

Parallel Complexity of Term Rewriting Systems*

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Abstract

In this workshop paper, we revisit the notion of parallel-innermost term rewriting. We provide a definition of parallel complexity and propose techniques to derive upper bounds on this complexity via the Dependency Tuple framework by Noschinski et al.

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1 Introduction

In this paper, we consider the problem of evaluating the potentiality of parallelisation in pattern-matching based recursive functions like the one depicted in Figure 1.

```
fn size(&self) -> int {
  match self {
    &Tree::Node { v, ref left, ref right }
    => left.size() + right.size() + 1,
    &Tree::Empty => 0 , } }
```

Figure 1 Tree size computation in Rust

In this particular example, the recursive calls to `left.size()` and `right.size()` can be done in parallel. Building on previous work on parallel-innermost rewriting [6, 4], and first ideas about parallel complexity [1], we propose a new notion of Parallel Dependency Tuples that capture such a behaviour, and a method to compute *parallel complexity bounds*.

2 Parallel-innermost Term Rewriting

The following definitions are mostly standard [3].

► **Definition 1** (Term rewrite system, innermost rewriting). $\mathcal{T}(\Sigma, \mathcal{V})$ denotes the set of terms over a finite signature Σ and the set of variables \mathcal{V} . For a term t , the set $\text{Pos}(t)$ of its positions is defined inductively as a set of strings of positive integers: (a) if $t \in \mathcal{V}$, then $\text{Pos}(t) = \{\varepsilon\}$, and (b) if $t = f(t_1, \dots, t_n)$, then $\text{Pos}(t) = \{\varepsilon\} \cup \bigcup_{1 \leq i \leq n} \{i\pi \mid \pi \in \text{Pos}(t_i)\}$. The position ε is called the root position of term t . The (strict) prefix order $<$ on positions is the strict partial order given by: $\pi < \tau$ iff there exists $\pi' \neq \varepsilon$ such that $\pi\pi' = \tau$. For $\pi \in \text{Pos}(t)$, $t|_\pi$ is the subterm of t at position π , and we write $t[s]_\pi$ for the term that results from t by replacing the subterm $t|_\pi$ at position π by the term s .

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32 For a term t , $\mathcal{V}(t)$ is the set of variables in t . If t has the form $f(t_1, \dots, t_n)$, $\text{root}(t) = f$
 33 is the root of t . A term rewrite system (TRS) \mathcal{R} is a set of rules $\{\ell_1 \rightarrow r_1, \dots, \ell_n \rightarrow r_n\}$
 34 with $\ell_i, r_i \in \mathcal{T}(\Sigma, \mathcal{V})$, $\ell_i \notin \mathcal{V}$, and $\mathcal{V}(r_i) \subseteq \mathcal{V}(\ell_i)$ for all $1 \leq i \leq n$. The rewrite relation of \mathcal{R}
 35 is $s \rightarrow_{\mathcal{R}} t$ iff there are a rule $\ell \rightarrow r \in \mathcal{R}$, a position $\pi \in \text{Pos}(s)$, and a substitution σ such
 36 that $s = s[\ell\sigma]_{\pi}$ and $t = s[r\sigma]_{\pi}$. Here, σ is called the matcher and the term $\ell\sigma$ is called the
 37 redex of the rewrite step. If $\ell\sigma$ has no proper subterm that is also a possible redex, $\ell\sigma$ is an
 38 innermost redex, and the rewrite step is an innermost rewrite step denoted by $s \xrightarrow{i}_{\mathcal{R}} t$.

39 $\Sigma_d^{\mathcal{R}} = \{f \mid f(\ell_1, \dots, \ell_n) \rightarrow r \in \mathcal{R}\}$ and $\Sigma_c^{\mathcal{R}} = \Sigma \setminus \Sigma_d^{\mathcal{R}}$ are the defined and constructor
 40 symbols of \mathcal{R} . We may omit the superscript and just write Σ_d and Σ_c if \mathcal{R} is not of importance
 41 or clear from the context. Finally, let $\text{Pos}_d(t) = \{\pi \mid \pi \in \text{Pos}(t), \text{root}(t|_{\pi}) \in \Sigma_d\}$.

