On the Cheeger constant of hyperbolic surfaces

Thomas Budzinski (CNRS and ENS de Lyon)

20 Septembre 2022
Séminaire de probabilités de Grenoble
Hyperbolic surface: Riemannian surface with constant curvature $-1$.

The Cheeger constant of a (compact) geometric object $X$ measures its "expansion":

$$h(X) = \inf \left\{ \frac{|\partial A|}{|A|} \mid A \subset X, |A| \leq \frac{|X|}{2} \right\}.$$

In general, $h(X)$ is bounded by the Cheeger constant of its universal cover, i.e. by 1 for hyperbolic surfaces.

Goal: prove that for a large hyperbolic surface $X$, we have $h(X) \leq \frac{2}{\pi} + o(1)$.

First: study the same phenomenon on $d$-regular graphs.
Connectivity of regular graphs

Let $G$ be a $d$-regular graph with $n$ vertices $V(G)$.

Various ways to measure the connectivity of a $d$-regular graph:

- Diameter: $\text{diam}(G) = \max\{d_G(x, y) | x, y \in V(G)\}$.
- Spectral gap: $d - \lambda_2$, where $d = \lambda_1 \geq \lambda_2 \geq \ldots$ is the spectrum of the adjacency matrix of $G$.
- Cheeger constant:

\[ h(G) = \inf \left\{ \frac{|\partial A|}{|A|} \left| A \subset V(G), |A| \leq \frac{|V(G)|}{2} \right\}, \]

where $\partial A$ is the set of edges with one end in $A$ and one end in $V(G) \setminus A$.

- Cheeger’s inequality:

\[ \frac{1}{2}(d - \lambda_2(G)) \leq h(G) \leq \sqrt{2d(d - \lambda_2(G))}. \]
Connectivity of regular graphs: diameter and spectral gap

- Diameter: at least \( \log_{d-1}(n) \) because the ball of radius \( r \) has size \( O((d-1)^r) \).
- For a uniform random graphs, the diameter is \((1 + o(1)) \log_{d-1}(n) \) w.h.p. [Bollobas–de la Vega 82].
  
  Reason: balls look like trees, so they grow "as quickly as possible".

- Spectral gap: comparison with the infinite \( d \)-regular tree [Alon–Boppana 91]:
  
  \[
  \lambda_2(G) \geq \lambda_2(\mathbb{T}_d) + o(1) = 2\sqrt{d-1} + o(1).
  \]

- This bound is optimal:
  
  - Uniform \( d \)-regular graphs satisfy \( \lambda_2(G) = 2\sqrt{d-1} + o(1) \) in probability [Friedman 03].
  
  - Arithmetic constructions of Ramanujan graphs, i.e. with \( \lambda_2(G) < 2\sqrt{d-1} \) [Lubotzky–Philipps–Sarnak 88, Margulis 88].
Cheeger constant:

\[ h(G) = \inf \left\{ \frac{|\partial A|}{|A|} \left| A \subset V(G), \ |A| \leq \frac{|V(G)|}{2} \right. \right\}, \]

\[ c_d = \limsup_{n \to +\infty} \max \{ h(G) \mid G \text{ is a } d\text{-regular graph with } n \text{ vertices} \}. \]

We have \( c_d \leq d - 2 = h(\mathbb{T}_d) \):

- Let \( A \) be a connected, not too small subset of \( V(G) \).
- If \( E(A) \) is the set of edges between vertices of \( A \), then

\[ d|A| = |\partial A| + 2|E(A)| \geq |\partial A| + 2(|A| - 1), \]

so \( \frac{|\partial A|}{|A|} \leq d - 2 + o(1) \).

On the other hand: various families of expanders show \( c_d > 0 \).

For example random graphs give \( c_d \geq \frac{d}{2} - O(\sqrt{d}) \) [Bollobas 88].
Actually \( c_d \leq \frac{d}{2} \) [Bollobas 88]:

- Choose \( A \subset V(G) \) randomly, by taking each vertex with probability \( \frac{1}{2} \) in an independent way.
- Then \( |A| \approx \frac{n}{2} \) with high probability...
- ...and \( \mathbb{E}[|\partial A|] = \frac{1}{2} \times \frac{dn}{2} \).
- So there is \( A \) with \( |A| \approx \frac{n}{2} \) and \( |\partial A| \leq \frac{dn}{4} \), so \( h(G) \leq \frac{d}{2} \).

Improvement [Alon 97]:

- First make \( 1 \ll k \ll n \) connected groups of size \( \frac{n}{k} \).
- Keep each group in \( A \) with probability \( \frac{1}{2} \).
- Then at least \( \approx n \) of the edges are inside a group, and \( \frac{d-2}{2} n \) of them are on the boundary between two groups.
- We obtain \( c_d \leq \frac{d-2}{2} \).

