does not have a general range of validity. Indeed, standard processes, such as Brownian motion for instance, provide "partial" counter-examples, see []. Because of this failure, which already takes place for the simplest possible model used in signal processing, a natural direction of research consisted in coining new scaling functions for which the range of validity of the multifractal formalism would be wider. This line of research was initiated by A. Arneodo and his collaborators, who introduced scaling functions based on the WTMM (Wavelet Transform Maxima Method) which indeed has been shown to yield experimentally the right spectra for a large collection of synthetic signals (FBM, cascade models of turbulence,...), but is, unfortunately, not backed by theoretical results. A variant of this idea consists in constructing a multifractal formalism based on wavelet leaders. Indeed, (36) implies that wavelet leaders are a pertinent quantiy to use in a structure function in order to have an intrepretation of the scaling function in terms of Hölder singularities. This possibility started to be explored in [1, 24] and we will now recall the main results concerning it, and also some of its limitations, which motivated the present paper.

4.1 The multifractal formalism based on wavelet leaders

SJ: mentioner que la motivation pour discriminer des modeles a toujours existe

An alternative scaling function based on wavelet leaders has been introduced in [24]. Its definition is similar to the Kolmogorov scaling function, except that increments have to be replaced by wavelet leaders. For any $p \in \mathbb{R}$, let

$$T_f(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p.$$

The *leader scaling function* is defined by

$$\forall p \in \mathbb{R}, \qquad \eta_f(p) = \liminf_{j \to +\infty} \frac{\log(T_f(p,j))}{\log(2^{-j})}.$$

Remarks: An important property of the leader scaling function is that one can prove that it has several robustness properties; first, it is proved in [24] that, if the wavelets belong to the Schwartz class, then $\forall p \in \mathbb{R}\ \zeta_f(p)$ is independent of the wavelet basis (and this result also holds for a range of $p > 0$ if the wavelets only have a finite smoothness).

Furthermore, one can show that the scaling function does not depend on the particular discretization of the time-scale half space which is chosen and, in particular, it is invariant under translations or dilations of f. Let us skech the proof of this point. First, one proves that the scaling function can be equivalently defined using the continuous wavelet transform. The proof follows similar ones of [39] where it is shown that the charactarization of Besov spaces on wavelets and Littlewood-Paley decomposition are the same. Then, the actions of translations and dilations on the continuous wavelet transform can be explicitely written, since they are just dilations and translations of the wavelet transform; therefore the corresponding wavelet leaders are of the same order of magnitude, and thus yield the same scaling function. Note that one can push this argument further and prove that the scaling function is invariant under a sufficiently smooth change of variable if $p > 0$ and a C^∞ change of variable for all values of p.

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SJ: Ces demonstrations n'ont vraiment été écrites nulle part. Je peux donner un peu plus de details. Par exemple donner l'idee en passant par la caracterisation sur la transformee en ondelettes continue 1D de la fonction d'echelle?

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Let us now show the following proposition, which shows that the wavelet leader scaling function can be alternatively defined through the "restricted leaders"

$$e_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|.$$

Proposition 11 *Let f be a uniform Hölder function, and*

$$S_f(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p.$$

Then

$$\forall p \in \mathbb{R}, \qquad \eta_f(p) = \liminf_{j \to +\infty} \frac{\log(S_f(p,j))}{\log(2^{-j})}.$$

Proof: Since

$$\sup_{\lambda' \subset \lambda} |c_{\lambda'}| \leq \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$

it follows that, if $p \geq 0$, then $S_f(p,j) \leq T_f(p,j)$, and, if $p \leq 0$, then $T_f(p,j) \leq S_f(p,j)$.

On the other hand, denote by μ the "father" of the cube λ (i.e. the cube twice as wide which contains λ, and $N(\mu)$ the 3^d "neighbours" of λ (i.e. the cubes of same width, whose boundary intersects the boundary of λ). Then

$$\sup_{\lambda' \subset 3\lambda} |c_{\lambda'}| \leq \sup_{\nu \in N(\mu)} \sup_{\lambda' \subset \nu} |c_{\lambda'}|.$$

It follows that, if $p \geq 0$, then $T_f(p,j) \leq 3^d S_f(p, j-1)$.

Finally, for any dyadic cube λ, there exists a "grandson" λ'' of λ such that $3\lambda'' \subset \lambda$. Therefore

$$\sup_{\lambda' \subset 3\lambda''} |c_{\lambda'}| \leq \sup_{\lambda' \subset \lambda} |c_{\lambda'}|;$$

therefore, if $p \leq 0$, then $S_f(p,j) \leq T_f(p, j+2)$. The proposition follows from these four estimates.

