On Cheeger constants of hyperbolic surfaces

Thomas Budzinski* & Nicolas Curien† & Bram Petri‡

Abstract

It is a well-known result due to Bollobas that the maximal Cheeger constant of large d-regular graphs cannot be close to the Cheeger constant of the d-regular tree. We prove analogously that the Cheeger constant of closed hyperbolic surfaces of large genus is bounded from above by $2/\pi \approx 0.63...$ which is strictly less than the Cheeger constant of the hyperbolic plane. The proof uses a random construction based on a Poisson–Voronoi tessellation of the surface with a vanishing intensity.

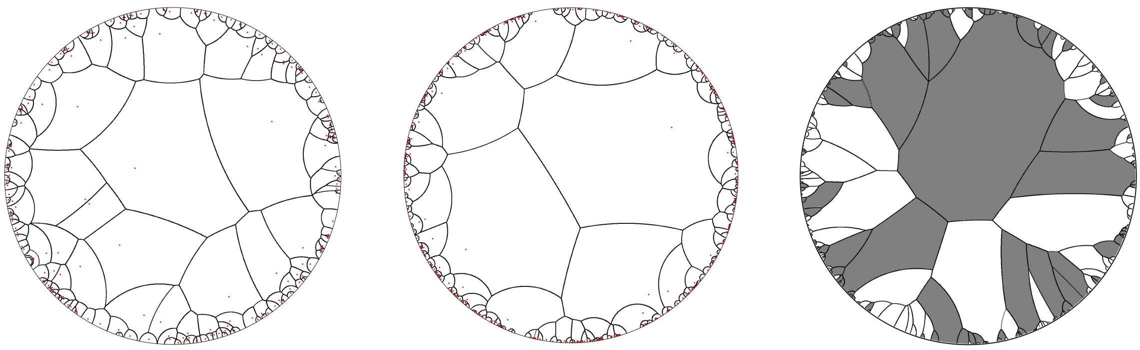

Figure 1 – From left to right: Poisson–Voronoi tessellations of the hyperbolic plane with decreasing intensity. Their limit (on the right) is the *pointless Voronoi tessellation* of the hyperbolic plane whose cells have been colored in black/white uniformly at random. This object has an average "linear" density equal to $2 \times \frac{2}{\pi}$ per unit of area.

1 Introduction

Let $\mathcal{S}$ be a closed hyperbolic surface. If $A \subset \mathcal{S}$ is a subset of $\mathcal{S}$, we define $h^*(A) = |\partial A|/|A|$ where $|\partial A|$ is the length of its boundary and $|A|$ is its area. If one of these quantities is not well defined, then $h^*(A) = +\infty$ by convention. The *Cheeger constant* of $\mathcal{S}$ is

$$h(\mathcal{S}) = \inf\{h^*(A) | A \subset \mathcal{S} \text{ with } |A| \leqslant |\mathcal{S}|/2\}. \quad (1)$$

Theorem 1. *For $g \geqslant 2$, let $\mathcal{M}_g$ be the moduli space of all isometry classes of closed hyperbolic surfaces of genus g. Then we have*

$$\limsup_{g\to\infty} \sup_{\mathcal{S}\in\mathcal{M}_g} h(\mathcal{S}) \leqslant \frac{2}{\pi}.$$

*ENS de Lyon. thomas.budzinski@ens-lyon.fr
†Université Paris-Saclay. nicolas.curien@gmail.com
‡Sorbonne Université. bram.petri@imj-prg.fr

We recall that the Cheeger constant of the hyperbolic plane $\mathbb{H}$ is equal to 1, which is asymptotically attained by large disks. As such, our result shows a gap between the maximal Cheeger constant of a large, closed hyperbolic surface and that of its universal cover. The presence of this gap was conjectured by Wright and Lipnowski [27] building upon [7] and inspired by similar results in graph theory. It also follows from our result that Cheeger's inequality [11] – the original reason for which the Cheeger constant was introduced – cannot be used to show that a closed hyperbolic surface of large genus has an optimal spectral gap. Finally, Theorem 1 implies recent results by Shen–Wu on random Belyĭ surfaces and random covers of the Bolza surface [24, 25].

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2 Context and sketch of proof

Before sketching the proof of Theorem 1, let us place it into context by telling the analogous story in the case of d-regular graphs. We will use similar notation as in the case of hyperbolic surfaces, so that the reader may follow the parallels more easily.

2.1 Warm-up: the graph case

Fix an integer $d \geqslant 3$ and consider the set $\mathcal{G}_d(n)$ of all connected simple graphs on n vertices having all degree d (we assume that $d \cdot n$ is even so that $\mathcal{G}_d(n)$ is not empty). If $\mathfrak{g}_n \in \mathcal{G}_d(n)$ and A is a subset of the vertices of $\mathfrak{g}_n$, the isoperimetric constant of A is defined as

$$h^*(A) = \frac{|\partial A|}{|A|},$$

where $|A|$ is the number of vertices of A and $|\partial A|$ is the number of edges having one endpoint in A and the other outside A. The Cheeger constant of $\mathfrak{g}_n$ is then

$$h(\mathfrak{g}_n) := \inf \{h^*(A) \mid A \subset \mathfrak{g}_n \text{ with } |A| \leqslant n/2\}.$$

It is easy to construct graphs $\mathfrak{g}_n \in \mathcal{G}_d(n)$ with small Cheeger constant, for example if they contain a large piece which looks roughly one-dimensional. On the other hand, we have

$$\mathbf{c}_d := \liminf_{n\to\infty} \sup_{\mathfrak{g}_n \in \mathcal{G}_d(n)} h(\mathfrak{g}_n) > 0,$$

since there are (families of) graphs, called *expanders*, whose Cheeger constant is uniformly bounded from below. The existence of such graphs has famously been proved by Margulis [19] through an explicit construction and Pinsker [22] using a probabilistic argument. Ramanujan graphs [18, 20] are very good expanders and their existence shows that $\mathbf{c}_d \geqslant \frac{d}{2} - O(\sqrt{d})$ when $d \to \infty$, which can also be proved using random graphs [4]. Conversely, an easy argument shows that the Cheeger constant of a large d-regular graph is asymptotically bounded from above by that of the d-regular tree $\mathbb{T}_d$, which is equal to $d-2$, so that in particular $\mathbf{c}_d \leqslant d-2$. This bound is not sharp and in fact Bollobas [4], later sharpened by Alon [1], proved that

$$\mathbf{c}_d \leqslant \frac{1}{2}(d-2) \quad \text{for all } d \geqslant 3 \tag{2}$$

(and even $\mathbf{c}_d = \frac{d}{2} - O(\sqrt{d})$ asymptotically as $d \to \infty$). This gap between the largest Cheeger constant of large d-regular graphs and that of $\mathbb{T}_d$ is the graph analog of our Theorem 1.

