Random walks

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Introduction

For $d \geq 1$, let $(e_i)_{1 \leq i \leq d}$ denote the canonical basis of $\mathbb{R}^d$. The *simple random walk* on $\mathbb{Z}^d$ is the random process $(S_n)_{n \geq 0}$ defined by

$$S_n = X_1 + \cdots + X_n,$$

where the variables X_i are independent and identically distributed with uniform law on

$$\{e_1, -e_1, \ldots, e_d, -e_d\}$$

. This process can also be described as a Markov chain on $\mathbb{Z}^d$ with a very simple transition matrix. In particular, it is irreducible and is recurrent for $d \in \{1, 2\}$ and transient for $d \geq 3$.

The are (at least!) two natural ways to widely generalize this process:

- we can replace the uniform measure on $\{e_1, -e_1, \ldots, e_d, -e_d\}$ by more general distributions on $\mathbb{Z}^d$ or even $\mathbb{R}^d$,
- or, if we view $\mathbb{Z}^d$ as a graph and (S_n) as a process choosing S_{n+1} uniformly at random among the neighbours of S_n.

In probability, *"random walks"* is a generic expression that encompasses both of these generalizations. In both cases, we can ask many natural questions such as recurrence or transience, typical displacement ("how far is S_n from S_0?"), or fine study of the set of points visited by the walk.

The first part of the course will be dedicated to the first generalization, i.e. random walks on $\mathbb{R}^d$. In the second part, we will study random walks on general graphs, which is the second generalization.

Finally, here are some bibliographical references:

- The lecture notes of Curien

Chapter 1

Random walks on $\mathbb{R}^d$

1.1 Recurrence and transience

1.1.1 Definitions and first properties

Let us start by giving the main definition for the first part of the course.

Definition 1. Let $d \geq 1$, and let μ be a probability measure on $\mathbb{R}^d$. A *random walk* on $\mathbb{R}^d$ with step distribution μ is a process $(S_n)_{n\geq 0}$ of the form

$$S_n = X_1 + \cdots + X_n,$$

where the X_i are independent and identically distributed with law μ.

Our goal will be tu understand the behaviour of these processes, and in particular how it depends on the dimension d and the step distribution μ.

Remark 2. It is possible (for example if μ has a density with respect to the Lebesgue measure) that the state space of S is uncountable. Therefore, the general theory of Markov chain does not apply to our setting. Therefore, the main goal of this Section 1.1 will be to reprove some of its results (strong Markov property, recurrence/transience dichotomy) with adapted definitions.

We denote by $\mathcal{F}_n$ the σ-algebra generated by $(X_1, \dots, X_n)$, so that $(\mathcal{F}_n)_{n\geq 0}$ is the natural filtration associated to the process (S_n).

Definition 3.

- A *stopping time* is a random variable $\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$ such that for all $n \geq 0$, we have $\{\tau \leq n\} \in \mathcal{F}_n$.
- For any stopping time τ, we denote by $\mathcal{F}_\tau$ the σ-algebra consisting of all the events A such that for all $n \geq 0$, we have $A \cap \{\tau \leq n\} \in \mathcal{F}_n$.

We can now formulate the strong Markov property.

Proposition 4 (Strong Markov property)**.** Let τ be a stopping time such that $\tau < +\infty$ almost surely, and let $S_n^{(\tau)} = S_{\tau+n} - S_\tau$. Then $S^{(\tau)}$ has the same law as S and is independent from $\mathcal{F}_\tau$.

This Markov property is not an immediate conseqquence of the general one of Markov chains because the state space may be uncountable. However, the proof is exactly the same.

Proof. Let $A \in \mathcal{F}_\tau$ and $k \geq 1$. Let also $f : \mathbb{R}^k \to \mathbb{R}$ be measurable and bounded. We can write

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_A f\left(S_1^{(\tau)}, \dots, S_k^{(\tau)}\right)\right] &= \sum_{n\geq 0} \mathbb{E}\left[\mathbb{1}_{A\cap\{\tau=n\}} f\left(X_{n+1}, \dots, X_{n+1} + \dots + X_{n+k}\right)\right] \\ &= \sum_{n\geq 0} \mathbb{P}\left(A \cap \{\tau = n\}\right) \mathbb{E}\left[f\left(X_{n+1}, \dots, X_{n+1} + \dots + X_{n+k}\right)\right] \\ &= \mathbb{P}(A)\mathbb{E}\left[f\left(S_1, \dots, S_k\right)\right].\end{aligned}$$

It follows that $\left(S_1^{(\tau)}, \dots, S_k^{(\tau)}\right)$ is independent from A for all k, so $S^{(\tau)}$ is independent from $\mathcal{F}_\tau$. Finally, by taking $A = \Omega$, we find that $S^{(\tau)}$ has the same law as S. □

The Hewitt–Savage 0-1 law. The usual Kolmogorv 0-1 law says that any event invariant under changing the values of finitely many increments has probability 0 or 1. Unfortunately, the assumption is too strong to apply to many natural events, such as "hitting an interval infinitely many times". Therefore, the following stronger statement will be very useful to us.

Theorem 1. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with values in a set E. Let $A \in \sigma\left(X_i | i \geq 1\right)$ be an event which is invariant under any permutation of the X_i with finite support (that is, any bijection $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ such that $\sigma(i) = i$ except for finitely many i). Then $\mathbb{P}(A) \in \{0, 1\}$.

Proof. The result is easy to prove if we further assume that A only depends on finitely many coordinates. Therefore, we will approximate A by events satisfying this property.

More precisely, let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. We first claim that there are events $A_n \in \mathcal{F}_n$ such that $\mathbb{P}(A\Delta A_n) \to 0$ as $n \to +\infty$, where the symetric difference $A\Delta A_n$ stands for $(A\backslash A_n) \cup (A_n\backslash A)$. Indeed, the result is obvious if A itself belongs to $\bigcup_n \mathcal{F}_n$. Moreover, it is straightforward to check that the set of events satisfying this approximation property form a σ-algebra[1], so it must be $\mathcal{F}_\infty$.