We denote by $\mathcal{L}(u)$ the Legendre transform of a concave function u, i.e.

$$\mathcal{L}(u)(H) = \inf_{p \in \mathbb{R}} \left(d + Hp - u(p) \right).$$

The *leader spectrum* of f is defined through a Legendre transform of the leader scaling function as follows

$$L_f(H) = \mathcal{L}(\eta_f)(H).$$

Of course, the leader spectrum of f has the same robustness properties as the leader scaling function. The following result, which is stronger than similar ones using alternative scaling functions, is proved in [24].

Theorem 3 *Let f be a uniform Hölder function, and assume that the wavelets belong to the Schwartz class; then,*

$$\forall H, \qquad d_f(H) \leq L_f(H). \tag{51}$$

Note that this result has the following consequence: Denote by $[H_{min}, H_{max}]$ the interval which is the support of $L_f(H)$; Then (51) implies that

$$supp(D_f) \subset supp(L_f).$$

Indeed,

$$supp(L_f) = \{H : L_f(H) \geq 0,$$

and, if $dim(A)$ is bounded by a negative number, then $A = \emptyset$.

Partial results concerning upper bounds for spectra can also be obtained when the wavelets have a finite smoothness, see [24]. It is however important to note that bounds for the decreasing part of the spectrum (i.e. obtained for negative ps in the Legendre transform) require that the wavelets belong to the Schwartz class.

The wavelet leaders multifractal formalism holds if

$$d_f(H) = L_f(H).$$

Let us now show why the motivation for studying how spectra are modified under fractional integrals which was mentioned in Section ??? meets our previous motivation of detecting the presence of oscillating singularities.

4.2 Spectra of fractional integrals

The purpose of this section is to investigate how spectra are transformed under fractional integration.

Definition 28 *Let f be a locally bounded function. The integrated spectra of singularities of f are*

$$d_f^s(H) := d_{f^{(-s)}}(H) = d_{\mathcal{I}^s f}(H).$$

The pointwise Hölder exponent of any locally bounded function f is shifted by an amount larger than or equal to s under a fractional integration of order s, see []:

$$\text{if} \quad s > 0, \qquad h_{I^s(f)}(x_0) \geq h_f(x_0) + s;$$

however, if f has a cusp singularities, at x_0 then

$$\forall x_0, \;\; \forall s > 0, \qquad h_{I^s(f)}(x_0) = h_f(x_0) + s.$$

Therefore, if a function f only displays cusps, then its integrated spectra of singularities are shifted according to:

$$d_f^s(H) = d_f(H - s). \tag{52}$$

This formula suggests one way to detect the presence of oscillating singularities in signals; recall that, for real-life data, the only access one has to their spectrum is through the multifractal formalism. Assume that f only has cusp singularities; then (52) will hold. Therefore, if we also assume that f and its fractional integrals satisfy the multifractal formalism, then the Legendre spectra of f and $I^s(f)$ will deduce from each other by a simple shift, according to (52). Therefore checking how the Legendre spectra behave under fractional integration will give an indication on the presence of oscillating singularities in a signal. These remarks motivate the following definition.

Definition 29 *The Integrated leader scaling function $\eta_f^s(p)$ is defined for any $p \in \mathbb{R}$ as*

$$\eta_f^s(p) = \eta_{f^{(-s)}}(p) = \liminf_{j\to+\infty} \frac{\log\left(2^{-dj}\sum_{\lambda\in\Lambda_j}|d_\lambda^s|^p\right)}{\log 2^{-j}},$$

The integrated Legendre spectrum of f is

$$L_f^s(H) = \inf_{p\in\mathbb{R}}\left(d + Hp - \eta_f^s(p)\right)$$

The following proposition shows that, in practice, one does not require to compute explicitely fractional integrals in order to derive Legendre spectra; indeed the pseudo-fractionals integrals will lead to the same scaling functions.

Proposition 12 *Let f be a uniform Hölder function;*

- *If the wavelet basis is sufficiently smooth, then, for $p > 0$, $\forall s > 0$, the leader scaling functions of $I^s(f)$ and $f^{(-s)}$ coincide,*
- *if the wavelets used belong to the Schwartz class, then the result is true for any $p \in \mathbb{R}$.*

Proof: l'argument est le meme que pour l'exposant de Hölder, mais il faut citer correctement le resultat de robustesse en distinguant $p > 0$ et $p < 0$; ca vaut sans doute le coup de donner la regularite exacte des ondelettes et de preciser la technique pratique: ondelettes de + en + regulieres.