The idea of the proof of (2) is surprisingly simple: fix a large graph $\mathfrak{g}_n \in \mathcal{G}_d(n)$ and color *uniformly at random* half of its vertices black and the other half white. Consider then the subset A of size $n/2$ consisting of the black vertices. Since an edge is counted in $|\partial A|$ if and only its endpoints have different colors, the probability that a given edge contributes to $|\partial A|$ is approximately $\frac{1}{2}$. We thus have

$$\mathbb{E}[|\partial A|] \approx \frac{1}{2}|\mathrm{Edges}(\mathfrak{g}_n)| = \frac{1}{2} \cdot \frac{dn}{2}, \tag{3}$$

and so $h(\mathfrak{g}_n) \leqslant \mathbb{E}[h^*(A)] \leqslant \frac{d}{2}$ asymptotically when n is large. To get the better bound $h(\mathfrak{g}_n) \leqslant \frac{d-2}{2}$, the idea is to first regroup the vertices of $\mathfrak{g}_n$ into connected "regions" $R_1, ..., R_k$ of vertices which are all large (i.e. $1 \ll |R_i|$), but not too large (i.e. $|R_i| \ll n$). In particular k must be large. In the graph case, one can perform such a splitting in various deterministic ways, e.g. using a spanning tree of $\mathfrak{g}_n$ as in [1, Section 3]. Since those regions are large and connected we have the crude bound valid on d-regular trees

$$h^*(R_i) \leqslant d - 2 + o(1). \tag{4}$$

Hence, the number of edges whose endpoints lie in two different regions $R_i \neq R_j$ is roughly $\frac{n(d-2)}{2}$. We can then proceed as above and color each region uniformly at random in black or white. Since the regions are not too large, a standard concentration argument shows that the black vertices form a subset $\tilde{A}$ of approximately $n/2$ vertices. Again, a given edge is counted in $|\partial \tilde{A}|$ if its endpoints lie in two different regions with different colors. By the same computation as in (3) for $\mathbb{E}\left[|\partial \tilde{A}|\right]$, we deduce

$$h^*(\tilde{A}) \preceq \frac{1}{2} \cdot \frac{n(d-2)}{2} \cdot \frac{2}{n} = \frac{d-2}{2}.$$

2.2 Hyperbolic surfaces

Let us now draw the parallel with the case of hyperbolic surfaces. A first difficulty is that the *a priori* crude bound (4) does not hold in the continuous setting: there is no upper bound on the Cheeger constant of a connected set since there are such sets with a nasty fractal boundary having a large one-dimensional measure. However, the global inequality

$$h(\mathcal{S}_g) \leqslant h(\mathbb{H}) + o(1) = 1 + o(1) \quad \text{as } g \to \infty$$

is still true for any hyperbolic surface $\mathcal{S}_g$ of genus $g \geqslant 2$. This can e.g. be derived by spectral considerations, combining works of Cheeger [11] and Cheng [12]. Here also, the existence of "expander surfaces" of large genus with Cheeger constant bounded away from 0 is known. The known constructions essentially fall into three (overlapping) categories. First of all, there are multiple ways to compare the Cheeger constant of a hyperbolic surface to that of a graph [8, 5]. Secondly, just like in the case of graphs, it is known, due to Buser [9], that spectral expansion implies isoperimetric expansion. As such, the many examples of surfaces with a spectral gap – the first examples are due to Selberg [23] and recently near optimal spectral expanders were found by Hide and Magee [15] – also give rise to surfaces with large Cheeger constants. Finally, multiple random constructions [21, 6] are known to provide examples.

Let us now try to mimic the proof of the graph case to prove a strict inequality in the last display. Obviously, taking half of the points of $\mathcal{S}_g$ does not make sense, and one wants to first split our deterministic surface $\mathcal{S}_g$ into regions $R_1, ..., R_k$ satisfying $1 \ll |R_i| \ll g$ before coloring them in black and white with equal probability. In [7] Brooks and Zuk roughly speaking postulated the existence of such a decomposition based on balls where the isoperimetric constants $h^*(R_i)$ of the regions in question are close to 1. Given those hypothetical decompositions, the coloring argument would yield

$$h(\mathcal{S}_g) \preceq \frac{1}{2}$$

when $g \to \infty$. We were not able to prove that such decompositions actually exist on every hyperbolic surface and rather use a *random splitting of $\mathcal{S}_g$ based on a Voronoi decomposition.* More precisely, we consider a Poisson point process with points $X_1, ..., X_N$ of intensity given by $\lambda \cdot \mu_{\mathcal{S}_g}$, where $\mu_{\mathcal{S}_g}$ is the hyperbolic area measure on $\mathcal{S}_g$ with total mass $4\pi(g-1)$, and $\lambda > 0$ is a small constant. Those points decompose $\mathcal{S}_g$ into the Voronoi regions $\mathrm{Vor}_\lambda(\mathcal{S}_g) := \{C_1, ..., C_N\}$ where

$$C_i = \left\{ x \in \mathcal{S}_g : d_{\mathcal{S}_g}(x, X_i) = \min_{1 \leqslant j \leqslant N} d_{\mathcal{S}_g}(x, X_j) \right\}.$$

When the intensity parameter λ is small, the typical area of a region is of order $1/\lambda$ and so one can expect (see Lemma 1 below) that indeed we have $1 \ll |C_i| \ll |\mathcal{S}_g|$ at least for most regions. Now recall that we have no *a priori* upper bound for the Cheeger constant of these regions (contrary to the graph case). The crux of the argument boils down to showing that, at least on average, we have $h^*(C_i) = \frac{4}{\pi}$. More precisely, we will show (see Proposition 2) that the expected length of the union $\partial \mathrm{Vor}_\lambda(\mathcal{S}_g)$ of the boundaries of the Voronoi cells $C_1, \ldots, C_N$ satisfies

$$\limsup_{\lambda \to 0} \sup_{g \geqslant 2} \sup_{\mathcal{S}_g \in \mathcal{M}_g} \frac{1}{|\mathcal{S}_g|} \mathbb{E}\left[|\partial \mathrm{Vor}_\lambda(\mathcal{S}_g)| \right] \leqslant \frac{2}{\pi}. \tag{5}$$

Given the above display, one can then run the same proof as in the graph case: we color uniformly the Voronoi cells in black and white and consider the resulting black component. Its volume is concentrated around $\frac{1}{2}|\mathcal{S}_g|$ and its boundary size is less than $|\mathcal{S}_g| \cdot \frac{1}{2} \cdot (\frac{2}{\pi} + \varepsilon)$ for g large when λ is small, so that $h(\mathcal{S}_g) \leqslant (\frac{2}{\pi} + \varepsilon)$ asymptotically as claimed in Theorem 1.

2.3 The pointless Voronoi tessellation of $\mathbb{H}$

Let us gain some intuition on the proof of (5) done in Proposition 2. Suppose first that the systole of $\mathcal{S}_g$ tends to ∞ as $g \to \infty$. In that situation, the neighborhood of each point in $\mathcal{S}_g$ looks like a piece of the hyperbolic plane and the Voronoi tessellation $\mathrm{Vor}_\lambda(\mathcal{S}_g)$ converges in distribution (in the local Hausdorff sense) towards the Voronoi tessellation $\mathrm{Vor}_\lambda(\mathbb{H})$ of the hyperbolic plane with intensity λ, see Figure 1.