Now, let us fix $n \geq 1$. We know that any event $B \in \mathcal{F}_\infty$ is of the form

$$\left\{(X_i)_{i\geq 1} \in \widetilde{B}\right\}$$

for some $\widetilde{B} \subset E^{\mathbb{N}^*}$ measurable. We then define the event

$$\phi_n(B) = \left\{(X_{n+1}, \dots, X_{2n}, X_1, \dots, X_n, X_{2n+1}, X_{2n+2}, \dots) \in \widetilde{B}\right\}.$$

Now, if A_n is the approximation of A described above, we can write

$$\mathbb{P}\left(A\Delta\Phi_n(A_n)\right) = \mathbb{P}\left(\phi_n(A)\Delta\phi_n(A_n)\right) = \mathbb{P}\left(\phi(A\Delta A_n)\right) = \mathbb{P}\left(A\Delta A_n\right) \xrightarrow[n\to+\infty]{} 0,$$

where the first equality uses the assumption on A and the third uses that the permuted process

$$(X_{n+1}, \dots, X_{2n}, X_1, \dots, X_n, X_{2n+1}, X_{2n+2}, \dots)$$

has the same law as $(X_i)_{i\geq 1}$. The last equation means that both A_n and $\phi_n(A_n)$ are good approximations of A, so their intersection $A_n \cap \phi_n(A_n)$ also is. More precisely, we have

$$\mathbb{P}\left(A\Delta(A_n \cap \phi_n(A_n))\right) \leq \mathbb{P}\left(A\Delta A_n\right) + \mathbb{P}\left(A\Delta\phi_n(A_n)\right) \xrightarrow[n\to+\infty]{} 0,$$

[1]Contrary to what I said in class, the monotone class theorem is not needed here.

so

$$\mathbb{P}(A) = \lim_{n\to+\infty} \mathbb{P}\left(A_n \cap \phi_n(A_n)\right) = \lim_{n\to+\infty} \mathbb{P}\left(A_n\right)\mathbb{P}\left(\phi_n(A_n)\right) = \mathbb{P}(A)^2,$$

so $\mathbb{P}(A) \in \{0,1\}$. □

Example 5. If $E \subset \mathbb{R}^d$, the event

$$A = \{\text{there are infinitely many times } n \text{ such that } S_n \in E\}$$

satisfies the assumption of the Hewitt–Savage 0-1 law, but not the assumption of the Kolmogorov version. Most of the upcoming applications will be events of this form.

1.1.2 Recurrence and transience

If for example our step distribution μ is included in the lattice $\mathbb{Z}^d$, then we can see (S_n) as a Markov chain on $\mathbb{Z}^d$, which is either recurrent or transient. On the other hand, if for example μ has a density with respect to the Lebesgue measure, then almost surely, we have $S_n \neq 0$ for all n, which would mean that S would be transient regardless of its "large scale" properties if we blindly copy the "discrete" definition. Therefore, it seems to make more sense to replace "visit a point infinitely often" by "approach a point infinitely often" in the definition of recurrence.

More precisely, for $x = (x_i)_{1\leq i\leq d} \in \mathbb{R}^d$, we write $|x| = \max_{1\leq i\leq d} |x_i|$.

Definition 6. Let $x \in \mathbb{R}^d$.

- We say that x is a *possible value* of S if

$$\forall \varepsilon > 0,\ \exists n \geq 0,\ \mathbb{P}\left(|S_n - x| < \varepsilon\right) > 0.$$

- We say that x is a *recurrent value* of S if

$$\forall \varepsilon > 0,\ \mathbb{P}\left(|S_n - x| < \varepsilon \text{ for infinitely many } n\right) > 0.$$

We write $\mathcal{P}_\mu$ for the set of possible values, and $\mathcal{R}_\mu$ for the set of recurrent values.

Remark 7. By the Hewitt–Savage law, the probability appearing in the definition of a recurrent value is either 0 or 1, so x is recurrent if and only if this probability is 1 for any $\varepsilon > 0$.

Example 8. For the simple random walk on $\mathbb{Z}^d$, we would have $\mathcal{P}_\mu = \mathbb{Z}^d$ and $\mathcal{R}_\mu$ is $\mathbb{Z}^d$ for $d \in \{1,2\}$ and is empty for $d \geq 3$.

The goal of the next result is to guarantee the equivalence between natural definitions of a recurrent/transient random walk on $\mathbb{R}^d$.

Theorem 2. Let μ be a probability measure on $\mathbb{R}^d$. Either $\mathcal{R}_\mu$ is empty, or $\mathcal{R}_\mu = \mathcal{P}_\mu$ is a closed subgroup of $\mathbb{R}^d$. We say that S is *transient* in the first case, and *recurrent* in the second.

Proof.

- We first notice that if $x \notin \mathcal{R}_\mu$, there is $\varepsilon > 0$ such that almost surely, the ball of radius ε around x is visited finitely many times. It follows that if $|x - y| < \frac{\varepsilon}{2}$, then the ball of radius $\frac{\varepsilon}{2}$ around y is visited finitely many times, so the complement of $\mathcal{R}_\mu$ is open, so $\mathcal{R}_\mu$ is closed.

- We now assume $\mathcal{R}_\mu \neq \emptyset$. We will show

$$\forall x \in \mathcal{P}_\mu, \, \forall y \in \mathcal{R}_\mu, \, y - x \in \mathcal{R}_\mu. \tag{1.1}$$

  The rough idea behind this equation is that S must approach y even if we condition it to hit a small ball around x, and to notice that by the Markov property, this means that a shifted version of S approaches $y-x$. More precisely, for any $z \in \mathbb{R}^d$ and any $m \geq 1$ and $\delta > 0$, we write

$$p_{\delta,m}(z) = \mathbb{P}\left(\forall n \geq m, \, |S_m - z| \geq \delta\right).$$

  We now fix $m \geq 1$ and $\varepsilon > 0$. Since $x \in \mathcal{P}_\mu$, let $k \geq 0$ be such that $\mathbb{P}\left(|S_k - x| < \varepsilon\right) > 0$. Then we have

$$\begin{aligned} p_{\varepsilon,m+k}(y) &\geq \mathbb{P}\left(|S_k - x| < \varepsilon \text{ and } \forall n \geq k+m, \, |S_n - S_k - (y-x)| \geq 2\varepsilon\right) \\ &= \mathbb{P}\left(|S_k - x| < \varepsilon\right) \times p_{2\varepsilon,m}(y-x) \end{aligned}$$

  by the fact that $(S_{k+i} - S_k)$ is a random walk and is independent of $\mathcal{F}_k$. Since $y \in \mathcal{R}_\mu$, the left-hand side is 0, so the right-hand side as well. But we know that the first factor is positive, so $p_{2\varepsilon,m}(y-x) = 0$ for any $m \geq 1$ and $\varepsilon > 0$, so $y - x \in \mathcal{R}_\mu$, which proves (1.1).