4.3 Conditions satisfied by Integrated Legendre Spectra

The Legendre spectrum of a function f is supported on an interval which is of the form $[H_{min}, H_{max}]$, see [29]; we won't discuss here the interpretation of H_{max}; on the other hand, the interpretation of the first point of the spectrum as being precisely the exponent H_{min} defined in (37) has an important consequence: As a consequence of the definition of H_{min} in terms of uniform Hölder regularity this exponent is exactly shifted by s under a fractional integration of order s, see [29].

We will see that more is true: For any function f, (55) identifies a whole portion at the beginning of its Legendre spectrum which is exactly shifted by s. Note that no such result holds for H_{max}: The example supplied by Lacunary Wavelet Series below shows that it can be shifted by an arbitrary amount. In order to obtain the general conditions satisfied by Integrated Legendre Spectra, we need to recall the function space interpretation of the Leader Scaling Function.

Definition 30 *Let $p > 0$, and $s, s' \in \mathbb{R}^d$; a function f belongs to $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$ if $f \in C^{s'}(\mathbb{R}^d)$ and if its integrated wavelet leaders satisfy*

$$\sup_{j\in\mathbb{N}} 2^{(sp-d)j} \sum_k \sup_{\lambda'\subset\lambda} |c_{\lambda'} 2^{s'j'}|^p < \infty. \tag{53}$$

The space $\mathcal{O}_p^{s,0}$ is simply denoted by $\mathcal{O}_p^s$.

It follows from the comparison we performed between fractional integration and pseudo-fractional integration that

$$f \in \mathcal{O}_p^{s,s'}(\mathbb{R}^d) \quad \iff I^{-s'} f \in \mathcal{O}_p^s.$$

We will need the following wavelet characterization of the Besov spaces (see []) :

$$f \in B_p^{s,\infty} \quad \text{if} \quad \exists C > 0, \;\; \forall j, \;\; \sum_k |C_\lambda|^p 2^{(sp-d)j} \leq C.$$

Since $\sup_{\lambda'\subset\lambda} |C_{\lambda'} 2^{j's'}| \geq |C_\lambda 2^{js'}|$ it follows that

$$\mathcal{O}_p^{s,s'} \hookrightarrow B_p^{s'+s,\infty}.$$

On the other hand,

$$\text{if} \;\; sp - d > 0, \quad \text{then} \;\; \mathcal{O}_p^s = B_p^{s,\infty},$$

see [24]; it follows that

$$\text{if} \;\; sp - d > 0, \quad \text{then} \;\; \mathcal{O}_p^{s,s'} = B_p^{s+s',\infty}.$$

These embeddings will allow us to express a part of the leader scaling function in terms of the wavelet scaling function. It follows from the wavelet characterization of Besov spaces and definition ??? that

$$\forall p > 0, \qquad \zeta_f(p) = \sup\{s' : f \in B_p^{s'/p,\infty}\}.$$

and, similarly, it follows from the definition of Oscillation spaces that

$$\forall p > 0, \qquad \eta_f^s(p) = \sup\{s' : f \in \mathcal{O}_p^{s'/p,-s}\}.$$

Since Besov spaces satisfy the property

$$f \in B_p^{s,\infty} \iff I^{s'} f \in B_p^{s-s',\infty},$$

It follows that

$$\forall p > 0, \;\; \forall s \in \mathbb{R}, \qquad \zeta_{I^s f}(p) = \zeta_f(p) + sp,$$

therefore, a similar property will also hold for the leader scaling function, but only when the corresponding oscillation spaces coincide with Besov spaces: The condition $sp - d > 0$ means that $\eta_f^s(p) > d$. Let us introduce the following definition.

Definition 31 *Let f be a uniform Hölder function; the index p_c is defined by the condition*

$$\eta_f^s(p_c) = d,$$

and

$$H_c = (\eta_f^s)'(p_c).$$

Then,

$$\text{if } \ p > p_c, \quad \text{then} \quad \eta_f^s(p) = \zeta_f(p) + sp. \tag{54}$$

This condition translates as follows on the Legendre transforms; denote by $\delta(H)$ the Legendre Wavelet spectrum defind by

$$\delta_f(H) = \inf_{p>0} \left(d + Hp - \zeta_f(p)\right).$$

Then the following result follows from (54). It shows a remarkable asymptotic property of the Legendre wavelet spectrum: It is the asymptotic limit of the increasing part of the integrated Legendre spectrum.