42 The notion of parallel-innermost rewriting dates back at least to [6]. Informally, in a
 43 parallel-innermost rewrite step, all innermost redexes are rewritten simultaneously. This cor-
 44 responds to executing all function calls in parallel on a machine with unbounded parallelism.

45 **► Definition 2** (Parallel-innermost rewriting [4]). A term s rewrites innermost in parallel to t
 46 with a TRS \mathcal{R} , written $s \xrightarrow{\parallel}_{\mathcal{R}} t$, iff $s \xrightarrow{i+}_{\mathcal{R}} t$, and either (a) $s \xrightarrow{i}_{\mathcal{R}} t$ with s an innermost
 47 redex, or (b) $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, and for all $1 \leq k \leq n$ either $s_k \xrightarrow{\parallel}_{\mathcal{R}} t_k$ or
 48 $s_k = t_k$ is a normal form.

49 **► Example 3** (size). Consider the TRS \mathcal{R} with the following rules modelling the code
 50 of Figure 1.

$$51 \begin{array}{l|l} \text{plus}(\text{Zero}, y) \rightarrow y & \text{size}(\text{Nil}) \rightarrow \text{Zero} \\ \text{plus}(\text{S}(x), y) \rightarrow \text{S}(\text{plus}(x, y)) & \text{size}(\text{Tree}(v, l, r)) \rightarrow \text{S}(\text{plus}(\text{size}(l), \text{size}(r))) \end{array}$$

52 Here $\Sigma_d^{\mathcal{R}} = \{\text{plus}, \text{size}\}$ and $\Sigma_c^{\mathcal{R}} = \{\text{Zero}, \text{S}, \text{Nil}, \text{Tree}\}$. We have the following parallel
 53 innermost rewrite sequence, where innermost redexes are underlined:

$$54 \begin{array}{l} \text{size}(\text{Tree}(\text{Zero}, \text{Nil}, \text{Tree}(\text{Zero}, \text{Nil}, \text{Nil}))) \\ \xrightarrow{\parallel}_{\mathcal{R}} \text{S}(\text{plus}(\text{size}(\text{Nil}), \text{size}(\text{Tree}(\text{Zero}, \text{Nil}, \text{Nil})))) \\ \xrightarrow{\parallel}_{\mathcal{R}} \text{S}(\text{plus}(\text{Zero}, \text{S}(\text{plus}(\text{size}(\text{Nil}), \text{size}(\text{Nil})))) \\ 55 \xrightarrow{\parallel}_{\mathcal{R}} \text{S}(\text{plus}(\text{Zero}, \text{S}(\text{plus}(\text{Zero}, \text{Zero})))) \\ \xrightarrow{\parallel}_{\mathcal{R}} \text{S}(\text{plus}(\text{Zero}, \text{S}(\text{Zero}))) \\ \xrightarrow{\parallel}_{\mathcal{R}} \text{S}(\text{S}(\text{Zero})) \end{array}$$

56 Note that in the second and in the third step, two innermost steps each are happening in
 57 parallel. A corresponding regular innermost rewrite sequence without parallel evaluation of
 58 redexes would have needed two more steps.

59 3 Finding Upper Bounds for Parallel Complexity

60 3.1 Notion of Parallel Complexity

61 We extend the notion of innermost runtime complexity to parallel-innermost rewriting.

62 **► Definition 4** ((Parallel) Innermost Runtime Complexity). The size $|t|$ of a term t is $|x| = 1$ if
 63 $x \in \mathcal{V}$ and $|f(t_1, \dots, t_n)| = 1 + \sum_{i=1}^n |t_i|$, otherwise. The derivation height of a term t w.r.t.
 64 a relation \rightarrow is the length of the longest sequence of \rightarrow -steps from t : $\text{dh}(t, \rightarrow) = \sup\{e \mid$
 65 $\exists t' \in \mathcal{T}(\Sigma, \mathcal{V}). t \rightarrow^e t'\}$ where \rightarrow^e is the e^{th} iterate of \rightarrow . If t starts an infinite \rightarrow -sequence,
 66 we write $\text{dh}(t, \rightarrow) = \omega$.