- We now show that $\mathcal{R}_\mu$ is a group: let $x, x' \in \mathcal{R}_\mu$. We first apply (1.1) with $y = x$ to get $0 \in \mathcal{R}_\mu$. We then apply (1.1) a second time with $y = 0$ to obtain $-x, -x' \in \mathcal{R}_\mu$. Finally, a last application of (1.1) with $y = -x'$ gives $-x - x' \in \mathcal{R}_\mu$, so $x + x' \in \mathcal{R}_\mu$.

- Finally, let $x \in \mathcal{P}_\mu$. By (1.1) with $y = 0$, we get $-x \in \mathcal{R}_\mu$, so $x \in \mathcal{R}_\mu$, so $\mathcal{R}_\mu = \mathcal{P}_\mu$ as soon as $\mathcal{R}_\mu \neq \emptyset$.

□

Remark 9. In dimensions 1, a closed subgroup of $\mathbb{R}$ is either $\mathbb{R}$ or $c\mathbb{Z}$ for some $c \geq 0$. In higher dimensions, a closed subgroup is of the form $\mathbb{Z}^i \times R^j \times \{0\}^k$ with $i + j + k = d$, up to the action of $GL_d(\mathbb{R})$.

A first characterization of recurrence/transience. For discrete Markov chain, we know that a point x is recurrent if and only if

$$\sum_{n \geq 0} \mathbb{P}_x\left(S_n = x\right) = +\infty.$$

Indeed, the left-hand side represents the expectation of the number of visits to 0, which is a geometric random variable by the strong Markov propety. Therefore, it is a.s. finite if and only if it has finite expectation. We now prove an analog of this result for random walks on $\mathbb{R}^d$.

Theorem 3. Fix $\varepsilon > 0$. A random walk S on $\mathbb{R}^d$ is transient if and only if

$$\sum_{n \geq 0} \mathbb{P}\left(|S_n| < \varepsilon\right) < +\infty.$$

Proof. We will first prove that the finiteness of the sum in the statement does not depend on the choice of $\varepsilon > 0$ (this is an immediate consequence of our Theorem, but we will need to prove this first). For this, we will prove the following inequality for any $\varepsilon > 0$ and integer $m \geq 1$:

$$\sum_{n \geq 0} \mathbb{P}\left(|S_n| < m\varepsilon\right) \leq (2m)^d \sum_{n \geq 0} \mathbb{P}\left(|S_n| < \varepsilon\right). \tag{1.2}$$

To show this inequality, we will partition the "big cube" $[-m\varepsilon, m\varepsilon)^d$ into $(2m)^d$ little cubes with side length ε (this is where the choice of the norm $|\cdot|$ is important). The little cubes will be indexed by $\mathbf{k} \in \{-m, \dots, m-1\}^d$. We get

$$\sum_{n\geq 0} \mathbb{P}\left(|S_n| < m\varepsilon\right) \leq \sum_{n\geq 0} \sum_{\mathbf{k}\in\{-m,\dots,m-1\}^d} \mathbb{P}\left(S_n \in \varepsilon\mathbf{k} + [0,\varepsilon)^d\right),$$

where the inequality comes from replacing the open ball $(-m\varepsilon, m\varepsilon)^d$ by $[-m\varepsilon, m\varepsilon)$. In order to study the contribution of a fixed $\mathbf{k} \in \{-m, \dots, m-1\}^d$, we write

$$T_{\mathbf{k}} = \inf\{\ell \geq 0 | S_\ell \in \varepsilon\mathbf{k} + [0,\varepsilon)^d\}.$$

We can now write

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{P}\left(S_n \in \varepsilon\mathbf{k} + [0,\varepsilon)^d\right) &= \sum_{n\geq 0} \sum_{\ell=0}^{n} \mathbb{P}\left(S_n \in \varepsilon\mathbf{k} + [0,\varepsilon)^d \text{ and } T_{\mathbf{k}} = \ell\right) \\
&\leq \sum_{\ell\geq 0} \sum_{n\geq \ell} \mathbb{P}\left(T_{\mathbf{k}} = \ell \text{ and } |S_n - S_\ell| < \varepsilon\right) \\
&= \sum_{\ell\geq 0} \sum_{j\geq 0} \mathbb{P}\left(T_{\mathbf{k}} = \ell\right) \mathbb{P}\left(|S_j| < \varepsilon\right) \qquad \leq \sum_{j\geq 0} \mathbb{P}\left(|S_j| < \varepsilon\right),
\end{aligned}$$

where the second line uses that two points in the same little cube must be close, and the third line uses independence. By summing over $\mathbf{k}$, we obtain (1.2). In particular, if the sum is finite for some $\varepsilon > 0$, then it is finite for all $\varepsilon > 0$.

We now finish the proof of the theorem. First, if $\sum_{n\geq 0} \mathbb{P}\left(|S_n| < \varepsilon\right) < +\infty$, then the expected number of vists of S in a certain ball is finite, so the number of visits is a.s. finite, which means that S is transient. On the other hand, if S is transient, we can find $\varepsilon > 0$ small such that

$$p_\varepsilon := \mathbb{P}\left(\forall n \geq 1,\ |S_n| > \varepsilon\right) > 0.$$

As in the usual discrete setting, we will try to compare the number of visits of a certain small ball with a geometric variable with parameter $1 - p_\varepsilon$. For this, we define the stopping times τ_i by $\tau_0 = 0$ and, for $i \geq 0$,

$$\tau_{i+1} = \inf\{n > \tau_i \,\big|\, |S_n| < \frac{\varepsilon}{2}\}.$$

For any i, we have

$$\mathbb{P}\left(\tau_{i+1} = +\infty | \tau_i < +\infty\right) \geq \mathbb{P}\left(\forall n \geq 1,\ \left|S_n^{(\tau_i)}\right| > \varepsilon\right) = p_\varepsilon,$$

where the inequality comes from the triangle inequality and the last equality from the strong Markov property. By induction on i, this becomes

$$\mathbb{P}\left(\tau_i < +\infty\right) \leq (1 - p_\varepsilon)^i$$

for all $i \geq 0$, so

$$\sum_{n\geq 0} \mathbb{P}\left(|S_n| < \varepsilon\right) = \mathbb{E}\left[\sum_{i\geq 0} \mathbb{1}_{\tau_i < +\infty}\right] < +\infty.$$

This proves the finiteness of the sum for some $\varepsilon > 0$, and the first part of the proof extends this to all $\varepsilon > 0$. □

1.1.3 The one-dimensional case

We conclude this section with a more precised study of the one-dimensional case. Note that the projection of a random walk on $\mathbb{R}^d$ on any of its coordinates is a one-dimensional random walk, so the results below also say something about higher-dimensional walks. We now assume $d = 1$ and $\mu \neq \delta_0$.