Proposition 13 *Let f be a uniform Hölder function.*

$$\textit{If } \ H < H_c, \quad \textit{then} \quad d_f^s(H) = \delta_f(H - s); \tag{55}$$

Let H_s^1 be the abscissa of the point where the gaph of $\mathcal{L}_f(H-s)$ is tangent to the line going through the origin.

$$\forall s \geq 0 \quad \textit{if} \quad H \leq H_s^1, \ \textit{then} \qquad L_f^s(H) = \mathcal{L}_f(H - s). \tag{56}$$

This means that a whole portion at the beginning of the spectrum is translated under fractional integration. Let us now identify geometrically which portion it is. For that, note that, since $d_f^s(H) = \inf_p(d + Hp - \eta_f^s(p))$, and since $\eta_f^s(p_c) = d$, it follows that $d_f^s(H_c) = H_c p_c$; therefore, the portion of the spectrum which is shifted can be geometrically determined as the part before the point $(H_c, H_c p_c)$ where a line starting from the origin is tangent to the Legendre spectrum. Not that this part increases as the order of integration increases. At the limit when $s \to +\infty$ one recovers the whole increasing part of the spectrum, which then coincides with the whole wavelet Legendre spectrum. As mentioned before, no such result holds for the decreasing part of the spectrum. The following properties also hold in all generality.

Proposition 14 *Let f be a uniform Hölder function. Then $\eta_f^s(p)$ and its Legendre transform are both concave in each variable. Furthermore*

$$H_{max}(f^{-s}) \geq H_{max}(f) + s.$$

The last condition implies that the support of the Integrated Legendre Spectrum widens under fractional integration. Note that, if f is a uniform cusp function, it is shifted, so that checking how the quantity $H_{max} - H_{min}$ behaves under fractional integration can also be used as evidence of the presence of oscillating singularities.

Proof:

Another motivation for introducing alternative scaling functions is to construct new multifractal formalisms which would yield informations on other quantities than the Hölder exponent. One possibility is to replace the Hölder exponent by the p-exponent already mentioned in Section 3.1. Another possibility is to take into account the oscillation exponent.

.

SJ a mettre ailleurs

The following result shows that the integrated Legendre spectra behave as expected for uniform cusp functions.

Proposition 15 *Let f be a uniform Hölder function, which is also a uniform cusp function. Then the Integrated Legendre Spectra $L^s_f(H)$ of f satisfy the shift property*

$$\forall s \geq 0 \;\; L^s_f(H) = L_f(H - s). \tag{57}$$

.

Ê **Mettre ici une illustration de spectres se translatant bien (FBM et/ou cascades?**

Proof:

5 Integrated spectra of functions displaying oscillating singularities

If the uniform cusp asumption does not hold, then $L^s_f(H)$ may change with s in a way which indicates the presence of oscillating singularities. We will first check that it can indeed be the case, by studying two examples of random processe which are known to dispaly oscillating singularities: *Lacunary Wavelet Series* (LWS) , and *Random Wavelet Series* (RWS).

.

Ê **Ajouter des resultats generaux , majorations de spectre, y compris dans le cas de fonctions de type cusp, et traiter les cas particuliers des RWS et LWS**

5.1 Lacunary wavelet series

Recall that the definition of Lacunary Wavelet Series (LWS) $X_{\alpha,\gamma}$ was given in Section ???. Note that, in [23], orthonormal wavelet bases are used in the construction; however one easlily checks that all results of [23] hold in the slightly more general setting supplied by biorthogonal wavelet bases. We will need this setting in order to prove additional results on LWS, because some arguments will be developped on fractional integrals of LWS, which are LWS only if we work the birothogonal setting.

Let us first recall the main results proved in [23]:

Theorem 4 *The spectrum of singularities of almost every sample path of the lacunary wavelet series $X_{\alpha,\gamma}$ is supported by the interval $[\alpha, d\alpha/\gamma]$, and*

$$\forall h \in \left[\alpha, \frac{d\alpha}{\gamma}\right], \qquad d(h) = \frac{h\gamma}{\alpha}.$$

The chirp spectrum of almost every sample path of $X_{\alpha,\gamma}$ is supported by the segment $h = \alpha(\beta+1)$, $h \in [\alpha, d\alpha/\gamma]$, and on this segment

$$d(h,\beta) = \gamma(\beta+1).$$

We will prove a remarkable property of LWS: At each point, their chirp exponent and their oscillation exponent coincide (therefore, the chirp spectrum and the oscillation spectrum also coincide) as a consequence of the following proposition.