67 A term $f(t_1, \dots, t_k)$ is basic (for a TRS \mathcal{R}) iff $f \in \Sigma_d^{\mathcal{R}}$ and $t_1, \dots, t_k \in \mathcal{T}(\Sigma_c^{\mathcal{R}}, \mathcal{V})$. $\mathcal{T}_{\text{basic}}^{\mathcal{R}}$ is
 68 the set of basic terms for a TRS \mathcal{R} . For $n \in \mathbb{N}$, we define the innermost runtime complexity
 69 function $\text{irc}_{\mathcal{R}}(n) = \sup\{\text{dh}(t, \xrightarrow{i}_{\mathcal{R}}) \mid t \in \mathcal{T}_{\text{basic}}, |t| \leq n\}$ and we introduce the parallel
 70 innermost runtime complexity function $\text{irc}_{\mathcal{R}}^{\parallel}(n) = \sup\{\text{dh}(t, \xrightarrow{\parallel}_{\mathcal{R}}) \mid t \in \mathcal{T}_{\text{basic}}, |t| \leq n\}$.

71 In the following, given a TRS \mathcal{R} , our goal shall be to infer (asymptotic) upper bounds
 72 for $\text{irc}_{\mathcal{R}}^{\parallel}$ fully automatically. As usual for runtime complexity, we are considering only basic
 73 terms as start terms, corresponding to a defined function called on data objects as arguments.
 74 An upper bound for (sequential) $\text{irc}_{\mathcal{R}}$ is also an upper bound for $\text{irc}_{\mathcal{R}}^{\parallel}$. We will introduce
 75 techniques to find upper bounds for $\text{irc}_{\mathcal{R}}^{\parallel}$ that are strictly tighter than these trivial bounds.

76 3.2 Complexity: the sequential case

77 We build on the Dependency Tuple framework [5], originally introduced to determine upper
 78 bounds for (sequential) innermost runtime complexity. A central idea is to group all function
 79 calls by a rewrite rule *together* rather than to regard them separately (as for termination [2]).

80 ► **Definition 5** (Sharp Terms \mathcal{T}^{\sharp}). *For every $f \in \Sigma_d$, we introduce a fresh symbol f^{\sharp} of the*
 81 *same arity. For a term $t = f(t_1, \dots, t_n)$ with $f \in \Sigma_d$, we define $t^{\sharp} = f^{\sharp}(t_1, \dots, t_n)$ and let*
 82 $\mathcal{T}^{\sharp} = \{t^{\sharp} \mid t \in \mathcal{T}(\Sigma, \mathcal{V}), \text{root}(t) \in \Sigma_d\}$.

83 To compute an upper bound for sequential complexity, we “count” how often each rewrite
 84 rule is used. The idea is that the cost of the function call to the lhs of a rule is 1 + the sum
 85 of the costs of all the function calls in the rhs, counted separately. To group k function
 86 calls together, we use “compound symbols” Com_k , which intuitively represent the sum of
 87 the runtimes of their arguments. Then, we can use polynomial interpretations $\mathcal{P}ol$ with
 88 $\mathcal{P}ol(\text{Com}_k(x_1, \dots, x_k)) = x_1 + \dots + x_k$ for all k to compute a complexity bound [5, Thm. 27].