Proposition 10. One of the following holds almost surely:

1. $S_n \xrightarrow[n\to+\infty]{} +\infty$,
2. $S_n \xrightarrow[n\to+\infty]{} -\infty$,
3. $\liminf_{n\to+\infty} S_n = -\infty$ and $\limsup_{n\to+\infty} S_n = +\infty$.

We say that S drifts to $+\infty$ in case 1, that S drifts to $+\infty$ in case 2 and that S oscillates in case 3.

Proof. By the Kolmogorov 0-1 law, each of the three cases above has probability 0 or 1, so it is enough to show that at least one occurs. If this is not the case, note that it means that either $\liminf S$ or $\limsup S$ is finite. Note that $\liminf S$ and $\limsup S$ are both deterministic by the Hewitt–Savage 0-1 law. Without loss of generality, assume that $\liminf S = c \in \mathbb{R}$ almost surely. Then $\mathcal{R}_\mu$ contains c, so it is nonempty, so $\mathcal{R}_\mu = \mathcal{P}_\mu$ is a subgroup of $\mathbb{R}$. But on the other hand $\mathcal{R}_\mu$ is bounded from below by c, so $\mathcal{R}_\mu = \{0\}$, so $\mathcal{P}_\mu = 0$. This implies $\mu = \delta_0$, which was excluded. □

Remark 11. A recurrent walk must oscillate, but a transient walk can either drift to $+\infty$ or $-\infty$, or oscillate (see examples below).

One-dimensional random walks with finite expectation. In the case of walks witgh finite expectation, the law of large numbers will allow us to get a very simple criterion for recurrence.

Theorem 4. Assume that $\mathbb{E}[|X_1|] < +\infty$.

- If $\mathbb{E}[X_1] > 0$, then S is transient and drifts to $+\infty$.
- If $\mathbb{E}[X_1] < 0$, then S is transient and drifts to $-\infty$.
- If $\mathbb{E}[X_1] = 0$, then S is recurrent and oscillates.

Proof. The transient case follows directly from the law of large numbers: we have $\frac{S_n}{n} \to \mathbb{E}[X_1] > 0$ almost surely, so $S_n > 1$ for n large enough, and similarly for the negative case.

For the recurrent case, a bit more work is needed. Roughly speaking, the argument will be that a walk with mean 0 at time n must be in $[-\varepsilon n, \varepsilon n]$ by the law of large numbers. On the other hand, if it is transient, it cannot spend too much time in a given interval of length 1. We will get a contradiction by cutting $[-\varepsilon n, \varepsilon n]$ into small intervals of length 1.

More precisely, let $\varepsilon > 0$. By the law of large number, we know that for n large enough (say, $n \geq N_0$ where N_0 is random), we have $|S_n| \leq \varepsilon n$. It follows that, for n large enough to have

$$n \geq \max\left(N_0, |S_1|/\varepsilon, \dots, |S_{N_0}|/\varepsilon\right),$$

we have

$$\forall 0 \leq i \leq n,\ |S_i| \leq \varepsilon n.$$

Therefore, with probability going to 1 as $n \to +\infty$, we have

$$\sum_{i=0}^{+\infty} \mathbb{1}_{|S_i| \leq \varepsilon n} \geq n,$$

so for n large enough, we can write

$$\mathbb{E}\left[\sum_{i=0}^{+\infty} \mathbb{1}_{|S_i| \leq \varepsilon n}\right] \geq \frac{n}{2}. \tag{1.3}$$

On the other hand, assume that S is transient. Then we can write $\sum_{i \geq 0} \mathbb{P}\left(|S_i| < 1\right) \leq C$ for some constant C (which, crucially, does not depend on ε). Therefore, for all $m \in \mathbb{Z}$, let $\tau_m = \inf\{i \geq 0 | S_i \in [m, m+1]\}$. By applying the strong Markov property to τ_m, we can write

$$\sum_{i \geq 0} \mathbb{P}\left(S_i \in [m, m+1]\right) = \mathbb{E}\left[\sum_{i \geq 0} \mathbb{1}_{S_i \in [m, m+1]}\right] \leq \mathbb{P}\left(\tau_m < +\infty\right) \mathbb{E}\left[\sum_{i \geq 0} \mathbb{1}_{|S_i| < 1}\right] \leq C.$$

Finally, by summing over $-\varepsilon n - 1 \leq m \leq \varepsilon n$, we find

$$\sum_{i \geq 0} \mathbb{P}\left(|S_i| \leq \varepsilon n\right) \leq (2\varepsilon n + 2)C.$$

This contradicts (1.3) if $\varepsilon > 0$ was chosen small enough. □

Heavy-tailed walks. We now want to say a few things about the case where $\mathbb{E}[|X_1|] = +\infty$, which is actually quite rich. We will first focus on the case where X is symetric and has a polynomial tail.

Proposition 12. Let μ be a probability measure on $\mathbb{Z}$ such that for any $k \geq 0$, we have $\mu(k) = \mu(-k) \sim k^{-\alpha}$ as $k \to +\infty$, where $\alpha \in (1, 2)$. Then the random walk S on $\mathbb{R}$ with step distribution μ is transient and oscillates.

Remark 13. We must have $\alpha > 1$ for the measure to be finite, and $\alpha < 2$ implies $\mathbb{E}[|X_1|] = +\infty$ (the case $\alpha = 2$ is a "limit" case).

Proof. First, note that S has the same law as $-S$ so S cannot drift to $+\infty$ or $-\infty$, so it oscillates. To prove transience, we will bound the probability $\mathbb{P}(S_n = 0)$ by using the fact that S makes at least one "large" jump before time n. More precisely, we fix $\varepsilon > 0$ and define

$$\tau = \inf\{i \geq 1 \big| |X_i| > n^{1+\varepsilon}\}.$$

By conditioning on τ, we can write

$$\mathbb{P}\left(S_n = 0\right) \leq \mathbb{P}\left(\tau > n\right) + \mathbb{E}\left[\mathbb{P}\left(S_n = 0 | \tau\right) \mathbb{1}_{\tau \leq n}\right]. \tag{1.4}$$