This classical object has been studied in stochastic geometry [16, 10] and in particular in relation to its percolation properties [3, 13, 14]. In particular, Isokawa [16] computed the mean characteristics of a typical[1] cell C_λ in $\mathrm{Vor}_\lambda(\mathbb{H})$: this cell is an almost surely finite convex hyperbolic polygon satisfying

$$\mathbb{E}[C_\lambda] = \frac{1}{\lambda} \quad \text{and} \quad \mathbb{E}[|\partial C_\lambda|] = \frac{8}{\sqrt{\pi\lambda}} \int_0^\infty \mathrm{e}^{-u} \sqrt{u + \frac{u^2}{4\pi\lambda}}\, \mathrm{d}u.$$

In particular, the "average Cheeger constant" of cells of $\mathrm{Vor}_\lambda(\mathbb{H})$ satisfies in the small intensity limit

$$\frac{\mathbb{E}[|\partial C_\lambda|]}{\mathbb{E}[|C_\lambda|]} \xrightarrow[\lambda\to 0]{} \frac{4}{\pi},$$

which explains (5) in the case when $\mathsf{Systole}(\mathcal{S}_g) \to \infty$. Note that recently percolation on hyperbolic Poisson–Voronoi tessellation with small intensity has been studied in [13]. Underneath the convergence of the above display as the intensity tends to 0 lies the fact that $\mathrm{Vor}_\lambda(\mathbb{H})$ converges (in distribution for the Hausdorff topology on compact sets of $\mathbb{H}$) towards a limiting object that we name the *pointless Poisson–Voronoi tessellation* of the hyperbolic disk (see Figure 1) and whose construction and properties will be studied in a forthcoming work.

Finally, in order to deal with the case in which the systole is not large, we will need to study $\partial\mathrm{Vor}_\lambda(\mathcal{S}_g)$ in the neighbourhood of a point x of $\mathcal{S}_g$. To do so, we will replace the hyperbolic plane by the *Dirichlet domain* – a specific fundamental domain – of $\mathcal{S}_g$ around x. We will prove that in order to prove (5), it will be enough to understand $\partial\mathrm{Vor}_\lambda(\mathcal{S})$ inside the Dirichlet domain, thus also reducing the general case to computations in the hyperbolic plane.

3 Proof

Let us recall some basic notions and fix notation before starting the proof.

3.1 Preliminaries

The hyperbolic plane. We recall that the *hyperbolic plane* $\mathbb{H}$ is the unit disk $\{z \in \mathbb{C} : ||z| < 1\}$, equipped with the Riemannian metric $\frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$. This metric has constant curvature -1 and is invariant under Möbius transformations. We will denote by $d_{\mathbb{H}}(x,y)$ the hyperbolic distance in $\mathbb{H}$. For several computations in the paper, it will be useful to use polar coordinates on $\mathbb{H}$. More precisely, for $r > 0$ and $\theta \in [0, 2\pi)$, we denote by $[r;\theta]$ the point of $\mathbb{H}$ of the form $z = \rho(r)\mathrm{e}^{i\theta}$, where $\rho(r) > 0$ is such that $d_{\mathbb{H}}(0,z) = r$. We note that the area measure on $\mathbb{H}$ can be written as $\sinh(r)\,\mathrm{d}r\,\mathrm{d}\theta$.

[1] By Palm calculus, such a cell can be obtained by adding the point 0 to the Poisson process and considering the associated region.

Hyperbolic surfaces and Dirichlet domains. We recall that a closed hyperbolic surface $\mathcal{S}$ of genus $g \geqslant 2$ is a Riemannian surface which is locally isometric to $\mathbb{H}$, or equivalently which is the quotient of $\mathbb{H}$ by a discrete, torsion-free, cocompact group G of isometries. We denote by $d_{\mathcal{S}}$ and $\mu_{\mathcal{S}}$ (or just d and μ if no confusion is possible) the hyperbolic metric and area measure on $\mathcal{S}$. If $A \subset \mathcal{S}$ is a Borel subset we write $|A|$ for $\mu(A)$ and $|\partial A|$ for the length of its boundary. If x is a point of a hyperbolic surface and $r \geqslant 0$, we denote by $B_r(x)$ the closed ball of radius r around x.

We will make important use of the *Dirichlet domain*, which is a particular choice of a fundamental domain for the action of G on $\mathbb{H}$, see [2, Section 9.4]. More precisely, let x be a point of a hyperbolic surface $\mathcal{S} = \mathbb{H}/G$, and let $p : \mathbb{H} \to \mathcal{S}$ be its universal cover, so that $p(0) = x$. We denote by $D(\mathcal{S}, x)$ the set of those points $y' \in \mathbb{H}$ such that $d_{\mathbb{H}}(0, y') = d_{\mathcal{S}}(x, p(y'))$. In other words, this means that the image under p of the geodesic from 0 to y' is still a shortest path in $\mathcal{S}$. Dirichlet domains are convex polygons of $\mathbb{H}$ [2, Theorem 9.4.2]. For (almost) every point $y \in S$, we will denote by $p^{-1}(y)$ the unique point of $D(\mathcal{S}, x)$ which is sent to y. Note that the maps $p : D(\mathcal{S}, x) \to \mathcal{S}$ and $p^{-1} : \mathcal{S} \to D(\mathcal{S}, x)$ are measure-preserving.

Poisson point processes. The surface $\mathcal{S}$ or the hyperbolic plane $\mathbb{H}$ both carry a Borel measure $\mu_{\mathcal{S}}$ which is of finite mass $4\pi(g-1)$ in the case of $\mathcal{S}$ and $\mu_{\mathbb{H}}$ which is σ-finite in the case of $\mathbb{H}$. They have no atoms, so one can define a Poisson point process on those spaces with intensity $\lambda \cdot \mu_{\cdot}$. This is a cloud of distinct random points that is characterized by the fact that for any disjoint measurable subsets $A_1, ..., A_k$, the number of points falling inside A_i are independent Poisson random variables of mean $|A_i|$, see [17] for details.

Poisson–Voronoi tessellation. Let $\mathcal{S}$ be a hyperbolic surface of (large) genus g. As announced above, to bound its Cheeger constant from above, we shall build a random subset A_λ of $\mathcal{S}$ in the following way. We first throw a Poisson point process $\Pi = \{X_1, ..., X_N\}$ on $\mathcal{S}$ with intensity $\lambda \cdot \mu_{\mathcal{S}}$, where $\lambda > 0$ is a small constant. In particular the random variable N follows a Poisson law with mean $\lambda \cdot 4\pi(g-1)$. We then consider the closed Voronoi cells $\mathrm{Vor}_\lambda(\mathcal{S}) = \{C_1, ..., C_N\}$ it defines and denote by $C(x)$ the Voronoi cell containing the point $x \in \mathcal{S}$ (with ties broken arbitrarily). Conditionally on Π, each cell is colored black or white independently with probability $\frac{1}{2}$ and we let $A_\lambda \subset \mathcal{S}$ be (the closure of) the union of the black cells. On the one hand, we will prove (Lemma 1) that unless $h(\mathcal{S})$ is very small (in which case our main result is trivial), the area $|A_\lambda|$ is close to $\frac{|\mathcal{S}|}{2}$ if the surface is large enough. On the other hand, we will show (Proposition 2) that $\mathbb{E}\left[|\partial A_\lambda|\right]$ is close to $\frac{1}{\pi}|\mathcal{S}|$ if the intensity λ is chosen small enough.