89 ► **Definition 6** (Dependency Tuple, DT [5]). *A dependency tuple (DT) is a rule of the*
 90 *form $s^{\sharp} \rightarrow \text{Com}_n(t_1^{\sharp}, \dots, t_n^{\sharp})$ where $s^{\sharp}, t_1^{\sharp}, \dots, t_n^{\sharp} \in \mathcal{T}^{\sharp}$. Let $\ell \rightarrow r$ be a rule with $\text{Pos}_d =$
 91 $\{\pi_1, \dots, \pi_n\}$ and $\pi_1 \triangleleft \dots \triangleleft \pi_n$ for a total order \triangleleft on positions. Then $\text{DT}(\ell \rightarrow r) = \ell^{\sharp} \rightarrow$
 92 $\text{Com}_n(r|_{\pi_1}^{\sharp}, \dots, r|_{\pi_n}^{\sharp})$. For a TRS \mathcal{R} , let $\text{DT}(\mathcal{R}) = \{\text{DT}(\ell \rightarrow r) \mid \ell \rightarrow r \in \mathcal{R}\}$.*

93 ► **Example 7.** For our running example, we get the following DTs:

$$\begin{aligned} & \text{plus}^{\sharp}(\text{Zero}, y) \rightarrow \text{Com}_0 \\ & \text{plus}^{\sharp}(\text{S}(x), y) \rightarrow \text{Com}_1(\text{plus}^{\sharp}(x, y)) \\ & \text{size}^{\sharp}(\text{Nil}) \rightarrow \text{Com}_0 \\ & \text{size}^{\sharp}(\text{Tree}(v, l, r)) \rightarrow \text{Com}_3(\text{size}^{\sharp}(l), \text{size}^{\sharp}(r), \text{plus}^{\sharp}(\text{size}(l), \text{size}(r))) \end{aligned}$$

95 The given polynomial interpretation, which orients all DTs with \succ and all rules from \mathcal{R}
 96 with \succsim , proves $\text{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n^2)$: $\mathcal{P}ol(\text{plus}^{\sharp}(x_1, x_2)) = \mathcal{P}ol(\text{size}(x_1)) = x_1$, $\mathcal{P}ol(\text{size}^{\sharp}(x_1)) =$
 97 $2x_1 + x_1^2$, $\mathcal{P}ol(\text{plus}(x_1, x_2)) = x_1 + x_2$, $\mathcal{P}ol(\text{Tree}(x_1, x_2, x_3)) = 1 + x_2 + x_3$, $\mathcal{P}ol(\text{S}(x_1)) =$
 98 $1 + x_1$, $\mathcal{P}ol(\text{Zero}) = \mathcal{P}ol(\text{Nil}) = 1$.

99 3.3 Computing Upper Bounds for Parallel Rewriting

100 To find upper bounds for runtime complexity of parallel-innermost rewriting, we can *reuse* the
 101 notion of DTs from Def. 6 for sequential innermost rewriting along with existing techniques [5]
 102 and implementations. We illustrate this in the following example.

103 ► **Example 8.** In the recursive size-rule, the two calls to $\text{size}(l)$ and $\text{size}(r)$ happen *in parallel*
 104 (this will be captured by the notion of *structural independency*). Thus, the cost for these two
 105 calls is not the *sum*, but the *maximum* of the calls. Regardless of which of these two calls
 106 has the higher cost, we still need to add the cost for the call to plus , which starts evaluating
 107 only after both calls to size have finished. With σ as the used matcher for the rule, we have:

$$\begin{aligned} & \text{dh}(\text{size}(\text{Tree}(v, l, r))\sigma, \dashv\vdash \rightarrow \mathcal{R}) \\ & = 1 + \max(\text{dh}(\text{size}(l)\sigma, \dashv\vdash \rightarrow \mathcal{R}), \text{dh}(\text{size}(r)\sigma, \dashv\vdash \rightarrow \mathcal{R})) + \text{dh}(\text{plus}(\text{size}(l), \text{size}(r))\sigma, \dashv\vdash \rightarrow \mathcal{R}) \end{aligned}$$