The first term is easy to bound: there is a constant $c > 0$ such that we have

$$\mathbb{P}\left(\tau > n\right) = \left(1 - \mathbb{P}\left(|X_1| \leq n^{1+\varepsilon}\right)\right)^n \leq \exp\left(-n \times (c + o(1))(n^{1+\varepsilon})^{1-\alpha}\right) \leq \exp(-n^{\varepsilon})$$

if ε was chosen small enough. This is summable. On the other hand, let us study the conditional distribution of $(X_i)_{i\leq 1}$ conditionally on τ. We find that the steps (X_i) are still independent and X_i has conditional law μ for $i > \tau$ and μ conditionned on being smaller than $n^{1+\varepsilon}$ for $i < \tau$. Finally, the conditional law of X_τ given τ is given by

$$\widetilde{\mu}(i) = \mathbb{1}_{|i|>n^{1+\varepsilon}} \frac{\mu(i)}{\sum_{|j|>n^{1+\varepsilon}} \mu(j)} \leq \frac{C(n^{1+\varepsilon})^{-\alpha}}{c\left(n^{1+\varepsilon}\right)^{1-\alpha}} = O\left(n^{-1-\varepsilon}\right)$$

for some constants $c, C > 0$. Therefore, if $\tau \leq n$, we can write

$$\mathbb{P}\left(S_n = 0|\tau\right) = \mathbb{P}\left(X_\tau = -(X_1 + \cdots + X_{\tau-1} + X_{\tau+1} + \cdots + X_n)|\tau \leq n\right) \leq \sup_i \widetilde{\mu}(i) = O\left(n^{-1-\varepsilon}\right).$$

By taking the expectation, we find that the second term of (1.4) is summable, so S is transient.

□

Remark 14. It is quite natural to ask what is the "slowest possible" rate of decrease for which the walk is still recurrent. It turns out that this question is quite difficult, and that some very irregular choices of μ can give rise to surprising counter-examples. For example, it was proved by Shepp [1] that for any function $\varepsilon(x) \to 0$ as $x \to +\infty$, we can find a symetric measure μ on $\mathbb{Z}$ such that $\mu\left(\mathbb{R}\backslash[-x, x]\right) \geq \varepsilon(x)$ for x large enough, but the associated random walk is still recurrent.

The Cauchy random walk. Finally, for symetric, heavy-tailed random walk with a "regular behaviour" like $\mu_k \sim k^{-\alpha}$, we have seen that the walk is recurrent for $\alpha > 2$ and transient for $\alpha < 2$. We conclude with a specific example which belongs to the "limiting" case $\alpha = 2$: the *Cauchy* random walk.

We denote by (B_t^1, B_t^2) a two-dimensional Brownian motion started from 0. We define $\tau_1 = \inf\{t \geq 0 | B_t^1 = 1\}$ and $X = B_{\tau_1}^2$. In other words, the variable X describes the point at which B first hits the vertical line $\{x = 1\}$. Then we can compute that X has density $\frac{1}{\pi(1+x^2)}$ on $\mathbb{R}$ with respect to the Lebesgue measure (this is the *Cauchy* distribution), which in particular implies $\mathbb{E}[|X|] = +\infty$.

Moreover, let $(X_i)_{i\geq 1}$ be i.i.d. copies of X. By applying the strong Markov property of Brownian motion to the hitting times τ_n of the vertical lines $\{x = n\}$, we find that $S_n = X_1 + \cdots + X_n$ follows the same law as $B_{\tau_n}^2$. By invariance of Brownian motion under scaling, this has the same law as nX_1, which shows

$$\mathbb{P}\left(|S_n| < 1\right) = \sum_{n\geq 1} \mathbb{P}\left(|X_1| < \frac{1}{n}\right) \underset{n\to+\infty}{\sim} \frac{1}{\pi n}.$$

This is summable, so the random walk S is recurrent. Moreover, for any fixed constant c, the same argument applies if we replace X by $X - c$. Therefore, we have built a one-dimensional random walk S such that for any $c \in \mathbb{R}$, the walk $(S_n - cn)_{n\geq 0}$ is recurrent. This means that for any c, there are infinitely many n such that S_n is "close" to cn, which is quite counterintuitive when compared to the finite expectation case!

Stable random variables. Cauchy variables are a particular case of the following class of random variables.

Definition 15. Let $\alpha \in (0,2]$. An α-stable law μ on $\mathbb{R}$ is a law such that if $X_1, \dots, X_n$ are i.i.d. with law μ, then $X_1 + \cdots + X_n$ has the same law as $n^{1/\alpha} X_1$.

Example 16. The most famous example is the Gaussian distribution ($\alpha = 2$). The Cauchy distribution is also stable with $\alpha = 1$, and so is the law of the hitting time of 1 for a Brownian motion (with $\alpha = 1/2$). These are the only examples of stable distributions with explicit densities with respect to Lebesgue. Stable distributions replace the Gaussian one in the central limit theorem when the variables have a heavy polynomial tail.

1.2 Characteristic function and applications

1.2.1 Motivation

Let μ be a probability measure on $\mathbb{R}^d$. Let $(X_i)_{i\geq 1}$ be i.i.d. variables with distribution μ and let $(S_n)_{n\geq 0} = (X_1 + \cdots + X_n)_{n\geq 0}$ be the corresponding random walk. For $\xi \in \mathbb{R}^d$, we define

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} \mu(\mathrm{d}x) = \mathbb{E}\left[e^{i\xi\cdot X}\right], \tag{1.5}$$

where X has law μ. The function $\hat{\mu}$ is called the *characteristic function* of μ. In particular, we have $\hat{\mu}(0) = 1$ and $|\hat{\mu}(\xi)| \leq 1$ for all $\xi \in \mathbb{R}^d$. Moreover, if $\mathbb{E}\left[|X|\right] < +\infty$, then differentiating (1.5) gives

$$\hat{\mu}(\xi) = 1 + i\xi \cdot \mathbb{E}[X] + o(|\xi|) \quad \text{as } \xi \to 0.$$

Similarly, if $\mathbb{E}[|X|^2] < +\infty$, by differentiating (1.5) twice, we find

$$\hat{\mu}(\xi) = 1 + i\xi \cdot \mathbb{E}[X] - \frac{\xi^t Q \xi}{2} + o\left(|\xi|^2\right) \quad \text{as } \xi \to 0,$$

where Q is the covariance matrix of X, i.e. $Q_{i,j} = \mathbb{E}\left[X_1^{(i)} X_1^{(j)}\right]$.

The reason why this characteristic function will play an important role for us is that we can easily access the characteristic function of S_n: for all $n \geq 0$ and $\xi \in \mathbb{R}^d$, we have

$$\mathbb{E}\left[e^{i\xi\cdot S_n}\right] = \mathbb{E}\left[e^{i\xi\cdot X_1} \dots e^{i\xi\cdot x_n}\right] = \hat{\mu}(\xi)^n.$$

Since the characteristic function characterizes the distribution, this is in theory sufficient to recover any information about the law of S_n. As an example, let us give an exact formula for random walks on $\mathbb{Z}^d$.