We have \( c_d \sim \frac{d}{2} \) as \( d \to +\infty \), but \( c_3 \) is still unknown.
The hyperbolic plane

The **hyperbolic plane** $\mathbb{H}$ can be seen as the unit disk, equipped with the metric

$$ds^2 = \frac{4 \, dx^2}{1 - |x|^2}.$$ 

**Curvature:**

$$|B(x, r)| = \pi \varepsilon^2 - \frac{\pi}{12} \varepsilon^4 K(x) + o(\varepsilon^4).$$

**Riemann uniformization theorem:** $\mathbb{H}$ is the unique simply connected surface with constant curvature equal to $-1$.

**Perimeter and volumes of balls:**

$$|B_\mathbb{H}(x, r)| = 2\pi (\cosh(r) - 1), \quad |\partial B_\mathbb{H}(x, r)| = 2\pi \sinh(r).$$
A compact hyperbolic surface $S$ is a 2d manifold equipped with a Riemannian metric with constant curvature $-1$. We consider closed surfaces, i.e. no boundary.

Gauss–Bonnet formula: $\int_S K(x)\,d\mathbf{x} = 2\pi(2 - 2g)$, where $g$ is the genus of the surface, i.e. the number of holes. So $g \geq 2$.

Equivalent definitions:
- $S$ is locally isometric to $\mathbb{H}$,
- $S$ is a quotient of $\mathbb{H}$ (by a nice enough group action),
- $S$ is a surface equipped with a conformal structure.

Hyperbolic surfaces with genus $g$ form a $(6g - 6)$-dimensional space $\mathcal{M}_g$. 
Diameter of a hyperbolic surface $S$: at least $(1 + o(1)) \log g$ (because balls are not larger in $S$ than in $\mathbb{H}$).

There is a random model of hyperbolic surfaces (random gluing of pants) where the diameter is $(1 + o(1)) \log g$ [B.–Curien–Petri 19].

Spectral gap: $\lambda_1(S)$ is the smallest nonzero eigenvalue of the Laplacian on $S$.

We have $\lambda_1(S) \leq \lambda_1(\mathbb{H}) + o(1) = \frac{1}{4} + o(1)$ as $g \to +\infty$ [Huber 74].

There is a random model of hyperbolic surfaces (random cover of a fixed, small surface) where the spectral gap is $\frac{1}{4} + o(1)$ [Hide–Maggee 21].

Selberg conjecture: $\lambda_1 > \frac{1}{4}$ for arithmetic surfaces (Selberg proved $\frac{3}{16}$).
The Cheeger constant of hyperbolic surfaces

- \( h(S) = \inf \left\{ \left| \frac{\partial A}{|A|} \right| : A \subset S, |A| \leq \frac{|S|}{2} \right\} \), where \(|A|\) is the area and \(|\partial A|\) the boundary length (maybe \(+\infty\)).

- Cheeger–Buser inequality: \( \frac{h(S)^2}{4} \leq \lambda_1(S) \leq 2h(S) + 10h(S)^2 \).

- Various families of hyperbolic surfaces with Cheeger constant bounded from below:
  - Random models: random surfaces built from 3-regular graphs [Brooks–Makover 04], Weil–Petersson random surfaces [Mirzakhani 13]...
  - Arithmetic surfaces: \( h(S) \geq 0, 168... \) [Brooks 99 + Kim–Sarnak 03].
The Cheeger constant of hyperbolic surfaces

- Hyperbolic plane: $h(\mathbb{H}) = 1$, attained for balls.
- If $h(S) > 1 + \varepsilon$, then for all $r \geq 0$ such that $|B_S(x, r)| < \frac{|S|}{2}$:

  $$\frac{d}{dr} |B_S(x, r)| = |\partial B_S(x, r)| \geq (1 + \varepsilon) |B_S(x, r)|,$$

  so $|B_S(x, r)| \geq c e^{(1+\varepsilon)r}$, absurd for $r$ large enough, so

  $$\limsup_{g \to +\infty} \sup_{S \in \mathcal{M}_g} h(S) \leq 1.$$

**Theorem (B.–Curien–Petri 22)**

We have

$$\limsup_{g \to +\infty} \sup_{S \in \mathcal{M}_g} h(S) \leq \frac{2}{\pi} \approx 0.637.$$
Strategy of proof

Like on graphs: cut the surface $S$ into $k$ regions ($1 \ll k \ll g$), color each region in black or white with probability $\frac{1}{2}$ and let $A$ be the union of black regions.