Equivalently, we can “factor in” the cost of calling `plus` into the maximum function:

$$\begin{aligned} & \text{dh}(\text{size}(\text{Tree}(v, l, r))\sigma, \dashv\vdash \rightarrow \mathcal{R}) \\ 110 \quad & = \max(1 + \text{dh}(\text{size}(l)\sigma, \dashv\vdash \rightarrow \mathcal{R}) + \text{dh}(\text{plus}(\text{size}(l), \text{size}(r))\sigma, \dashv\vdash \rightarrow \mathcal{R}), \\ & \quad 1 + \text{dh}(\text{size}(r)\sigma, \dashv\vdash \rightarrow \mathcal{R}) + \text{dh}(\text{plus}(\text{size}(l), \text{size}(r))\sigma, \dashv\vdash \rightarrow \mathcal{R})) \end{aligned}$$

Intuitively, this would correspond to evaluating `plus(size(l), size(r))` twice, in two parallel threads of execution, which costs the same amount of time as evaluating `plus(size(l), size(r))` once. We can represent this maximum of the execution times of two threads by introducing *two* DTs for our recursive `size`-rule:

$$\begin{aligned} 115 \quad & \text{size}^\sharp(\text{Tree}(v, l, r)) \rightarrow \text{Com}_2(\text{size}^\sharp(l), \text{plus}^\sharp(\text{size}(l), \text{size}(r))) \\ & \text{size}^\sharp(\text{Tree}(v, l, r)) \rightarrow \text{Com}_2(\text{size}^\sharp(r), \text{plus}^\sharp(\text{size}(l), \text{size}(r))) \end{aligned}$$

To express the cost of a concrete rewrite sequence, we would non-deterministically choose the DT that corresponds to the “slower thread”.

In other words, the cost of the function call to the lhs of a rule is $1 +$ the sum of the costs of all the function calls in the rhs *that are in structural dependency with each other*. The actual cost of the function call to the lhs in a concrete rewrite sequence is the *maximum* of all the possible costs of such *chains* (in the prefix order $<$ on positions) of structural dependency. Thus, *structurally independent* function calls are considered in separate DTs, whose non-determinism models the parallelism of these function calls.

The notion of *structural dependency* of functions calls is captured by Def. 9. Basically, it comes from the fact that a term cannot be evaluated before all its subterms have been reduced to normal forms (innermost rewriting/*call by value*).

► **Definition 9** (Structural dependency). *The term t_1 structurally depends on t_2 iff there exists a rule $\ell \rightarrow r \in \mathcal{R}$ such that $r|_{\tau_1} = t_1$ and $r|_{\tau_2} = t_2$ with $\tau_1 < \tau_2$ in the prefix order $<$.*

We thus revisit the notion of DTs, which now embed structural dependencies.

► **Definition 10** (Parallel Dependency Tuples DT^\parallel , PDTs). *For a rewrite rule $\ell \rightarrow r$, we define the set of its Parallel Dependency Tuples (PDTs) $DT^\parallel(\ell \rightarrow r)$: if $\text{Pos}_d(r) = \emptyset$, then $DT^\parallel(\ell \rightarrow r) = \{\ell^\sharp \rightarrow \text{Com}_0\}$; otherwise, $DT^\parallel(\ell \rightarrow r) = \{\ell^\sharp \rightarrow \text{Com}_k(r|_{\pi_1}^\sharp, \dots, r|_{\pi_k}^\sharp) \mid k > 0, \pi_1 > \dots > \pi_k \text{ is a maximal } >\text{-chain in } \text{Pos}_d(r)\}$. For a TRS \mathcal{R} , let $DT^\parallel(\mathcal{R}) = \bigcup_{\ell \rightarrow r \in \mathcal{R}} DT^\parallel(\ell \rightarrow r)$.*

► **Example 11.** For our recursive `size`-rule $\text{lhs} \rightarrow \text{rhs}$, we have $\text{Pos}_d(\text{rhs}) = \{1, 11, 12\}$. The two maximal $>$ -chains are $11 > 1$ and $12 > 1$. With $\text{rhs}|_1 = \text{plus}(\text{size}(l), \text{size}(r))$, $\text{rhs}|_{11} = \text{size}(l)$, and $\text{rhs}|_{12} = \text{size}(r)$, we get the PDTs from Ex. 8.

