Random Walks
ENS de Lyon, 2025-2026

Random walks: Partial exam

2025, October 20th
13h30-15h30

Course notes are allowed. You can write in English or in French.

Exercise 1 (4 points) We fix two parameters $a, b \in \mathbb{R}$. Let μ be the probability measure on $\mathbb{R}^3$ given by

$$\mu(1,a,b) = \mu(2,1,a) = \mu(-1,a,a) = \mu(-2,-1,-b) = \frac{1}{4}.$$

Let S be the random walk on $\mathbb{R}^3$ with step distribution μ. Depending on the values of a and b, is S recurrent or transient?

<u>*Sketch of solution:*</u> Corollaries of the Chung-Fuchs criterion show that the walk is recurrent if and only if it has mean 0 and its support is included in a linear hyperplane. The mean is $(0, a/2, a/2)$ so S is transient if $a \neq 0$. If $a = 0$, the vector space spanned by the support contains $(1,0,0)$ and $(0,1,0)$, so it is $\mathbb{R}^3$ if $b \neq 0$. If $a = b = 0$, the walk is recurrent.

Exercise 2 (7+2 points) The goal of this exercise is to build a recurrent random walk S on $\mathbb{Z}$ with the property that there is a constant $c > 0$ such that

$$\mathbb{P}(|S_1| \geq n) \geq c \times n^{-2/3} \tag{1}$$

for infinitely many values of n.

1. Why does the existence of such a walk seem a bit surprising? **(0.5 point)**

   For all n, we write $f(n) = 2^{2^n}$ (for example $f(3) = 2^8 = 256$). We fix a constant $c > 0$ and define the measure μ on $\mathbb{Z}$ by

   $$\mu(i) = \begin{cases} \frac{c}{f(n)^{2/3}} & \text{if } i = f(n) \text{ or } i = -f(n) \text{ for some } n \geq 1, \\ 0 & \text{if not.} \end{cases}$$

2. Show that there is a choice of c such that this is a probability measure. We work with this choice of c from now on. **(0.5 point)**

3. Prove that the characteristic function of μ is real and is given by
$$\phi(u) = 2c \sum_{n\geq 1} \frac{1}{f(n)^{2/3}} \cos\left(f(n)u\right).$$
(1 point)

4. For any $m \geq 1$, let $u_m = f(m)^{-2/3}$. Prove that there is a constant K such that for any $m \geq 1$ and any $u_m \leq u \leq 2u_m$, we have
$$\phi(u) \geq 1 - \frac{K}{f(m)^{2/3}}.$$
(2.5 points) *Indication: treat separately the terms $n \leq m-1$ and the terms $n \geq m$.* *Sketch of solution:* Use the inequalities $\cos(x) \geq 1 - \frac{x^2}{2}$ for $n \leq m-1$ and $\cos(x) \geq -1$ for $n \geq m$.

5. Deduce that the random walk S with step distribution μ is recurrent.**(2.5 points)** *Sketch of solution:* Use the strong Chung–Fuchs criterion, and write
$$\int_0^1 \frac{1}{1-\phi(u)} \mathrm{d}u \geq \sum_m \int_{u_m}^{2u_m} \frac{1}{1-\phi(u)} \mathrm{d}u.$$

6. (Only if you have done all the rest) How could we adapt the construction to replace the exponent $\frac{2}{3}$ in (1) by any exponent $\alpha > 0$? **(2 points)** *Sketch of solution:* Take $f(m) = 2^{b^m}$ with b larger than 2, and replace the exponent $2/3$ by $\alpha > 0$. We find that he argument still works for $\alpha = \frac{2}{b+1}$, which can be made arbitrarily small by taking b large.

Exercise 3 (9 points) A *leaf* in a plane tree is a vertex with no child. The goal of the exercise is to show that most plane trees with n vertices have approximately $\frac{n}{2}$ leaves. In all the exercise, we denote by ν the geometric distribution with parameter $\frac{1}{2}$, i.e. $\nu(i) = \frac{1}{2^{i+1}}$ for $i \geq 0$. We can admit that ν has mean 1.