3.2 Area estimate

We start with the area estimate. We believe that Lemma 1 below should be true even without the Cheeger constant assumption, but we could not find a short argument to prove the general statement.

Lemma 1 (Area estimate)**.** *For any $\lambda > 0$ and $\delta > 0$, there is $g_0 \geqslant 2$ with the following property. For every hyperbolic surface $\mathcal{S}$ of genus $g \geqslant g_0$ such that $h(\mathcal{S}) \geqslant \delta$, if the random subset A_λ of $\mathcal{S}$ is built* *as described above, we have*

$$\mathbb{P}\left(\left|\frac{|A_\lambda|}{|\mathcal{S}|}-\frac{1}{2}\right|>\delta\right)<\delta.$$

Proof. Let $\mathcal{S}$ be a hyperbolic surface with Cheeger constant at least δ. It follows immediately from our setup that $\mathbb{E}\left[|A_\lambda|\right]=\frac{|\mathcal{S}|}{2}$, so we only need to bound the variance of $|A_\lambda|$. By conditioning on $\mathrm{Vor}_\lambda(\mathcal{S})$, we have

$$\mathrm{Var}\left(|A_\lambda|\right)=\mathbb{E}\left[\frac{1}{4}\sum_{i=1}^{N}|C_i|^2\right]=\frac{1}{4}\int_{\mathcal{S}^2}\mathbb{P}\left(C(x)=C(y)\right)\mu(\mathrm{d}x)\,\mu(\mathrm{d}y).$$

To bound $\mathbb{P}\left(C(x)=C(y)\right)$ from above, we will first argue that the assumption $h(\mathcal{S})\geqslant\delta$ implies a lower bound on the volume of balls around x and y for "most" points $x,y\in S$.

More precisely, for all $x\in S$ and $r>0$, by the Cheeger constant assumption we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left|B_r(x)\right|=\left|\partial B_r(x)\right|\geqslant\delta\left|B_r(x)\right|.$$

Therefore, let $r_1>r_0>0$ (the values of r_0 and r_1, depending only on δ and λ, will be specified later), and assume that g is large enough to have $2\pi(\cosh(r_1)-1)<\frac{|\mathcal{S}|}{2}$. We have

$$\left|B_{r_1}(x)\right|\geqslant\mathrm{e}^{\delta(r_1-r_0)}\left|B_{r_0}(x)\right|.$$

On the other hand, by the collar lemma, we know that there is an absolute constant $C>0$ such that, if r_0 is small enough, we have

$$|\{x\in\mathcal{S}\mid\mathrm{InjRad}(x)\leqslant r_0\}|\leqslant Cr_0\cdot|\mathcal{S}|.$$

But if the injectivity radius around x is larger than r_0, then $|B_{r_0}(x)|=2\pi\left(\cosh(r_0)-1\right)$ and we get a lower bound on $|B_{r_1}(x)|$. In particular, by taking r_0 small enough, we find

$$\left|\left\{x\in\mathcal{S}\mid\ |B_{r_1}(x)|<\mathrm{e}^{\delta(r_1-r_0)}2\pi\left(\cosh(r_0)-1\right)\right\}\right|\leqslant\delta^3|\mathcal{S}|. \tag{6}$$

We denote by $K_{r_1}\subset\mathcal{S}^2$ the set of pairs (x,y) such that $d(x,y)>2r_1$ and neither x nor y satisfies the event in the left-hand side of (6). Let $(x,y)\in K_{r_1}$. If x and y belong to the same Voronoi cell, since $B_{r_1}(x)$ and $B_{r_1}(y)$ are disjoint, at least one of these two balls contains none of the points X_i. It follows that

$$\begin{aligned}\mathbb{P}\left(C(x)=C(y)\right)&\leqslant\exp\left(-\lambda|B_{r_1}(x)|\right)+\exp\left(-\lambda|B_{r_1}(y)|\right)\\&\leqslant2\exp\left(-2\pi\lambda\mathrm{e}^{\delta(r_1-r_0)}\left(\cosh(r_0)-1\right)\right).\end{aligned}$$

In particular, if we have chosen a large enough value for r_1, this probability is smaller than δ^3. Therefore, we get

$$\mathrm{Var}\left(|A_\lambda|\right)\leqslant\frac{\delta^3|\mathcal{S}|^2}{4}+\frac{|\mathcal{S}^2\backslash K_{r_1}|}{4}\leqslant\frac{\delta^3|\mathcal{S}|^2}{4}+2\frac{\delta^3|\mathcal{S}|^2}{4}+\frac{2\pi}{4}\left(\cosh(2r_1)-1\right)|\mathcal{S}|,$$

where the second term comes from (6), and the third term counts the pairs (x,y) with $d(x,y)<2r_1$. In particular, for $|\mathcal{S}|=4\pi(g-1)$ large enough, the variance of $|A_\lambda|$ is smaller than $\delta^3|\mathcal{S}|^2$ and the conclusion follows by the Chebychev inequality. □

3.3 Perimeter estimate

Let us now pass to the perimeter estimate which is the most technical part of the proof.

Proposition 2 (Perimeter estimate)**.** *Recall that* $\partial \mathrm{Vor}_\lambda(\mathcal{S})$ *is the union of the sides of the Poisson-Voronoi cells* $C_1, ..., C_N$ *with intensity* $\lambda \cdot \mu_{\mathcal{S}}$*. Then we have*

$$\limsup_{\lambda\to 0} \sup_{g\to\infty} \sup_{\mathcal{S}\in\mathcal{M}_g} \frac{1}{|\mathcal{S}|}\mathbb{E}\left[|\partial \mathrm{Vor}_\lambda(\mathcal{S})|\right] \leqslant \frac{2}{\pi}.$$

An optimization lemma. Before proving this estimate, we state two intermediate results that will be useful for us. The first is a nice optimization lemma, for which a very short and direct proof is provided in the last page of [26].

Lemma 3. *Let* ν *be a probability measure on* $[0, 2\pi]$*. Then we have*

$$\int_0^{2\pi}\int_0^{2\pi}\left|\sin\frac{\theta_1-\theta_2}{2}\right|\nu(\mathrm{d}\theta_1)\nu(\mathrm{d}\theta_2) \leqslant \frac{2}{\pi},$$

with equality if ν *is the uniform measure.*

The proof of Lemma 3 consists of assuming by density that ν has a smooth density f with respect to Lebesgue, and expressing the integral in terms of the Fourier coefficients of f. We will use this lemma to handle the fact that a Dirichlet domain $D(\mathcal{S}, x)$ is not as isotropic as $\mathbb{H}$.