Strategy already used on specific expander models, with a clever choice of the regions:

- Arithmetic surfaces: $h(S) \lesssim 0.44$ [Brooks–Zuk 02],
- A natural model built from a random 3-regular graph: $h(S) \leq \frac{2}{3} + o(1)$ [Shen–Wu 22].

If $S$ is partitionned into regions $C_i$ with $\max(|C_i|) \ll |S|$, then $|A| \approx \frac{|S|}{2}$.

We have

$$\mathbb{E}[|\partial A|] = \frac{1}{2} |\partial C| := \frac{1}{2} \left| \bigcup_i \partial C_i \right|,$$

so $h(S) \leq \frac{|\partial C|}{|S|}$. 
Let \((x_i)\) be a Poisson point process with intensity \(\lambda\) on \(S\), i.e. for all \(R \subset S\), the number of points \(x_i\) in \(R\) has law \(Poisson(\lambda|R|)\), with independence between disjoint regions.

Voronoi tessellation:

\[
C_i = \{z \in S \mid \forall j, \, d_S(x_i, z) \leq d_S(x_j, z)\}.
\]

Finally, take \(\lambda\) small. We need to prove that for \(\lambda\) small enough:

\[
\limsup_{g \to +\infty} \sup_{S \in M_g} \mathbb{E} \left[ \left| \bigcup_i \partial C_i \right| \right] \leq \left( \frac{2}{\pi} + \delta \right) |S|.
\]
Let $\partial^\varepsilon C$ be the $\varepsilon$-neighbourhood of $\partial C$. Then

$$|\partial C| = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} |\partial^\varepsilon C|,$$

so

$$\mathbb{E} [|\partial C|] \leq \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_S \mathbb{P} (x \in \partial^\varepsilon C) \, dx,$$

so we want $\mathbb{P} (x \in \partial^\varepsilon C) \leq \frac{2}{\pi} + \delta$.

Easy version of the argument: assume that $S$ has a large injectivity radius around $x$, i.e. $B_S(r, x) = B_{\mathbb{H}}(r, x)$ for some $r \gg 1$. We first get the bound 1 instead of $\frac{2}{\pi}$.
Poisson–Voronoi tesselation of the hyperbolic plane

- Local picture around $x$: Poisson–Voronoi tesselation of the hyperbolic plane (for decreasing values of $\lambda$).

- Large injectivity radius: we can do all the computations in $\mathbb{H}$ instead of $S$.

- For a typical cell $C_i$ [Isokawa 00]:

$$
\mathbb{E} [ | C_i | ] = \frac{1}{\lambda}, \quad \mathbb{E} [ | \partial C_i | ] = \frac{8}{\sqrt{\pi \lambda}} \int_0^{\infty} e^{-u} \sqrt{u + \frac{u^2}{4\pi \lambda}} \, du \sim \frac{4}{\pi \lambda},
$$

but not robust enough for the general case.
The closest point to $x$

- Condition on the closest point of the Poisson process to $x$, say $x_{i_0}$, with $d_{\mathbb{H}}(x, x_{i_0}) = r$.

- If $x \in \partial^\varepsilon C$, then there is $j \neq i_0$ such that the bisector between $x_{i_0}$ and $x_j$ intersects $B_{\mathbb{H}}(x, \varepsilon)$, so $r \leq d_{\mathbb{H}}(x, x_j) \leq r + 2\varepsilon$, so

$$\mathbb{P}(x \in \partial^\varepsilon C \mid x_{i_0}) \lesssim \lambda \times 2\varepsilon \times |\partial B_{\mathbb{H}}(x, r)|.$$
On the other hand, law of $d_{\mathbb{H}}(x, x_{i_0})$:

$$\mathbb{P}(d_{\mathbb{H}}(x, x_{i_0}) \geq r) = \exp (-\lambda |B_{\mathbb{H}}(x, r)|).$$

For $\lambda$ small, we have $r$ large so $|\partial B_{\mathbb{H}}(r)| \approx |B_{\mathbb{H}}(r)|$, so

$$\mathbb{P}(x \in \partial^\varepsilon C) \leq 2\varepsilon \lambda^2 \int_0^{+\infty} \left( \frac{d}{dr} |B_{\mathbb{H}}(r)| \right) \times |\partial B_{\mathbb{H}}(r)| \times e^{-\lambda |B_{\mathbb{H}}(r)|} \, dr \approx 2\varepsilon \lambda^2 \int_0^{+\infty} \left( \frac{d}{dr} |B_{\mathbb{H}}(r)| \right) \times |B_{\mathbb{H}}(r)| \times e^{-\lambda |B_{\mathbb{H}}(r)|} \, dr\approx 2\varepsilon \lambda^2 \left[ \left( -\frac{|B_{\mathbb{H}}(r)|}{\lambda} + \frac{1}{\lambda^2} \right) e^{-\lambda |B_{\mathbb{H}}(r)|} \right]_0^{+\infty} = 2\varepsilon.$$

So $\mathbb{E}[|\partial C|] \lesssim |S|$ and $h(S) \lesssim 1$. 