Proposition 16 *For each j let F_j denote the set of points $k2^{-j}$ such that one of the $C_{j,k}^{(i)}$ is not vanishing. Let $\delta \in [0,1]$. Denote by $B_{j,k}^{\delta}$ the ball centered at $k2^{-j}$ ($k \in F_j$) and of radius $2^{-\delta j}$. Let*

$$E_\delta = \limsup_{j\to\infty} \bigcup_k B_{j,k}^{\delta}, \quad G_\delta = \bigcap_{\delta'<\delta} E_{\delta'} - \bigcup_{\delta'>\delta} E_{\delta'} \quad \textit{if} \quad \delta < 1, \quad \textit{and} \quad G_1 = \bigcap_{\delta'<1} E_{\delta'}.$$

Note that the E_δ are decreasing (in δ).

If $\delta < \gamma/d$, $E_\delta = T$ a.s.

If $x \in G_\delta$, the chirp exponents and the oscillation exponent of the LWS $X_{\alpha,\gamma}$ at x coincide and are given by $(h,\beta) = (\alpha/\delta, 1/\delta - 1)$.

Proof: It has been proved in [23] that the Hölder exponent at a point $x \in G_\delta$ is $h = \alpha/\delta$. After a pseudo-fractional integration of order s, the LWS of type (α,γ) is transformed in another LWS of type $(\alpha+s,\gamma)$, using the biorthogonal wavelet basis generated by the wavelet ψ^{-s}. Therefore the same result, applied to this new LWS, implies that its Hölder exponent at x is given by $h = (\alpha+s)/\delta$. Since this resut holds for any $s \geq 0$, it follows that the chirp and oscillation exponent both take the value $1/\delta - 1$.
Ê

As a corollary the first part of the following theorem holds.

Theorem 5 *The oscillating singularity spectrum of almost every sample path of the lacunary wavelet series $X_{\alpha,\gamma}$ is supported by the segment $h = \alpha(\beta+1)$, $h \in [\alpha, d\alpha/\gamma]$, and on this segment*

$$d(h,\beta) = \gamma(\beta+1).$$

The β-spectrum of $X_{\alpha,\gamma}$ is supported by the interval $\left[0, \frac{d}{\gamma} - 1\right]$ on which it is given by

$$\forall \beta \in \left[0, \frac{d}{\gamma} - 1\right], \qquad \mathcal{D}(\beta) = \gamma(\beta+1).$$

The second part of the theorem follows directly from the fact that $dim(G_\delta) = \gamma/\delta$, see [23].

The following proposition shows how the spectra of LWS evolve through fractional integration. Such processes display oscillating singularities and, as expected, this feature is reflected in their integrated spectra: These spectra are not derived form each other by a shift.

Proposition 17 *Let $X_{\alpha,\gamma}$ be a Lacunary Wavelet Series. The Legendre and Hölder integrated spectra of $X_{\alpha,\gamma}$ coincide and are given by*

$$\begin{aligned} d^s_{\alpha,\gamma}(H) = L^s_{\alpha,\gamma}(H) &= \frac{\gamma H}{\alpha+s} \quad \text{if } H \in \left[\alpha+s, \frac{d(\alpha+s)}{\gamma}\right] \\ &= -\infty \qquad \text{else}. \end{aligned} \tag{58}$$

Proof: First, note that the spectrum of singularities of $X_{\alpha,\gamma}$ has been computed in [23] and shows that (58) yields the correct value for $d^s_{\alpha,\gamma}(H)$ if $s = 0$. A fractional integration of order s transforms a wavelet basis, into a vaguelette basis and multiplies the coefficients by 2^{-sj}; since one passes from a wavelet basis to a biorthogonal wavelet basis by applying an operator in $\mathcal{M}^\infty$, and since this operation does not alter the values taken by the Hölder exponent, it follows that the result of [23] which yields the spectrum of a LWS also applies to each fractional integral, but with a new LWS of parameters $(\alpha+s, \gamma)$; and, therefore (58) yields the correct value for $d^s_{\alpha,\gamma}(H)$, for any $s > 0$.

Let us now consider the leader scaling function. Since the wavelets used belong to the Schwartz class, this scaling function is independent of the wavelet used, and therefore the wavelet used in the construction of the LWS can also be used as analyzing wavelet. Let us estimate

$$T_f(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p.$$

We first obtain an upper bound for this quantity:

- At the scale j there are exactly $2^{\gamma j}$ leaders of size $2^{-\alpha j}$.
- The scale $j+1$ adds at most $2^{\gamma(j+1)}$ new leaders of size $2^{-\alpha(j+1)}$.
- ...

We can continue this enumeration untill we reach the scale l such that $l = dj/\gamma + \log j$, at which, by standard results of random coverings of the cube, we obtain that all leaders have been computed.