Intersection of thin rings. The second lemma will help ruling out some pathological behaviours of Dirichlet domains. For $0 < a < b$ and $x \in \mathbb{H}$, we denote by $R_a^b(x)$ the set of points $z \in \mathbb{H}$ such that

$$a \leqslant d(x,z) \leqslant b.$$

Lemma 4. *Let* $K > 0$ *and let* x, y *be two distinct points of* $\mathbb{H}$*. Then for all* $a > 0$*, we have*

$$\left|R_a^{a+\varepsilon}(x)\cap R_a^{a+\varepsilon}(y)\right| = o(\varepsilon)$$

as $\varepsilon \to 0$*. Moreover, the* $o(\varepsilon)$ *is uniform in* (x, y, a) *provided* $a \leqslant K$ *and* $d(x, y) \geqslant K^{-1}$*.*

Proof. We first note that if $d(x, y) > 2K + 2$, then the intersection $R_a^{a+\varepsilon}(x) \cap R_a^{a+\varepsilon}(y)$ is empty as soon as $\varepsilon < 1$, so we may assume $d(x, y) \leqslant 2K + 2$. For the same reason, since $d(x, y) \geqslant K^{-1}$, we may assume $a \geqslant \frac{1}{2K}$. Moreover, by invariance under Möbius transformations, we may assume that $x = 0$ and that $y = [d; 0]$ in polar coordinates, with $K^{-1} \leqslant d \leqslant 2K + 2$.

Let us express $R_a^{a+\varepsilon}(0) \cap R_a^{a+\varepsilon}(y)$ in polar coordinates. First, if $z = [r; \theta] \in R_a^{a+\varepsilon}(0) \cap R_a^{a+\varepsilon}(y)$, then we must have $r \in [a, a + \varepsilon]$. Second, by the hyperbolic law of cosines, we have

$$\cosh(d(y,z)) = \cosh(d)\cosh(r) - \cos(\theta)\sinh(d)\sinh(r).$$

Since $a \leqslant d(y, z) \leqslant a + \varepsilon$, we deduce

$$\frac{\cosh(d)\cosh(r)-\cosh(a+\varepsilon)}{\sinh(d)\sinh(r)} \leqslant \cos(\theta) \leqslant \frac{\cosh(d)\cosh(r)-\cosh(a)}{\sinh(d)\sinh(r)}.$$

That is, $\cos(\theta)$ lies in an interval $I(a,d,r,\varepsilon)$ of length $\frac{\cosh(a+\varepsilon)-\cosh(a)}{\sinh(d)\sinh(r)} \leqslant K\frac{\varepsilon\sinh(a+\varepsilon)}{\sinh(r)} \leqslant C(K)\varepsilon$ by the assumption $r \geqslant \frac{1}{2K}$. This implies that θ must lie in a set $S(a,d,r,\varepsilon) \subset [0,2\pi]$ of measure at most $C(K)\sqrt{\varepsilon}$. Therefore, we have

$$\begin{aligned}\left|R_a^{a+\varepsilon}(x) \cap R_a^{a+\varepsilon}(y)\right| &= \int_a^{a+\varepsilon} |S(a,d,r,\varepsilon)| \sinh(r)\,\mathrm{d}r \\ &\leqslant \sinh(a+\varepsilon)C(K)\varepsilon^{3/2} \leqslant \sinh(K+1)C(K)\varepsilon^{3/2},\end{aligned}$$

which proves the lemma. □

Proof of Proposition 2. In the rest of the proof we write $\partial C \equiv \partial \mathrm{Vor}_\lambda(\mathcal{S})$ to lighten notation. For all $\varepsilon > 0$, we denote by $\partial^\varepsilon C$ the set of points of $\mathcal{S}$ lying at hyperbolic distance at most ε from ∂C. Since ∂C is a.s. the union of finitely many geodesic segments, we have

$$|\partial C| = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} |\partial^\varepsilon C|.$$

It follows from Fatou's lemma that

$$\mathbb{E}\left[|\partial C|\right] \leqslant \liminf_{\varepsilon\to 0} \frac{1}{2\varepsilon} \mathbb{E}\left[|\partial^\varepsilon C|\right] = \liminf_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\mathcal{S}} \mathbb{P}\left(x \in \partial^\varepsilon C\right)\, \mu_{\mathcal{S}}(\mathrm{d}x). \tag{7}$$

Hence, let $0 < \varepsilon < 1$ and $x \in \mathcal{S}$, and let i_1 be such that that $C(x) = C_{i_1}$. We first note that if $x \in \partial^\varepsilon C$, then there are at least two Voronoi cells which intersect the ball $B_\varepsilon(x)$, and one of these cells is C_{i_1}. Hence, there is an $i_2 \neq i_1$ such that the bisector between X_{i_1} and X_{i_2} intersects $B_\varepsilon(x)$. Therefore we introduce, for any point y, the set $A^\varepsilon_{\mathcal{S}}(x,y) \subset \mathcal{S}$ consisting of those points z such that $d(x,z) \geqslant d(x,y)$ and such that the bisector between y and z intersects the closed ball $B_\varepsilon(x)$. We first note that if $z \in A^\varepsilon_{\mathcal{S}}(x,y)$, then by the triangle inequality, we have[2]

$$d(x,y) \leqslant d(x,z) \leqslant d(x,y) + 2\varepsilon. \tag{8}$$

The event $x \in \partial^\varepsilon C$ is equivalent to saying that at least one other point of the Poisson process lands in the region $A^\varepsilon_{\mathcal{S}}(x, X_{i_1})$. Therefore, by conditioning on the point X_{i_1} closest to x, we have

$$\mathbb{P}\left(x \in \partial^\varepsilon C\right) = \lambda \int_{\mathcal{S}} \mu_{\mathcal{S}}(\mathrm{d}y)\ \exp\left(-\lambda |B_{d(x,y)}(x)|\right) \times \left(1 - \exp\left(-\lambda |A^\varepsilon_{\mathcal{S}}(x,y)|\right)\right) \tag{9}$$

$$\leqslant \lambda^2 \int_{\mathcal{S}} \mu_{\mathcal{S}}(\mathrm{d}y)\ \exp\left(-\lambda |B_{d(x,y)}(x)|\right) |A^\varepsilon_{\mathcal{S}}(x,y)|. \tag{10}$$

If the injectivity radius at $x \in \mathcal{S}$ is much larger than $d(x,y)$, then $|A^\varepsilon_{\mathcal{S}}(x,y)|$ can be computed as in the hyperbolic plane (and one would recover the estimates of Isokawa [16]). In the general case, we will work with the Dirichlet domain $D(\mathcal{S},x) \subset \mathbb{H}$ defined above. We recall that $D(\mathcal{S},x)$ is a fundamental domain for the projection $p : \mathbb{H} \to S$ with $p(0) = x$. We claim that for $y \in \mathcal{S}$, the subset $A^\varepsilon_{\mathcal{S}}(x,y)$ of $\mathcal{S}$ is very close to the subset $A^\varepsilon_{D(\mathcal{S},x)}(0, p^{-1}(y))$ of $\mathbb{H}$.

[2] Using this inequality to crudely bound $|A^\varepsilon_{\mathcal{S}}(x,y)|$, we would obtain Proposition 2 (and therefore Theorem 1) with a constant 1 instead of $\frac{2}{\pi}$. This is why we will need the more accurate description given by Lemma 6.

