Random Walks
ENS de Lyon, 2024-2025

Random walks: Partial exam

2024, October 21st

Course notes are allowed. You can write in English or in French.

Exercise 1 Let $d \geq 1$ and let (S_n) be a random walk on $\mathbb{R}^d$ with i.i.d. increments (X_i). We assume that S is symmetric, i.e. S has the same law as $-S$. Let $(Z_i)_{i\geq 1}$ be i.i.d. variables independent from (S_n) with $\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = 2) = \frac{1}{2}$. We write

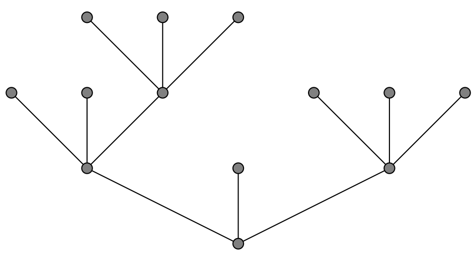
$$S'_n = \sum_{i=1}^{n} Z_i X_i.$$

1. Let φ_X be the characteristic function of X_1. Show that the characteristic function of $Z_1 X_1$ has real values and is given by

$$\varphi_{ZX}(u) = \frac{1}{2}\varphi_X(u) + \frac{1}{2}\varphi_X(2u).$$

2. Deduce that if S is transient, then S' is also transient.

Exercise 2 We denote by t_0 the following tree:



1. Draw the graphs of the contour function and of the Lukasiewicz walk of t_0.
2. Let μ be the probability measure on $\mathbb{N}$ defined by $\mu(0) = \mu(3) = \frac{1}{2}$ and $\mu(i) = 0$ for $i \neq 0, 3$. Let T be a Galton–Watson tree with offspring distribution μ. Compute the conditional probability

$$\mathbb{P}(T = t_0 | T \text{ has exactly 13 vertices}).$$

Exercise 3 Let (S_n) be an aperiodic random walk on $\mathbb{Z}$ with i.i.d. increments (X_i) such that $\mathbb{E}[X_1] = 0$ and $\sigma^2 := \mathbb{E}[X_1^2] \in (0, +\infty)$. The goal of the exercise is to estimate the probability

$$\mathbb{P}\left(S_n = S_1 + \cdots + S_n = 0\right).$$

We denote by φ the characteristic function of X_1. We recall that $\varphi(u) = 1 - \frac{\sigma^2}{2}u^2 + o(u^2)$ as $u \to 0$ and that there is a constant $c > 0$ such that $|\varphi(u)| \le 1 - cu^2$ for all $u \in [-\pi, \pi]$.

1. Show that for all $n \ge 1$, the characteristic function of the pair $(S_n, S_1 + \cdots + S_n)$ is given by

$$\psi_n(u, v) = \prod_{k=1}^{n} \varphi(u + kv).$$

2. We fix $\alpha, \beta > 0$. Show that

$$\mathbb{P}\left(S_n = S_1 + \cdots + S_n = 0\right) = \frac{n^{-\alpha-\beta}}{(2\pi)^2} \int_{-\pi n^\alpha}^{\pi n^\alpha} \int_{-\pi n^\beta}^{\pi n^\beta} \psi_n\left(\frac{x}{n^\alpha}, \frac{y}{n^\beta}\right) \mathrm{d}x \, \mathrm{d}y.$$

3. We would like $\psi_n\left(\frac{x}{n^\alpha}, \frac{y}{n^\beta}\right)$ to converge to a limit which is neither 0 nor 1 as $n \to +\infty$. Explain informally why $\alpha = 1/2$ and $\beta = 3/2$ seem like a reasonable choice.

4. Prove that for any $x, y \in \mathbb{R}$, we have

$$\psi_n\left(\frac{x}{n^{1/2}}, \frac{y}{n^{3/2}}\right) \xrightarrow[n \to +\infty]{} \exp\left(-\frac{\sigma^2}{2} \int_0^1 (x + ty)^2 \mathrm{d}t\right).$$

5. Conclude that there is a constant $c' > 0$ such that

$$\mathbb{P}\left(S_n = S_1 + \cdots + S_n = 0\right) \underset{n \to +\infty}{\sim} \frac{c'}{n^2}.$$

   It is not asked to compute c', but you can if you have some time left.