Lemma 17. If $\mu(\mathbb{Z}^d) = 1$, then for all $n \geq 0$ and $x \in \mathbb{Z}^d$, we have

$$\mathbb{P}\left(S_n = x\right) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} e^{-i\xi\cdot x} \hat{\mu}(\xi)^n \mathrm{d}\xi.$$

Proof. The result is an example of application of the Fourier inversion formula. More precisely, let us compute the right-hand side using the Fubini theorem:

$$\begin{aligned}\frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} e^{-i\xi\cdot x} \mathbb{E}\left[e^{i\xi\cdot S_n}\right] \mathrm{d}\xi &= \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \sum_{y\in\mathbb{Z}^d} e^{-i\xi\cdot x} e^{i\xi\cdot y} \mathbb{P}(S_n = y) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d} \sum_{y\in\mathbb{Z}^d} \mathbb{P}(S_n = y) \int_{(-\pi,\pi)^d} e^{i\xi\cdot(y-x)} \mathrm{d}\xi.\end{aligned}$$

The integral is 0 if $x \neq y$ and $(2\pi)^d$ if $x = y$, so we finally find $\mathbb{P}(S_n = x)$. □

Our goals in the enxt pages will be to turn this formula into a criterion for recurrence or transience, and more generally into quantitative estimates on the quantity $\mathbb{P}(S_n = x)$.

1.2.2 The Chung–Fuchs criterion

We will prove the following general result for random walks on $\mathbb{R}^d$, which gives a characterization of recurrent random walks in terms of the characteristic function of μ.

Theorem 5. Let S be a random walk on $\mathbb{R}^d$ with step distribution μ, and fix $\alpha > 0$. Then S is recurrent if and only if

$$\lim_{r\to 1} \int_{(-\alpha,\alpha)^d} \mathrm{Re}\left(\frac{1}{1 - r\hat{\mu}(\xi)}\right) = +\infty. \tag{1.6}$$

Proof for $\mu(\mathbb{Z}^d)$ and $\alpha = \pi$. As suggested by Lemma 17, the proof is simpler for random waks with integer values. Indeed, using Lemma 17, for $|r| < 1$, we have

$$\sum_{n\geq 0} r^n \mathbb{P}\left(S_n = 0\right) = \frac{1}{(2\pi)^d} \sum_{n\geq 0} \int_{(-\pi,\pi)^d} r^n \hat{\mu}(\xi)^n \mathrm{d}\xi = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} \frac{\mathrm{d}\xi}{1 - r\hat{\mu}(\xi)},$$

where we could use the Fubini formula because $|r| < 1$. The left-hand side is real, so we can take the real part on the right-hand side. When $r \to 1$, the left-hand side goes to $\sum_{n\geq 0} \mathbb{P}(S_n = 0)$. We know that S is recurrent if and only if this limit is infinite and the result follows. □

Remark 18. Our proof does actually better: in the transient case (1.6) gives the expected number of vists of S at 0. Since this is a geometric random variable by the strong Markov property, by taking the inverse, we recover the parameter of this geometric variable, which also the probability for the walk to never come back to 0. For example, for the simple random walk on $\mathbb{Z}^3$, we find

$$\begin{aligned}
\mathbb{P}\left(\forall n \geq 1,\, S_n \neq 0\right) &= \frac{(2\pi)^3}{3} \left(\int_{(-\pi,\pi)^3} \frac{\mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_3}{3 - \cos\xi_1 - \cos\xi_2 - \cos\xi_3} \right)^{-1} \\
&= \frac{256\pi^6}{\sqrt{6}\,\Gamma(1/24)\,\Gamma(5/24)\,\Gamma(7/24)\,\Gamma(11/24)} \\
&\approx 0.3405...
\end{aligned}$$

The first inequality follows directly from the proof, and I have no idea how to obtain the second.

We now move on to the proof for the general case. We will only treat the case $d = 1$ in details, but the general case is very similar. We want to estimate the sum $\sum_{n\geq 0} \mathbb{P}\left(|S_n| < x\right)$ for some fixed $x > 0$. The reason why the argument for $\mathbb{Z}^d$ cannot be directly adapted is that the Fourier transform of the indicator $\mathbb{1}_{|x|<1}$ is not integrable. Therefore, it is not directly possible to compute $\mathbb{P}\left(|S_n| < x\right)$ using Fourier inversion as in Lemma 17. In order to solve this problem, we will approximate this indicator by a smoother function. Therefore, we fiw a constant $\beta > 0$ and for $x \in \mathbb{R}$ we write

$$f(x) = \left(1 - \frac{|x|}{\beta}\right)^+.$$

Lemma 19. • For any $t \in \mathbb{R}$, we have

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{itx} \, \mathrm{d}x = \beta \left(\frac{\sin \beta t/2)}{\beta t/2} \right)^2.$$

• For any $x \in \mathbb{R}$, we have

$$\hat{\hat{f}}(x) = \int_{\mathbb{R}} \hat{f}(t) e^{itx} \, \mathrm{d}t = 2\pi f(x).$$

Proof. The first item is a direct computation using an integration by parts, and the second item follows from the Fourier inversion formula, which applies because $\hat{f}$ is integrable. □

Since we are only interested in the finiteness of an integral, we can afford to lose constant factors, so we will crudely bound $\mathbb{1}_{|x|<\beta}$ by either $\hat{f}$ (for the upper bound) or f (for the lower bound).

More precisely, let us fix $\beta > 0$. There is a constant $c > 0$ such that fro all x, we have $\mathbb{1}_{|x|<1/\beta} \leq c\hat{f}(x)$, so we can write

$$\mathbb{P}\left(|S_n| < \frac{1}{\beta} \right) \leq c\mathbb{E}\left[\hat{f}(S_n) \right] = c \int_{\mathbb{R}} \mathbb{E}\left[e^{itS_n} \right] f(t) \, \mathrm{d}t = c \int_{\mathbb{R}} f(t) \hat{\mu}(t)^n \, \mathrm{d}t,$$

where we could use the Fubini formula because f is integrable. We can now take the real part of the right-hand side, multiply by r^n for some $|r| < 1$ and sum over n to get

$$\sum_{n \geq 0} r^n \mathbb{P}\left(|S_n| < \frac{1}{\beta} \right) \leq c \int_{\mathbb{R}} \mathrm{Re}\left(\frac{f(t)}{1 - r\hat{\mu}(t)} \right) \mathrm{d}t \leq c \int_{-\beta}^{\beta} \mathrm{Re}\left(\frac{1}{1 - r\hat{\mu}(t)} \right) \mathrm{d}t,$$

where in the end we used that f has compact support in $[-\beta, \beta]$ (this is where it is important that we bounded the indicator using $\hat{f}$ and not f). In particular, if the walk S is recurrent, the left-hand side goes to $+\infty$ as $r \to 1$, so the right-hand side also does, which proves one direction of the criterion by taking $\beta = \alpha$.