Lemma 5. *For $x, y \in \mathcal{S}$, we have*

$$|A^{\varepsilon}_{\mathcal{S}}(x,y)| \leqslant \left|A^{\varepsilon}_{D(\mathcal{S},x)}(0, p^{-1}(y))\right| + o(\varepsilon)$$

as $\varepsilon \to 0$, uniformly in $x, y \in \mathcal{S}$.

Proof. Let us fix x and y, and write $r = d_{\mathcal{S}}(x,y)$. Since $p^{-1} : \mathcal{S} \to D(\mathcal{S},x)$ is measure-preserving, it is sufficient to prove

$$\left|p^{-1}\left(A^{\varepsilon}_{\mathcal{S}}(x,y)\right) \setminus A^{\varepsilon}_{D(\mathcal{S},x)}(0, p^{-1}(y))\right| = o(\varepsilon), \tag{11}$$

uniformly in $(x,y) \in \mathcal{S}^2$. We fix a Fuchsian group G such that $\mathcal{S} = G\backslash\mathbb{H}$. Moreover, to avoid heavy notation involving the projection map, we will use $w' \in D(\mathcal{S},x)$ to denote the unique (up to a set of measure 0) pre-image of $w \in \mathcal{S}$ under p. In particular, $x' = 0$.

Now assume that $\varepsilon < \frac{1}{2}\mathsf{Systole}(\mathcal{S})$ so that $B_\varepsilon(0) \subset D(\mathcal{S},x)$, and let z' be in the difference of sets of (11). This implies that there is a point $a \in B_\varepsilon(x)$ such that

$$d_{\mathcal{S}}(a,z) < d_{\mathcal{S}}(a,y) \quad \text{but} \quad d_{\mathbb{H}}(a',z') > d_{\mathbb{H}}(a',y').$$

Now let $g \cdot a'$ (for some $g \in G$) be the translate of a' such that $d_{\mathcal{S}}(a,z) = d_{\mathbb{H}}(g \cdot a', z')$. The last display means that $d_{\mathbb{H}}(g \cdot a', z') < d_{\mathbb{H}}(a', y)$. Observe that $g \cdot a' \in B_\varepsilon(g \cdot x')$. We get

$$d_{\mathbb{H}}(g \cdot x', z') \leqslant d_{\mathbb{H}}(g \cdot x', g \cdot a') + d_{\mathbb{H}}(g \cdot a', z') \leqslant \varepsilon + d_{\mathcal{S}}(a,z) \leqslant 2\varepsilon + d_{\mathcal{S}}(x,z) \leqslant r + 4\varepsilon,$$

where we have used (8) for the last inequality. On the other hand, by the definition of the Dirichlet domain, we have

$$d_{\mathbb{H}}(g \cdot x', z') \geqslant d_{\mathbb{H}}(x', z') = d_{\mathcal{S}}(x,z) \geqslant d_{\mathcal{S}}(x,y) = r.$$

Finally, we have

$$d_{\mathbb{H}}(x', g \cdot x') \leqslant d_{\mathbb{H}}(x', z') + d_{\mathbb{H}}(z', g \cdot x') = d_{\mathcal{S}}(x,z) + d_{\mathbb{H}}(z', g \cdot x') \leqslant 2r + 6\varepsilon,$$

where we have used (8) again. Putting the last three inequalities together, we find that the set described in the left-hand side of (11) is contained in

$$\bigcup_{\substack{w \in G \cdot 0 \setminus \{0\} \\ d_{\mathbb{H}}(0,w) \leqslant 2r+1}} \left\{ z' \in \mathbb{H} \middle| \begin{array}{c} r \leqslant d(w,z') \leqslant r + 4\varepsilon \text{ and} \\ r \leqslant d(0,z') \leqslant r + 2\varepsilon \end{array} \right\}.$$

The sets in this union are intersections of two annuli of width 4ε, centered around 0 and a translate w of 0. Such an intersection has area $o(\varepsilon)$, by Lemma 4. Note that in order to apply this lemma, we use the fact that $r \leqslant \mathsf{Diameter}(\mathcal{S})$ and that a translate $w = g \cdot 0$ satisfies $d(w,0) \geqslant \mathsf{Systole}(\mathcal{S})$.

Finally, the number of sets in the union can be bounded in terms of r, and hence in terms of $\mathsf{Diameter}(\mathcal{S})$. Indeed, two points in $G \cdot 0$ are least $\mathsf{Systole}(S)$ apart and as such only so many of them fit in a disk of radius $2r + 1$. This concludes the proof of Lemma 5. ☐

The next step is to give a precise description of the set $A^{\varepsilon}_{D(\mathcal{S},x)}(0,y)$. It will be particularly natural to express this description in polar coordinates. For all $r > 0$, we denote by $I_r(x)$ the set of angles $\theta \in [0,2\pi)$ such that the point $[r;\theta]$ belongs to $D(\mathcal{S},x)$. We note that $I_r(x)$ is a finite union of intervals, and that $I_{r_2}(x) \subset I_{r_1}(x)$ when $r_1 \leqslant r_2$ by convexity of $D(\mathcal{S},x)$. Moreover, the area measure on $D(\mathcal{S},x)$ can be written as $\sinh(r)\mathbb{1}_{\theta\in I_r(x)}\,\mathrm{d}r\,\mathrm{d}\theta$. Then we have the following good approximation of $A^{\varepsilon}_{D(\mathcal{S},x)}(0,y)$ (see also Figure 2).

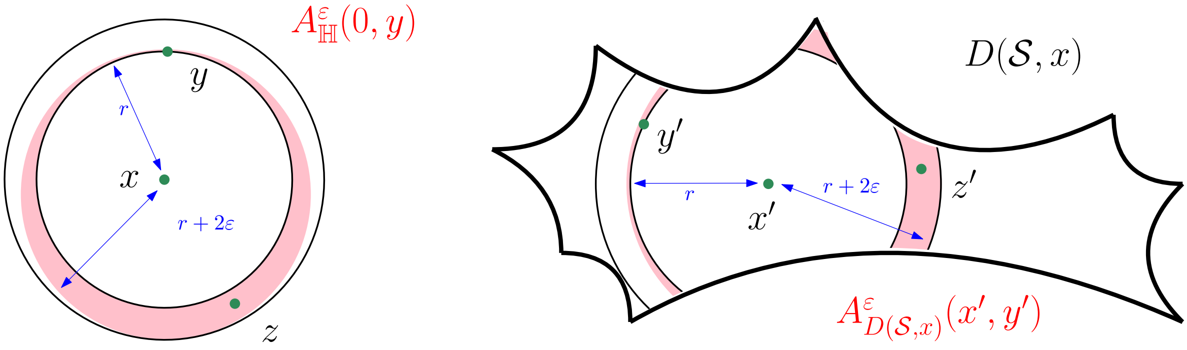

Figure 2 – On the left, in pink the set $A^{\varepsilon}_{\mathbb{H}}(x,y)$, see Lemma 6. On the right the set $A^{\varepsilon}_{D(\mathcal{S},x)}(x',y')$ which by Lemma 5 is a very good approximation of $A^{\varepsilon}_{\mathcal{S}}(x,y)$.