SJ: on a le choix entre donner l'argument de recouverments aleatoires marteau pilon, ou faire un calcul non optimal, mais qui suffit et rend ce morceau autosuffisant

We have therefore obtained an upper bound for $T_f(p,j)$:

$$T_f(p,j) \leq 2^{-dj} \sum_{l=j}^{dj/\gamma+\log j} 2^{\gamma l} \left(2^{-\alpha l}\right)^p$$

We note that we sum geometric sequences, which are either increasing or decreasing depending whether $p > \gamma/\alpha$ or $p < \gamma/\alpha$. Recall that the *leader scaling function* is

$$\forall p \in \mathbb{R}, \qquad \zeta_f(p) = \liminf_{j\to+\infty} \frac{\log(T_f(p,j))}{\log(2^{-j})}.$$

We obtain that

$$\zeta_f(p) \geq \inf\left(d - \gamma + \alpha p, \frac{dp\alpha}{\gamma}\right)$$

Let us show how to obtain lower bounds for $T_f(p,j)$. First, we note that

$$T_f(p,j) \geq 2^{\gamma j} \left(2^{-\alpha j}\right)^p \tag{59}$$

(by restricting the sum to the $2^{\gamma j}$ indices corresponding to nonvanishing wavelet leaders at scale j) . We also note that, for $l \leq dj/\gamma - \log j$ at most one half of the wavelet leaders have been attained; therefore, by restricting the sum to the other dyadic cubes, where the supremum will be attained between the scales $dj/\gamma - \log j$ and $dj/\gamma + \log j$, we obtain that

$$\begin{aligned} T_f(p,j) &\geq 2^{-dj} \sum_{dj/\gamma-\log j}^{dj/\gamma+\log j} 2^{\gamma l} \left(2^{-\alpha l}\right)^p \\ &\leq 2^{-dj} j^A 2^{dj} \left(2^{-\alpha dj/\gamma}\right)^p. \end{aligned}$$

It follows from this lower bound, together with (59) that

$$\zeta_f(p) = \inf\left(d - \gamma + \alpha p, \frac{dp\alpha}{\gamma}\right). \tag{60}$$

By taking a Legendre transform, we obtain that the Legendre leader spectrum of LWS has the required form stated in Proposition 17.

All the above computations remain valid in the biorthogonal setting, and therefore when performing a pseudo-fractional integral instead of a fractional integral. Since a pseudo-fractional integral amounts to replace α by $\alpha + s$ in the definition of LWS, therefore, after performing this shift, all results proved aboce also holds for integrated spectra.

5.2 Random wavelet series

We now consider the setting supplied by *Random wavelet series*, introduced in [9].

Definition 32 *A Stochastic process $X : \mathbb{R} \to \mathbb{R}$ is a Random Wavelet Series (R.W.S.) if its wavelet coefficients $C_{j,k}$ in a birothogonal wavelet basis in the schwartz class satisfy the following requirements:*

1. *$\forall j$, the $C_{j,k}$ ($k \in 0, \ldots, 2^j - 1$) are identically distributed random variables; the probability distribution of $-\frac{\log_2(|C_{j,k}|)}{j}$ is denoted by $d\boldsymbol{\rho}_j$; it is defined on $\mathbb{R} \cup \{+\infty\}$;*
2. *the $C_{j,k}$ ($j \in \mathbb{N}$, $k \in 0, \ldots, 2^j - 1$) are independent;*
3. *there exists $\gamma > 0$ such that*

$$\boldsymbol{\rho}(\alpha) := \inf_{\epsilon > 0} \limsup_{j \to +\infty} \frac{\log_2 \left(2^j \int_{\alpha-\epsilon}^{\alpha+\epsilon} \boldsymbol{\rho}_j(t) dt \right)}{j} \tag{61}$$

   is strictly negative for $\alpha < \gamma$,

4. *there exists α such that $\rho(\alpha) > 0$.*

Remarks:

- The third requirement is necessary in order to be sure that the series $\sum c_{j,k}\psi_{j,k}$ is convergent and that its sum has some uniform Hölder regularity.
- LWS are not exactly a subcase of RWS, since the histograms of wavelet coefficients of LWS are deterministic at each scale, while their are random for RWS.
- The function $\boldsymbol{\rho}$ thus defined is called the *spectrum of large deviation* of the process X. It is an upper semi-continuous function, but needs not be monotonous nor concave. By definition, the *support* of $\boldsymbol{\rho}$ is

$$supp(\boldsymbol{\rho}) = \{\alpha : \quad \boldsymbol{\rho}(\alpha) \geq 0\}.$$

- We do not make any assumption on the probability measures $d\boldsymbol{\rho}_j$; Note that $\boldsymbol{\rho}_j(\{+\infty\})$ is the probability that $C_{j,k} = 0$.

.