We define the probability measure μ on $\{-1, 0, 1, 2, \ldots\}$ by $\mu(i) = \nu(i+1)$ for all $i \geq -1$. Let $(X_n)_{n\geq 1}$ be i.i.d. random variables with law μ. For all $n \geq 0$, we write
$$S_n = \sum_{i=1}^n X_i \quad \text{and} \quad Y_n = \sum_{i=1}^n \mathbb{1}_{X_i=-1}.$$

1. We write $\tau = \inf\{n \geq 0 | S_n = -1\}$. Show carefully that for any $0 \leq x \leq n-1$, we have
$$\mathbb{P}\left(\tau = n \text{ and } Y_n = x\right) = \frac{1}{n}\mathbb{P}\left(S_n = -1 \text{ and } Y_n = x\right).$$
(2.5 points)

Sketch of solution: Use the cyclic lemma. Adapt the proof of the Kemperman formula using the cyclic lemma, and use the fact that Y_n is not affected by cyclic shifts.

2. Let $\varepsilon > 0$. Using a variance argument, prove that

$$\mathbb{P}\left(\left|Y_n - \frac{n}{2}\right| \geq \varepsilon n\right) = O\left(\frac{1}{n}\right).$$

(2 points)

Sketch of solution: Compute $\mathbb{E}[Y_n] = n/2$ and $\mathrm{Var}(Y_n) = \sum_i \mathrm{Var}(\mathbb{1}_{X_i=-1}) = n/4$, and use Bienaymé-Chebychev.

3. Deduce that

$$\mathbb{P}\left(\left|Y_n - \frac{n}{2}\right| \geq \varepsilon n \,|\, \tau = n\right) \xrightarrow[n\to+\infty]{} 0.$$

(2 points)

Sketch of solution: Using question 1 and the Kemperman formula, the left-hand side rewrites as

$$\begin{aligned}
\frac{\mathbb{P}\left(\left|Y_n - \frac{n}{2}\right| \geq \varepsilon n \text{ and } \tau = n\right)}{\mathbb{P}(\tau = n)} &= \frac{\frac{1}{n}\mathbb{P}\left(\left|Y_n - \frac{n}{2}\right| \geq \varepsilon n \text{ and } S_n = -1\right)}{\frac{1}{n}\mathbb{P}(S_n = -1)} \\
&= \mathbb{P}\left(\left|Y_n - \frac{n}{2}\right| \geq \varepsilon n \,|\, S_n = -1\right) \\
&\leq \frac{\mathbb{P}\left(\left|Y_n - \frac{n}{2}\right| \geq \varepsilon n\right)}{\mathbb{P}(S_n = -1)}.
\end{aligned}$$

We showed that the numerator is $O(1/n)$. On the other hand S is aperiodic hand has mean 0, so the local CLT shows that the denominator is of order $n^{-1/2}$.

4. Let T_n be a uniform random plane tree with n vertices. Prove that T_n has the same distribution as a Galton–Watson tree with offspring distribution ν, conditioned on having exactly n vertices. **(1 point)**

Sketch of solution: For any tree t with n vertices, the probability for a Galton-Watson tree with offspring distribution ν to be equal to t is

$$\prod_{v\in t} 2^{-1-k_v(t)} = 2^{-2n+1}$$

using $\sum_v k_v(t) = n - 1$. This does not depend on t, so the Galton–Watson tree conditioned on having size n is uniform.

5. Let L_n be the number of leaves of T_n. Using the Lukasiewicz path, conclude that we have the convergence in probability

$$\frac{L_n}{n} \xrightarrow[n\to+\infty]{(P)} \frac{1}{2}.$$

(0.5 point)

6. To prove this last result, why was it more convenient to rely on the Lukasiewicz path than on the contour function? **(1 point)**

   Sketch of solution: The number of leaves is not as convenient to read on the contour function. It is the number of $+1$ step followed by a -1 steps. In particular, this introduces dependances that make the variance computation of squestion 2 more complicated, and it is not immediately invariant by cyclic shifts, which would make question 1 more difficult as well.