Lemma 6. *Let $\delta > 0$. Then there are $r_0(\delta) > 1$ and $\varepsilon_0(\delta) > 0$ with the following property. For all $0 < \varepsilon < \varepsilon_0(\delta)$ and any point $y = [r;\theta] \in \mathbb{H}$ such that $d(0,y) = r > r_0(\delta)$, we have the inclusion*

$$A^{\varepsilon}_{\mathbb{H}}(0,y) \subset \left\{ [r';\theta'] \in \mathbb{H} \middle| r \leqslant r' \leqslant r + 2(1+\delta)\left|\sin\frac{\theta'-\theta}{2}\right|\varepsilon \right\}. \tag{12}$$

In particular, under those assumptions, if $x \in \mathcal{S}$ and $y \in D(\mathcal{S},x)$, we have

$$\left|A^{\varepsilon}_{D(\mathcal{S},x)}(0,y)\right| \leqslant 2(1+\delta)\varepsilon\sinh(r+3\varepsilon)\int_{I_r(x)}\left|\sin\frac{\theta'-\theta}{2}\right|\mathrm{d}\theta'.$$

Proof. This is just a calculation using hyperbolic trigonometry. We write $z = [r';\theta'] \in A^{\varepsilon}_{\mathbb{H}}(0,y)$. We want to prove that $r' \leqslant r + \left(2(1+\delta)\left|\sin\frac{\theta'-\theta}{2}\right|\right)\varepsilon$. Because $z \in A^{\varepsilon}_{\mathbb{H}}(0,y)$, there is a point a of the form $[\varepsilon;\varphi]$ with $\varphi \in [0,2\pi)$ which is closer to z than to y. Using the hyperbolic cosine law, we can write down the distances $d(a,y)$ and $d(a,z)$ in terms of r, r', ε, θ, θ' and φ. We find

$$\cosh(r')\cosh(\varepsilon) - \cos(\varphi-\theta')\sinh(r')\sinh(\varepsilon) \leqslant \cosh(r)\cosh(\varepsilon) - \cos(\varphi-\theta)\sinh(r)\sinh(\varepsilon),$$

or equivalently

$$\cos(\varphi-\theta')\sinh(r') - \cos(\varphi-\theta)\sinh(r) \geqslant \coth(\varepsilon)\left(\cosh(r')-\cosh(r)\right) \geqslant \frac{\cosh(r')-\cosh(r)}{\varepsilon}. \tag{13}$$

Moreover, by the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&\cos(\varphi-\theta')\sinh(r')-\cos(\varphi-\theta)\sinh(r)\\
&=\big(\cos(\theta')\sinh(r')-\cos(\theta)\sinh(r)\big)\cos(\varphi)+\big(\sin(\theta')\sinh(r')-\sin(\theta)\sinh(r)\big)\sin(\varphi)\\
&\leqslant\sqrt{\big(\cos(\theta')\sinh(r')-\cos(\theta)\sinh(r)\big)^2+\big(\sin(\theta')\sinh(r')-\sin(\theta)\sinh(r)\big)^2}\\
&=\sqrt{\sinh^2(r')+\sinh^2(r)-2\cos(\theta'-\theta)\sinh(r)\sinh(r')}\\
&=\sqrt{\big(\sinh(r')-\sinh(r)\big)^2+2(1-\cos(\theta'-\theta))\sinh(r)\sinh(r')}.
\end{aligned}
$$

Hence (13) squared becomes

$$
\varepsilon^2\left(\sinh(r')-\sinh(r)\right)^2+4\varepsilon^2\sin^2\frac{\theta'-\theta}{2}\sinh(r)\sinh(r')\geqslant\left(\cosh(r')-\cosh(r)\right)^2.
$$

If ε is small enough and r large enough (depending only on δ), the first term in the left-hand side is small compared to the right-hand side, so we have

$$
4\varepsilon^2\sin^2\frac{\theta'-\theta}{2}\sinh(r)\sinh(r')\geqslant(1+\delta/2)^{-2}\left(\cosh(r')-\cosh(r)\right)^2,
$$

which becomes

$$
2\varepsilon\left|\sin\frac{\theta'-\theta}{2}\right|\geqslant\left(1+\frac{\delta}{2}\right)^{-1}\frac{\cosh(r')-\cosh(r)}{\sqrt{\sinh(r)\sinh(r')}}\geqslant\left(1+\frac{\delta}{2}\right)^{-1}\frac{(r'-r)\sinh(r)}{\sqrt{\sinh(r)\sinh(r')}},
$$

which implies

$$
r'-r\leqslant\left(1+\frac{\delta}{2}\right)2\varepsilon\left|\sin\frac{\theta'-\theta}{2}\right|\sqrt{\frac{\sinh(r')}{\sinh(r)}}.
$$

Finally, recalling $r'\leqslant r+2\varepsilon$, if r is large enough (depending only on δ), this last expression is smaller than $(1+\delta)2\varepsilon\left|\sin\frac{\theta'-\theta}{2}\right|$, which concludes the proof of (12).

Let us move on to the second point. We know that $A^{\varepsilon}_{D(\mathcal{S},x)}(0,y)\subset D(\mathcal{S},x)\cap A^{\varepsilon}_{\mathbb{H}}(0,y)$. Using the expression of the area measure in polar coordinates, Equation (12) translates into

$$
\begin{aligned}
\left|A^{\varepsilon}_{D(\mathcal{S},x)}(0,y)\right|&\leqslant\int_0^{2\pi}\int_r^{r+2(1+\delta)|\sin\frac{\theta'-\theta}{2}|\varepsilon}\mathbb{1}_{\theta'\in I_{r'}(x)}\sinh(r')\,\mathrm{d}r'\,\mathrm{d}\theta'\\
&\leqslant\int_{I_r(x)}\int_r^{r+2(1+\delta)|\sin\frac{\theta'-\theta}{2}|\varepsilon}\sinh(r')\,\mathrm{d}r'\,\mathrm{d}\theta'\\
&\leqslant 2(1+\delta)\varepsilon\sinh(r+3\varepsilon)\int_{I_r(x)}\left|\sin\frac{\theta'-\theta}{2}\right|\mathrm{d}\theta',
\end{aligned}
$$

where the second inequality uses the inclusion $I_{r'}(x)\subset I_r(x)$. This concludes the proof of Lemma 6. □