For the other direction, we need to bound the indicator from *below*, so we write $\mathbb{1}_{|x|<\beta} \geq f(x)$ (the lower bound must have compact support, so it is important to use f and not $\hat{f}$).We can then write

$$\mathbb{P}\left(|S_n| < \beta \right) \geq \mathbb{E}\left[f(S_n) \right] = \frac{1}{2\pi} \mathbb{E}\left[\hat{\hat{f}}(S_n) \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) \hat{\mu}(t) \, \mathrm{d}t.$$

Again, we can take the real part, sum over n and apply Fubini to get, for $|r| < 1$:

$$\sum_{n \geq 0} r^n \mathbb{P}\left(|S_n| < \beta \right) \geq \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{Re}\left(\frac{\hat{f}(t)}{1 - r\hat{\mu}(t)} \right) \mathrm{d}t.$$

The function that we integrate is nonnegative and is bounded from below by a constant $c > 0$ on the interval $\left[-\frac{1}{\beta}, \frac{1}{\beta} \right]$, so we can write

$$\sum_{n \geq 0} r^n \mathbb{P}\left(|S_n| < \beta \right) \geq \frac{c}{2\pi} \int_{-1/\beta}^{1/\beta} \mathrm{Re}\left(\frac{1}{1 - r\hat{\mu}(t)} \right) \mathrm{d}t.$$

If the walk is transient, the left-hand side is bounded as $r \to 1$, so the right-hand side as well, which proves the second direction by taking $\beta = \frac{1}{\alpha}$.

Finally, the proof in any dimension $d \geq 1$ is exactly the same. We just need to replace the function $t \to f(t)$ on $\mathbb{R}$ by the function $(t_1, \ldots, t_d) \to \prod_{i=1}^{d} f(t_i)$ on $\mathbb{R}^d$, and all the proof adapts well.

Strong Chung–Fuchs criterion. We could wonder if the additional parameter r is really necessary in the Chung–Fuchs criterion. It turns out that we can get rid of it, but then the proof becomes much more complicated, and out of the scope of this course.

Theorem 6. Let $\alpha > 0$. The random walk S is recurrent if and only if

$$\int_{(-\alpha,\alpha)^d} \mathrm{Re}\left(\frac{1}{1-\hat{\mu}(\xi)}\right) \mathrm{d}\xi = +\infty.$$

Note that if S is transient, since $\mathrm{Re}\left(\frac{1}{1-r\hat{\mu}(\xi)}\right) \geq 0$, the Fatou lemma shows

$$\int_{(-\alpha,\alpha)^d} \mathrm{Re}\left(\frac{1}{1-\hat{\mu}(\xi)}\right) \mathrm{d}\xi \leq \liminf_{r\to 1} \int_{(-\alpha,\alpha)^d} \mathrm{Re}\left(\frac{1}{1-r\hat{\mu}(\xi)}\right) \mathrm{d}\xi < +\infty,$$

so the reverse implication is easy. The difficult implication is the direct one, and the proof can be found in [2, Chapter 8]. It is not purely Fourier-analytic, and relies on additional random walk estimates.

1.2.3 Applications

We can now apply the Chung–Fuchs criterion to obtain general results about the behaviour of random walks depending on the dimension.

Theorem 7.
- For $d = 1$, if $\mathbb{E}[|X|] < +\infty$ and $\mathbb{E}[X] = 0$, then S is recurrent.
- For $d = 2$, if $\mathbb{E}\left[|X|^2\right] < +\infty$ and $\mathbb{E}[X] = 0$, then S is recurrent.
- For $d \geq 3$, if the step distribution μ is not supported by a linear hyperplane of $\mathbb{R}^d$, then S is transient.

Note that for the one-dimensionl part, we recover a result proved in Section 1.1.3.

Proof.
- Let us start with the one-dimensional statement. We write $\hat{\mu}(t) = a(t) + ib(t)$ with $a(t), b(t) \in [-1, 1]$. Then we have

$$\mathrm{Re}\left(\frac{1}{1-r\hat{\mu}(t)}\right) = \frac{1-r\cdot a(t)}{(1-r\cdot a(t))^2 + r^2 b(t)^2} \geq \frac{1-r}{(1-r\cdot a(t))^2 + r^2 b(t)^2}.$$

But since $\mathbb{E}[X] = 0$, we know that $\hat{\mu}(t) = 1 + o(t)$ as $t \to 0$, so let us fix $\varepsilon > 0$. Then we can find $\delta > 0$ such that for $0 \leq t \leq \delta$, we have $|1 - a(t)| \leq \varepsilon t$ and $|b(t)| \leq \varepsilon t$. This implies

$$(1 - r\cdot a(t))^2 = ((1-r) + r(1-a(t)))^2 \geq 2(1-r)^2 + 2\varepsilon^2 r^2 t^2,$$

so

$$\mathrm{Re}\left(\frac{1}{1-r\hat{\mu}(t)}\right) \geq \frac{1-r}{2(1-r)^2 + 3\varepsilon^2 r^2 t^2}.$$

Since $\mathrm{Re}\left(\frac{1}{1-r\hat{\mu}(t)}\right) \geq 0$ for all $t \in \mathbb{R}$, this becomes

$$\begin{aligned}\int_1^{-1} \mathrm{Re}\left(\frac{1}{1-r\hat{\mu}(t)}\right) \mathrm{d}t &\geq \int_0^\delta \frac{1-r}{2(1-r)^2 + 3\varepsilon^2 r^2 t^2} \mathrm{d}t \\ &= \frac{1}{1-r} \int_0^\delta \frac{\mathrm{d}t}{2 + 3\left(\frac{t\varepsilon r}{1-r}\right)^2} \\ &= \frac{1}{1-r} \int_0^{\frac{\delta\varepsilon r}{1-r}} \frac{1}{2+3u^2} \frac{1-r}{\varepsilon r} \mathrm{d}u.\end{aligned}$$

As $r \to 1$, this converges to $\frac{1}{\varepsilon} \int_0^{+\infty} \frac{\mathrm{d}u}{2+3u^2}$. Since this holds for any $\varepsilon > 0$, we have proved that the left-hand side goes to $+\infty$ as $r \to 1$.

