Random Walks
ENS de Lyon, 2024-2025

Random walks: Final exam

2024, December 16th

Course notes are allowed. You can write in English or in French.

Exercise 1 (6 points) We denote by $\mathbb{T}_3$ the infinite tree where all vertices have degree 3, and by d the graph distance between vertices of v. Let S be the simple random walk on $\mathbb{T}_3$. Let v_0 be a fixed vertex of $\mathbb{T}_3$. For any vertex v of $\mathbb{T}_3$, we write

$$h(v) = \mathbb{P}_v \left(\forall n \geq 0,\ S_n \neq v_0 \right).$$

1. Check that h is harmonic on $\mathbb{T}_3 \backslash \{v_0\}$ and that $h(v)$ only depends on $d(v, v_0)$.
2. Using the transience of $\mathbb{T}_3$ and the weak Markov property, show that $\sup_{v \in \mathbb{T}_3} h(v) = 1$.
3. Conclude that for any vertex v, we have

$$h(v) = 1 - \frac{1}{2^{d(v,v_0)}}.$$

4. We denote by $\widetilde{S}$ the random walk S started from $v_1 \neq v_0$ and conditionned on never hitting v_0. Show that $\widetilde{S}$ is a Markov chain and compute its transitions.
5. Let $S^{(1)}$ and $S^{(2)}$ be two independent simple random walks on $\mathbb{T}_3$ started from the same vertex. We denote by $\mathcal{I}$ the set of vertices visited by both $S^{(1)}$ and $S^{(2)}$. Show that $\mathcal{I}$ is almost surely finite.

Exercise 2 (5 points) Let $d \geq 1$ and let G be a graph with vertex degrees bounded by d. We fix a vertex a of G. Let S be a (possibly infinite) set of edges of G such that:

- each vertex of G is the endpoint of at most one edge of S,
- a is not the endpoint of an edge of S.

Finally, let G' be the graph obtained from G by contracting each edge of S into one vertex. In particular, we can identify the edges of G' with the edges of G that are not in S. For any edge e of G, we will denote by $N(e)$ the set of edges of G which have one common endpoint with e. The goal is to prove that G' is transient if and only if G is transient.

1. Explain why one of the two implications is immediate.
2. Let θ' be a flow on G' from a to ∞. Show that there is a unique flow θ on G from a to ∞ such that $\theta(\vec{e}) = \theta'(\vec{e})$ for any oriented edge $\vec{e}$ of G'.

3. Prove that for any edge $e \in S$, this flow θ satisfies

$$|\theta(e)| \leq \sum_{e' \in N(e)} |\theta'(e')|.$$

(since the inequality only involves the absolute value of the flow, we can forget about the orientation).

4. Deduce that the energy $\mathcal{E}(\theta)$ of θ satisfies

$$\mathcal{E}(\theta) \leq (1+4d)\mathcal{E}(\theta')$$

and conclude.

Exercise 3 (5 **points**) Let $2 < \alpha < 3$, and let $(p_k)_{k \geq 1}$ be a sequence of positive real numbers such that $\sum_{k \geq 1} p_k = \frac{1}{4}$ and $p_k \sim k^{-\alpha}$ as $k \to +\infty$. Let μ be the probability measure on $\mathbb{Z}^2$ defined by

$$\mu(k,0) = \mu(-k,0) = \mu(0,k) = \mu(0,-k) = p_k$$

for all $k \geq 1$, and $\mu(x,y) = 0$ if $x, y \neq 0$ or if $x = y = 0$. Finally, we denote by S the random walk on $\mathbb{Z}^2$ with step distribution μ.

1. Compute $\mathbb{E}[S_1]$ and $\mathbb{E}\left[\|S_1\|^2\right]$. From the results of the course, is it possible to conclude directly if S is recurrent or transient?

2. We denote by $\widehat{\mu}$ the characteristic function of μ. Check that for all $(\xi_1, \xi_2) \in \mathbb{R}^2$, we have

$$\widehat{\mu}(\xi_1, \xi_2) = \sum_{k \geq 1} \frac{\cos(k\xi_1) + \cos(k\xi_2)}{2}.$$

3. We denote by $\|\xi\|$ the Euclidean norm of ξ. Show that there is $c > 0$ such that if $\|\xi\|$ is small enough, then

$$\widehat{\mu}(\xi) \leq 1 - c\|\xi\|^{\alpha-1}.$$

Hint: Separate the sum in two, depending on whether $k \leq \|\xi\|^{-1}$ or not.

4. Deduce that S is transient.

Exercise 4 ($4+2$ **points**) Let H be a subset of $\mathbb{N}$, and let $\mathbb{T}^h$ be the infinite tree with one root such that all vertices of generation h have two children each if $h \in H$, and one child each if $h \notin H$. We denote by $h_1 < h_2 < \dots$ the elements of H.

1. Prove that the simple random walk S on $\mathbb{T}^h$ is recurrent if and only if

$$\sum_{i \geq 0} \frac{h_{i+1} - h_i}{2^i} = +\infty.$$

Hint: It may be useful to introduce that quantity $n_h = \# H \cap \{0, \dots, h\}$ for $h \geq 0$.

2. Prove that $\mathbb{T}^h$ has a positive Cheeger constant if and only if the sequence $(h_{i+1} - h_i)$ is bounded.

3. (Bonus question). We assume that $h_i \sim i^\alpha$ for some $\alpha > 1$. We admit that the distance between the root and S_n is typically of order n^β. What should be the value of β in terms of α? (no proof needed, we just ask for a heuristic argument).