To connect PDTs with our parallel-innermost rewrite relation $\dashv\vdash \rightarrow \mathcal{R}$, we need the notion of *chain tree*, which is an extension of dependency chains [2], and its complexity.

► **Definition 12** (Chain Tree, *Cplx* [5]). *Let \mathcal{D} be a set of DTs and \mathcal{R} be a TRS. Let T be a (possibly infinite) tree whose nodes are labelled with a DT from \mathcal{D} and a substitution. Let the root node be labelled with $(s^\sharp \rightarrow \text{Com}_n(\dots) \mid \sigma)$. Then T is a $(\mathcal{D}, \mathcal{R})$ -chain tree for $s^\sharp\sigma$ iff we have the following for all nodes of T : if a node is labelled with $(u^\sharp \rightarrow \text{Com}_m(v_1^\sharp, \dots, v_m^\sharp) \mid \mu)$, then $u^\sharp\mu$ is in normal form w.r.t. \mathcal{R} . Moreover, if this node has the children $(p_1^\sharp \rightarrow \text{Com}_{m_1}(\dots) \mid \delta_1), \dots, (p_k^\sharp \rightarrow \text{Com}_{m_k}(\dots) \mid \delta_k)$, then there are pairwise different $i_1, \dots, i_k \in \{1, \dots, m\}$ with $v_{i_j}^\sharp\mu \dashv\vdash^*_{\mathcal{R}} p_j^\sharp\delta_j$ for all $j \in \{1, \dots, k\}$.*

Let $\mathcal{S} \subseteq \mathcal{D}$ and $s^\sharp \in \mathcal{T}^\sharp$. For a chain tree T , $|T|_{\mathcal{S}} \in \mathbb{N} \cup \{\omega\}$ is the number of nodes in T labelled with a DT from \mathcal{S} . We define $\text{Cplx}_{(\mathcal{D}, \mathcal{S}, \mathcal{R})}(s^\sharp) = \sup\{|T|_{\mathcal{S}} \mid T \text{ is a } (\mathcal{D}, \mathcal{R})\text{-chain tree for } s^\sharp\}$. For terms s^\sharp without a $(\mathcal{D}, \mathcal{R})$ -chain tree, we define $\text{Cplx}_{(\mathcal{D}, \mathcal{S}, \mathcal{R})}(s^\sharp) = 0$.

We can now make our main correctness claim:

150 ► **Proposition 13** (*Cplx* bounds Derivation Height for $\parallel \rightarrow_{\mathcal{R}}$). Let \mathcal{R} be a TRS, let $t =$
 151 $f(t_1, \dots, t_n) \in \mathcal{T}(\Sigma, \mathcal{V})$ such that all t_i are in normal form. Then we have $\text{dh}(t, \parallel \rightarrow_{\mathcal{R}}) \leq$
 152 $\text{Cplx}_{\langle DT^{\parallel}(\mathcal{R}), DT^{\parallel}(\mathcal{R}), \mathcal{R} \rangle}(t^{\#})$. If \mathcal{R} is confluent, then $\text{dh}(t, \parallel \rightarrow_{\mathcal{R}}) = \text{Cplx}_{\langle DT^{\parallel}(\mathcal{R}), DT^{\parallel}(\mathcal{R}), \mathcal{R} \rangle}(t^{\#})$.

153 Thus, in particular we can use polynomial interpretations in the DT framework for our
 154 PDTs to get upper bounds for $\text{irc}_{\mathcal{R}}^{\parallel}$.