Thomas Budzinski
Cheeger constant of hyperbolic surfaces
Let $A^\varepsilon(x, y)$ be the set of points $z \in \mathbb{H}$ such that the bisector between $y$ and $z$ intersects $B_\mathbb{H}(x, \varepsilon)$.

Good approximation in polar coordinates (for $r$ large):

$$\left\{ (r'; \theta') \mid r \leq r' \leq r + 2\varepsilon \left| \sin \frac{\theta'}{2} \right| \right\} .$$

So

$$|A^\varepsilon(x, y)| \approx 2\varepsilon \times \frac{2}{\pi} \times |\partial B_\mathbb{H}(x, r)| .$$

$$\mathbb{P}(x \in \partial^\varepsilon C \mid x_{i_0}) \approx \lambda |A^\varepsilon(x, x_{i_0})| \approx 2\varepsilon \lambda \times \frac{2}{\pi} |\partial B_\mathbb{H}(x, r)| .$$
Reminder: $S$ is a quotient of $\mathbb{H}$ by the action of a discrete isometry group $G$, i.e. there is a surjective isometry $p : \mathbb{H} \to S$ such that $p(x') = p(y')$ iff $\exists g \in G, y' = g \cdot x'$.

Fundamental domain: $D \subset \mathbb{H}$ such that $p$ is a bijection from $D$ to $S$.

Let $x \in S$, and assume $p(0) = x$. The Dirichlet domain of $S$ around $x$ is

$$D = D(S, x) = \{ x' \in \mathbb{H} \mid \forall g \in G, d_{\mathbb{H}}(0, x) \leq d_{\mathbb{H}}(0, g \cdot x) \}.$$ 

In other words, it is the Voronoi cell around 0 of the point set $\{ g \cdot 0 \mid g \in G \}$. 
Roughly speaking, the argument still works, replacing $|B_\mathbb{H}(x, r)|$ by $|B_D(x, r)| = |B_\mathbb{H}(x, r) \cap D|$.

Now, when we evaluate the contribution of points at distance $r$ from $x$, we need to integrate over the set

$$I_r = \{ \theta \in [0, 2\pi] | [r, \theta] \in D \},$$

and not just over $[0, 2\pi]$:

$$P \left( x \in \partial^\varepsilon C \mid x_{i_0} = [r, \theta] \right) \approx \lambda |A^\varepsilon(x, x_{i_0}) \cap D|$$

$$\approx \int_{I_r} 2\varepsilon \sinh(r) \left| \sin \frac{\theta' - \theta}{2} \right| \, d\theta'$$

and similarly, express the law of $x_{i_0}$ in polar coordinates:

$$\exp \left( -\lambda |B_D(x, r)| \right) 1_{\theta \in I_r} \sinh(r) \, dr \, d\theta.$$
The general case

In the contribution of \( \{ d(x, x_{i_0}) = r \} \), the following integral appears:

\[
\int_{I_r \times I_r} \left| \sin \frac{\theta - \theta'}{2} \right| \, d\theta \, d\theta'.
\]

Useful lemma: for all finite measure \( \mu \) on \([0, 2\pi]\),

\[
\int_0^{2\pi} \int_0^{2\pi} \left| \sin \frac{\theta_1 - \theta_2}{2} \right| \mu(\mathrm{d}\theta_1) \mu(\mathrm{d}\theta_2) \leq \frac{2}{\pi} \mu([0, 2\pi])^2,
\]

with equality for the uniform measure \([\text{Toth 56}].\)

We find again

\[
\mathbb{P}(x \in \partial^\varepsilon C) \leq 2\varepsilon \lambda^2 \times \frac{2}{\pi} \int_0^{+\infty} \left( \frac{d}{dr} |B_D(r)| \right)^2 \times e^{-\lambda |B_D(r)|} \, dr
\]

and finish the computation as before.
Further questions

- The bound $\frac{2}{\pi}$ should not be sharp, as the interfaces are not straight geodesics. Can we get to $\frac{1}{2}$ or below?
- On $\mathbb{H}$, Poisson–Voronoi has a nontrivial limit when $\lambda \to 0$: points go to infinity but interfaces stay there ("pointless Voronoi diagram"). Study this object?
  - Already known: $p_c \sim \frac{\pi}{3} \lambda$ as $\lambda \to 0$ [Hansen–Müller 20].
THANK YOU!