IL faudrait dire ce qui a ete fait autour du spectre de grande deviation: Riedi? Levy-Vehel? autres depuis?

Let us define

$$h_{\min} := \sup \{\alpha, \gamma < \alpha \Rightarrow \boldsymbol{\rho}(\gamma) < 0\}$$

and

$$h_{\max} := \left(\sup_{\alpha > 0} \frac{\boldsymbol{\rho}(\alpha)}{\alpha} \right)^{-1}.$$

The following result of [9] yields the spectrum of singularities of R.W.S.

Theorem 6 *Let X be a R.W.S. The Hölder exponent of X is almost-surely almost-everywhere $h_{\max}$.*

The spectrum of singularities of X is supported by $[h_{\min}, h_{\max}]$ and

$$\forall h \in [h_{\min}, h_{\max}], \qquad d(h) = h \sup_{\alpha \in (0,h]} \frac{\boldsymbol{\rho}(\alpha)}{\alpha}$$

Since it is shown in [9] that the multifractal proerties of X only depend on the function ρ (and not on the particular sequence ρ_j), with a slight abuse of notations, from now on, we will denote RWS by X_ρ.

We will check the following proposition which shows how the spectra of RWS evolve through fractional integration. Such processes display oscillating singularities and, as expected, this feature is reflected in their integrated spectra, which are not derived form each other by a shift. Let

$$h^s_{max} := \left(\sup_{\alpha > 0} \frac{\boldsymbol{\rho}(\alpha - s)}{\alpha} \right)^{-1}.$$

Theorem 7 *Let X_ρ be a R.W.S. Almost-surely, for all $s > 0$, the integrated spectrum of singularities of X_ρ is supported by $[h_{\min} + s, h^s_{max}]$ where it satisfies*

$$\forall h \in [h_{\min} + s, h^s_{max}], \qquad d^s_X(h) = h \sup_{\alpha \in (0,h]} \frac{\boldsymbol{\rho}(\alpha - s)}{\alpha}; \tag{62}$$

Note that the starting point of the spectrum h_{min} is shifted by s, but it is not the case with the largest possible Hölder exponent h^s_{max}, which is usually shifted by more than s.

This theorem follows immediately from the previous one by noticing that a fractional integration of order s transforms a wavelet basis, into a vaguelette basis and multiplies the coefficients by 2^{-sj}; since one passes from a wavelet basis to a biorthogonal wavelet basis by applying an opertator in $\mathcal{M}^\infty$, and since this operation does not alter the values taken by the Hölder exponent, it follows that the result which yields the spectrum of a RWS also applies to each fractional integral, but with a new RWS whose large deviation spectrium is $\rho(\alpha - s)$; and, therefore (62) yields the correct value ofr $d^s_{\alpha,\gamma}(H)$ for any $s > 0$.

Since the spectrum of singularities of a RWS is, in general, not a concave fuction, we cannot expect that the multifractal formalism will hold. However, we will show that a weaker result holds.

Theorem 8 *With probability 1, for every s, the integrated Legendre spectrum of a RWS is the concave hull of the Hölder spectrum $d^s_X(h)$.*

Proof: Let us compute the leader scaling function of a RWS. Since the wavelets used belong to the Schwartz class, it it is independent of the wavelet used, this scaling function and therefore we can use the wavelet used in the the definition of the LWS as analyzing wavelet.

SJ: A finir

We now consider the spectra of oscillating singularities of R.W.S.

Theorem 9 *Let f be a random wavelet series With probability one, f has the following properties:*

The spectrum of oscillating singularities of a RWS X_ρ is supported by the domain

$$A_\rho = \left\{ (h,\beta) : \quad h \in [h_{min}, h_{max}],\ \beta \geq 0 \ \ and \ \frac{h}{\beta+1} \in supp(\rho) \right\}.$$

and, if $(h,\beta) \in A_\rho$,

$$d(h,\beta) = (1+\beta)\boldsymbol{\rho}\left(\frac{h}{1+\beta}\right). \tag{63}$$

The β-spectrum of a RWS X_ρ is supported by the interval $\left[0, \frac{d}{\gamma} - 1\right]$ on which it is given by

$$\forall \beta \in \left[0, \frac{d}{\gamma} - 1\right], \qquad\qquad \mathcal{D}(\beta) = \gamma(\beta + 1).$$

The first part of the theorem, which yields the scillating singularities spectrum was proved in [9].