We can now finish the proof of Proposition 2. We re-express (9) as an integral over $D(\mathcal{S},x)$ (since the projection p preserves the measure), and write it down in polar coordinates:

$$
\begin{aligned}
\mathbb{P}\left(x\in\partial^{\varepsilon}C\right)&\leqslant\lambda^2\int_0^{+\infty}\exp\left(-\lambda|B_r(x)|\right)\int_{I_r(x)}|A^{\varepsilon}_{\mathcal{S}}(x,p([r;\theta]))|\sinh(r)\,\mathrm{d}\theta\,\mathrm{d}r\\
&=o(\varepsilon)+\lambda^2\int_0^{+\infty}\exp\left(-\lambda|B_r(x)|\right)\int_{I_r(x)}\left|A^{\varepsilon}_{D(\mathcal{S},x)}(0,[r;\theta])\right|\sinh(r)\,\mathrm{d}\theta\,\mathrm{d}r,
\end{aligned}
$$

where the $o(\varepsilon)$ is uniform in x, and the last part comes from Lemma 5 and the fact that $D(\mathcal{S}, x)$ is bounded. We now assume that ε is smaller than the $\varepsilon_0(\delta)$ of Lemma 6. For r larger than the $r_0(\delta)$ of Lemma 6, we bound $\left|A^{\varepsilon}_{D(\mathcal{S},x)}(0, [r;\theta])\right|$ using Lemma 6. For $r \leqslant r_0(\delta)$, we use the crude bound $\left|A^{\varepsilon}_{D(\mathcal{S},x)}(0, [r;\theta])\right| \leqslant 4\pi\varepsilon \sinh(r+2\varepsilon)$ coming from (8). We obtain

$$\begin{aligned}\mathbb{P}\left(x \in \partial^{\varepsilon} C\right) \leqslant o(\varepsilon) + \lambda^2 \int_0^{r_0(\delta)} \int_0^{2\pi} 4\pi\varepsilon \sinh(r+2\varepsilon)\sinh(r)\,\mathrm{d}\theta\,\mathrm{d}r \\ + 2\lambda^2(1+\delta)\varepsilon \int_{r_0}^{+\infty} \exp\left(-\lambda|B_r(x)|\right) \int_{I_r(x)^2} \left|\sin\frac{\theta'-\theta}{2}\right| \sinh(r)\sinh(r+3\varepsilon)\,\mathrm{d}\theta\,\mathrm{d}\theta'\mathrm{d}r.\end{aligned}$$

The first integral is bounded by $C(\delta)\lambda^2\varepsilon$, so if λ is chosen smaller than some $\lambda_0(\delta)$ it is smaller than $\delta\varepsilon$. Moreover, up to increasing the value $r_0(\delta)$, we may assume $\sinh(r+3\varepsilon) \leqslant (1+\delta)\sinh(r)$. By these remarks and Lemma 3 to handle the integral over $I_r(x)^2$, we find

$$\mathbb{P}\left(x \in \partial^{\varepsilon} C\right) \leqslant o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}\lambda^2(1+\delta)^2\varepsilon \int_{r_0}^{+\infty} \exp\left(-\lambda|B_r(x)|\right)|I_r(x)|^2\sinh^2(r)\,\mathrm{d}r. \tag{14}$$

Our goal is now to write this in a form which can be directly integrated. For this, we notice that

$$|I_r(x)|\sinh(r) = |\partial B_r(x)| = \frac{\mathrm{d}}{\mathrm{d}r}\left|B_r(x)\right|.$$

Therefore, for all $r \geqslant r_0(\delta)$, we have

$$|B_r(x)| = \int_0^r |I_s(x)|\sinh(s)\,\mathrm{d}s \geqslant |I_r(x)|(\cosh(r)-1).$$

Up to increasing the value $r_0(\delta)$, we may assume $\cosh(r) - 1 \geqslant (1+\delta)^{-1}\sinh(r)$, so that $|I_r(x)|\sinh(r) \leqslant (1+\delta)|B_r(x)|$. Therefore, replacing one of the two factors $|I_r(x)|\sinh(r)$ in (14), we obtain

$$\begin{aligned}\mathbb{P}\left(x \in \partial^{\varepsilon} C\right) &\leqslant o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}\lambda^2(1+\delta)^3\varepsilon \int_0^{+\infty} \left(\frac{\mathrm{d}}{\mathrm{d}r}\left|B_r(x)\right|\right)|B_r(x)|\exp\left(-\lambda\left|B_r(x)\right|\right)\,\mathrm{d}r \\ &= o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}\lambda^2(1+\delta)^3\varepsilon\left[-\left(\frac{1}{\lambda^2} + \frac{|B_r(x)|}{\lambda}\right)\exp\left(-\lambda\left|B_r(x)\right|\right)\right]_0^{+\infty} \\ &= o(\varepsilon) + \delta\varepsilon + \frac{4}{\pi}(1+\delta)^3\varepsilon,\end{aligned}$$

where the $o(\varepsilon)$ is still uniform in x. Plugging this into (7), we obtain

$$\mathbb{E}\left[|\partial C|\right] \leqslant \frac{2}{\pi}(1+\delta)^4|\mathcal{S}|$$

for $\lambda < \lambda_0(\delta)$, which finally proves Proposition 2. □

Remark. Let us briefly compare our approach for Proposition 2 to the computations of Isokawa [16] in the hyperbolic plane. The main difference is that Isokawa takes a point of the Poisson process as the center of polar coordinates, whereas we center them around a typical point of $\mathcal{S}$. In particular, Isokawa does not need to consider an ε-thickening of the boundary. However, if we tried to adapt the arguments of [16] to our Dirichlet domains, we would need to carefully study the interplay between the sets $I_r(x)$ and $I_s(x)$ for some $r \neq s$, which seems complex. On the other hand, our argument only uses the interplay between $I_r(x)$ and itself (this is the role of Lemma 3).

3.4 Proof of Theorem 1

Let $0 < \delta < \frac{1}{2} < \frac{2}{\pi}$ and let $\mathcal{S}$ be a surface with genus g and Cheeger constant larger than δ (otherwise our result is trivial). Let $\lambda > 0$ be small enough so that we have by Proposition 2

$$\mathbb{E}\left[|\partial \mathrm{Vor}_\lambda(\mathcal{S})|\right] \leqslant \frac{2}{\pi}(1+\delta)|\mathcal{S}|.$$

Recall that A_λ is obtained by coloring each cell of $\mathrm{Vor}_\lambda(\mathcal{S})$ with probability $1/2$ independently. In particular, each side of $\partial\mathrm{Vor}_\lambda(\mathcal{S})$ is in ∂A_λ with probability $1/2$ (conditionally on $\mathrm{Vor}_\lambda(\mathcal{S})$), so

$$\mathbb{E}\left[|\partial A_\lambda|\right] \leqslant \frac{1}{\pi}(1+\delta)|\mathcal{S}|.$$

Therefore, by the Markov inequality

$$\mathbb{P}\left(|\partial A_\lambda| \leqslant \frac{1}{\pi}(1+\delta)^2|\mathcal{S}|\right) \geqslant 1 - \frac{(1+\delta)|\mathcal{S}|/\pi}{(1+\delta)^2|\mathcal{S}|/\pi} = \frac{\delta}{1+\delta}.$$

Now suppose that g is large enough so that Lemma 1 holds with δ replaced by δ^2. In this case, since $\frac{\delta}{1+\delta} + (1-\delta^2) > 1$ there is a positive probability that both $(1-\delta)\frac{|\mathcal{S}|}{2} \leqslant |A_\lambda| \leqslant (1+\delta)\frac{|\mathcal{S}|}{2}$ and $|\partial A_\lambda| \leqslant \frac{1}{\pi}(1+\delta)^2|\mathcal{S}|$. This implies

$$h(\mathcal{S}) \leqslant \frac{(1+\delta)^2}{1-\delta}\frac{2}{\pi},$$

which proves Theorem 1. Et voilà.

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