155 ► **Example 14** (Ex. 8 continued). For our TRS \mathcal{R} computing the size function on trees, we
 156 get the set $DT^{\parallel}(\mathcal{R})$ with the following PDTs:

$$157 \begin{array}{l} \text{plus}^{\#}(\text{Zero}, y) \rightarrow \text{Com}_0 \\ \text{plus}^{\#}(\text{S}(x), y) \rightarrow \text{Com}_1(\text{plus}^{\#}(x, y)) \end{array} \left| \begin{array}{l} \text{size}^{\#}(\text{Nil}) \rightarrow \text{Com}_0 \\ \text{size}^{\#}(\text{Tree}(v, l, r)) \rightarrow \text{Com}_2(\text{size}^{\#}(l), \text{plus}^{\#}(\text{size}(l), \text{size}(r))) \\ \text{size}^{\#}(\text{Tree}(v, l, r)) \rightarrow \text{Com}_2(\text{size}^{\#}(r), \text{plus}^{\#}(\text{size}(l), \text{size}(r))) \end{array} \right.$$

158 The interpretation $\mathcal{P}ol$ from Ex. 7 implies $\text{irc}_{\mathcal{R}}^{\parallel}(n) \in \mathcal{O}(n^2)$. This bound is tight: consider
 159 $\text{size}(t)$ for a comb-shaped tree t where the first argument of Tree is always Zero and the
 160 third is always Nil . The function plus , which needs time linear in its first argument, is called
 161 linearly often on data linear in the size of the start term. Due to the structural dependencies,
 162 these calls do not happen in parallel (so call $k + 1$ to plus must wait for call k).

163 ► **Example 15.** Note that $\text{irc}_{\mathcal{R}}^{\parallel}(n)$ can be asymptotically lower than $\text{irc}(n)$, for instance in:

$$164 \begin{array}{l} \text{doubles}(\text{Zero}) \rightarrow \text{Nil} \\ \text{doubles}(\text{S}(x)) \rightarrow \text{Cons}(\text{d}(\text{S}(x)), \text{doubles}(x)) \end{array} \left| \begin{array}{l} \text{d}(\text{Zero}) \rightarrow \text{Zero} \\ \text{d}(\text{S}(x)) \rightarrow \text{S}(\text{S}(\text{d}(x))) \end{array} \right.$$

165 The upper bound $\text{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n^2)$ is tight: from a term $\text{doubles}(\text{S}(\text{S}(\dots \text{S}(\text{Zero}) \dots)))$,
 166 we get linearly many calls of the linear-time function d on arguments of size linear in the
 167 start term. However, the Parallel Dependency Tuples in this example are:

$$168 \begin{array}{l} \text{doubles}^{\#}(\text{Zero}) \rightarrow \text{Com}_0 \\ \text{doubles}^{\#}(\text{S}(x)) \rightarrow \text{Com}_1(\text{d}^{\#}(\text{S}(x))) \\ \text{doubles}^{\#}(\text{S}(x)) \rightarrow \text{Com}_1(\text{doubles}^{\#}(x)) \end{array} \left| \begin{array}{l} \text{d}^{\#}(\text{Zero}) \rightarrow \text{Com}_0 \\ \text{d}^{\#}(\text{S}(x)) \rightarrow \text{Com}_1(\text{d}^{\#}(x)) \end{array} \right.$$

169 Then the given polynomial interpretation, which orients all DTs with \succ and all rules from
 170 \mathcal{R} with \succ , proves $\text{irc}_{\mathcal{R}}^{\parallel} \in \mathcal{O}(n)$: $\mathcal{P}ol(\text{doubles}^{\#}(x_1)) = \mathcal{P}ol(\text{d}(x_1)) = 2x_1$, $\mathcal{P}ol(\text{d}^{\#}(x_1)) = x_1$,
 171 $\mathcal{P}ol(\text{doubles}(x_1)) = \mathcal{P}ol(\text{Cons}(x_1, x_2)) = \mathcal{P}ol(\text{Zero}) = \mathcal{P}ol(\text{Nil}) = 1$, $\mathcal{P}ol(\text{S}(x_1)) = 1 + x_1$.

172 4 Conclusion

173 We have come up with a notion of parallel runtime complexity and a concrete algorithm to
 174 compute upper bounds on this complexity on TRSs. Future work includes practical design
 175 of *parallel rewriting engines* as well as the formalisation of complexity w.r.t. term *height*
 176 (considering terms as trees), which seems to be more practical for our parallelisation needs.

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