Let us now come back to the problem of determining when the presence of oscillating singularities can be inferred from the inspection of integrated spectra. First, note that Proposition 15 does not allow to conclude that, if Integrated Legendre Spectra satisfy (57), then the signal only displays cusp singularities. We will now study simple counterexamples of this where the integrated spectra of functions displaying oscillating singularities are exactly shifted by s. They are obtained as a perturbation of the previous example: One considers the superposition of a lacunary wavelet series and a *Fractional Brownian Motion* (FBM).
Ê

Let $a > 0$ be such that $a \notin \mathbb{N}$; denote by B_a the FBM of order a; let us make precise the definition that we use outside of the classical case where $d = 1$ and $0 < a < 1$. If $d = 1$ and if $a > 1$ and a is not a integer, B_a is defined by induction on $[a]$ as the primitive which vanishes at the origin of B_{a-1}. If $d > 1$, we use the following definition for the Fractional Brownian Sheet: The Gaussian white noise is defined as the random distribution $\sum \chi_n e_n(x)$ where e_n is an orthonormal basis of $L^2(\mathbb{R}^d)$; a famous theorem of P. Lévy states that this construction is independent of the particular orthonormal basis chosen, so that we will pick we will pick a wavelet basis. The Fractional Brownian Sheet of exponent a is obtained formally by taking fractional integral of this expansion, (and substractiong the values at 0 when one meets the range of functions). Note that this definition is coherent with the usual definition of the FBM in 1D and $a \in (0, 1)$ since, in this case, the sample paths of FBM also deduce from each other by tractional integrals. Let us now consider

$$Y_{a,\alpha,\gamma} = X_{\alpha,\gamma} + B_a$$

where the two processes $X_{\alpha,\gamma}$ and B_a are independent. The spectra of these processes and of their fractional integrals can be determined as a consequence of the following general result concerning the pointwise regularity of superposition of FBM and deterministic functions, see [29]: if f is a uniform Hölder function and $a > 0$, then,

$$\text{a.s.,} \ \forall x_0 \in \mathbb{R}, \qquad h_{f+B_a}(x_0) = \inf(a, h_f(x_0)).$$

The integrated spectra of singularities of $Y_{a,\alpha,\gamma}$ follows directly from this result.

Corollary 2 *If $a \in [\alpha, \frac{d\alpha}{\eta}]$, then the spectrum of singularities of $Y^{(-s)}_{a,\alpha,\gamma}$ is*

$$\left.\begin{aligned} d^s_{a,\alpha,\gamma}(H) &= \frac{\eta H}{\alpha + s} && \textit{if } H \in [\alpha + s, a + s] \\ &= d && \textit{if } H = a + s \\ &= -\infty && \textit{else.} \end{aligned}\right\} \tag{64}$$

.

SJ:

donner la forme

$$f = \sum \chi_{j,k} 2^{dj/2} \psi(2^j x - k).$$

Faire de meme pour les RWS

Il faudrait faire des dessins pour illustrer ces differents spectres

On peut ici encore montrer que le $h^s_f(x_0)$ est dans ce cas le raccord de fonctions affines formant un coude .

Furthermore, the singularities of $Y_{a,\alpha,\gamma}$ for $H \in [\alpha, a]$ are oscillating singularities, since they correspond to the superposition of an oscillating singularity (the LWS) and a process which is strictly smoother (the FBM).

Corollary 3 *T he Legendre spectrum is a straight segment between the points* $(\alpha + s, \eta)$ *and* $(a + s, d)$.

Proof: It is a straightforward consequence of the lemma.

Lemma 3 *If* f *and* g *are two uniform Hölder functions, and if* $p > 0$*, then*

$$\eta_f(p) \neq \eta_g(p) \Longrightarrow \eta_{f+g}(p) = \inf(\eta_f(p), \eta_g(p)).$$

This lemma follows directly from the function space interpretation of $\eta_f(p)$, see [], and the fact that this interpetation is either based on a normed space if $p \geq 1$ or a semi-normed space if $0 < p < 1$.

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SJ: le faire

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il reste la cas $p < 0$ **Il faut se placer dans la base qui diagonale le FBM et utiliser le fait que la gaussienne decentree est plus grande que la gaussienne**

Zuhai disait que si f **est gaussienne centree et** g **est deterministe,** $h_{f+G} = min(h_f, h_g)$
On doit avoir qqc de similaire sur les fts d'echelle
A FAIRE

Mettre ici une illustration de ces processus? a defaut on peut aussi faire la fonction de Riemann + FBM ou Riemann + Weierstrass.

Therefore the integrated Legendre spectra satisfy the shift property (57). It follows that one cannot detect the presence of oscillating singularities on $Y_{a,\alpha,\gamma}$ by inspecting its integrated Legendre spectra; this observation supplies a motivation for a method, whose purpose is to detect the presence of oscillating singularities in such situations. This will the subject of the second part of the present paper.

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