- We move on to the two-dimensional part. We assume $\mathbb{E}\left[|X|^2\right] < +\infty$, and denote by Q the covariance matrix of X. By the Fatou lemma, we have

$$\liminf_{r \to 1} \int_{(-\pi,\pi)^2} \mathrm{Re}\left(\frac{1}{1 - r\hat{\mu}(t)}\right) \mathrm{d}\xi \geq \int_{(-\pi,\pi)^2} \mathrm{Re}\left(\frac{1}{1 - \hat{\mu}(\xi)}\right) \mathrm{d}\xi.$$

When $\xi \to 0$, we know that $\frac{1}{1-\hat{\mu}(\xi)} \sim \frac{2}{\xi^t Q \xi}$, so $\mathrm{Re}\left(\frac{1}{1-\hat{\mu}(\xi)}\right) \sim \frac{2}{\xi^t Q \xi}$. In particular, there is $c > 0$ such that $\mathrm{Re}\left(\frac{1}{1-\hat{\mu}(\xi)}\right) \geq \frac{c}{|\xi|^2}$ for all $\xi \in (-\pi, \pi)$. Therefore, we can write

$$\int_{(-\pi,\pi)^2} \mathrm{Re}\left(\frac{1}{1 - \hat{\mu}(\xi)}\right) \mathrm{d}\xi \geq \int_{B(0,1)} \frac{\mathrm{d}\xi}{|\xi|^2} = \int_0^1 \frac{2\pi s \, \mathrm{d}s}{s^2} = +\infty,$$

where the second integral is over the ball of radius 1 around 0 and the third is obtained by taking polar coordinates. This concludes the proof.

- We now prove transience for $d \geq 3$. For this, we will need a general estimate showing that the characteristic function does not stay too close to 1.

Lemma 20. Assume that the support of μ is not included in an affine hyperplane. Then there is a constant $c > 0$ such that, for ξ in a neighbourhood of 0, we have

$$|\hat{\mu}(\xi)| \leq 1 - c|\xi|^2.$$

Proof. We first prove the lemma in the case where μ has bounded support. We can then denote by m its mean and by Q its covariance matrix. As $\xi \to 0$, we have

$$\hat{\mu}(\xi) = 1 + im \cdot \xi - \frac{\xi^t Q \xi}{2} + o\left(|\xi|^2\right).$$

It follows that

$$\begin{aligned}
|\hat{\mu}(\xi)|^2 &= 1 - \xi^t Q \xi + |m \cdot \xi|^2 + o\left(|\xi|^2\right) \\
&= 1 - \sum_{i,j=1}^{d} \xi_i \xi_j \mathbb{E}\left[X^{(i)} X^{(j)}\right] + \mathbb{E}[X \cdot \xi]^2 + o\left(|\xi|^2\right) \\
&= 1 - \mathrm{Var}(X \cdot \xi) + o\left(|\xi|^2\right) \\
&= 1 - |\xi|^2 \mathrm{Var}\left(X \cdot \frac{\xi}{|\xi|}\right) + o\left(|\xi|^2\right).
\end{aligned}$$

However, the support of μ is not included in any hyperplane. this means that for any u with $|u| = 1$, the variable $X \cdot u$ is not deterministic so $\mathrm{Var}(X \cdot u) > 0$. Therefore, we have

$$|\hat{\mu}(\xi)| \leq 1 - \frac{|\xi|^2}{2} \min_{|u|=1} \mathrm{var}(X \cdot u) + o\left(|\xi|^2\right),$$

where $\min_{|u|=1} \mathrm{var}(X \cdot u) > 0$ by compactness, which proves the lemma if μ is compactly supported.

For a general μ, let $A \subset \mathbb{R}^d$ be a bounded subset such that the restriction of μ to A is not supported by a hyperplane. This implies $\mu(A) > 0$, so we can define the probability measure $\mu_A = \frac{\mu(A\cap \cdot)}{\mu(A)}$ on A. Then can write

$$\begin{aligned}
|\hat{\mu}(\xi)| &= |\mu(A)\hat{\mu}_A(\xi) + \mu(A^c)\hat{\mu}_{A^c}(\xi)| \\
&\leq \mu(A)\,|\hat{\mu}_A(\xi)| + (1-\mu(A))\,|\hat{\mu}_{A^c}(\xi)| \\
&\leq \mu(A)\left(1 - c|\xi|^2\right) + 1 - \mu(A) \\
&= 1 - c\mu(A)|\xi|^2,
\end{aligned}$$

where we applied our lemma to the compactly supported measure μ_A. □

We can now conclude the proof of transience for $d \geq 3$. We know that there is $c > 0$ such that for any ξ in some neighbourhood of 0 and for any $0 < r < 1$, we have

$$\mathrm{Re}\left(\frac{1}{1 - r\hat{\mu}(\xi)}\right) = \frac{\mathrm{Re}\,(1 - r\hat{\mu}(\xi))}{\mathrm{Re}\,(1 - r\hat{\mu}(\xi))^2 + \mathrm{Im}\,(1 - r\hat{\mu}(\xi))^2} \leq \frac{1}{\mathrm{Re}\,(1 - r\hat{\mu}(\xi))}.$$

We know that there is a neighbourhood of 0 where Re $(\hat{\mu}(\xi)) > 0$. In this neghbourhood, we can therefore write

$$\mathrm{Re}\left(\frac{1}{1 - r\hat{\mu}(\xi)}\right) \leq \frac{1}{\mathrm{Re}\,(1 - \hat{\mu}(\xi))} = \frac{1}{1 - Re\,(\hat{\mu}(\xi))} \leq \frac{1}{1 - |\hat{\mu}(\xi)|} \leq \frac{1}{|\xi|^2}.$$

This is integrable in a neighbourhood of 0 for $d \geq 3$, which proves transience in the case where μ is not supported in any affine hyperplane.

Finally, if μ is supported in an affine hyperplane which is not linear, there is a vector $v \in \mathbb{R}^d$ and a constant $a \neq 0$ such that $v \cdot X_1 = a$ almost surely. It follows that $v \cdot S_n = an$ almost surely for all n, so $|S_n| \to +\infty$ and the walk is transient.

□

Bibliography

[1] L. A. Shepp. Recurrent random walks with arbitrarily large steps. *Bulletin of the American Mathematical Society*, 70(4):540 – 542, 1964.

[2] F. Spitzer. *Principles of Random Walk*. Graduate texts in Mathematics. Springer, 1964.