Recurrence of the UIHPM via duality of resistances

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A planar map is a gluing of polygons homeomorphic to the sphere (finite case) or to the plane (infinite case).

We consider rooted maps (distinguished oriented edge).
Local limits of uniform random maps

- Natural idea for probabilists: pick a random map uniformly at random among all maps of fixed finite size (in a certain class).
- Motivation: theoretical physics (two-dimensional quantum gravity).
- To take the limit when the size goes to $+\infty$, local convergence: two maps are close if they have large isomorphic balls around the root.
- Limiting objects are infinite planar maps such as:
  - the UIPT for triangulations [Angel–Schramm 2003],
  - the UIPQ for quadrangulations [Krikun 2005],
  - the UIPM for general planar maps [Ménard–Nolin 2013].
- Metric properties are well understood:
  - Volume growth $\approx r^4$ [Angel 2004]
  - Scaling limit: Brownian plane [Curien–Le Gall 2014]
  - Links with Liouville Quantum Gravity [Miller–Sheffield 2015, 2016...]

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Recurrence of the UIHPM
Historically, the simple random walk on these models has been much harder to understand.

Recurrence of the SRW on a wide class of models [Gurel-Gurevich–Nachmias 2012, Lee 2017].

Speed: the distance of the SRW at time $n$ to the root is $n^{1/4+o(1)}$ [Gwynne–Hutchcroft 2018].

Bounds on the resistance between the root and the boundary of the ball of radius $r$:

- At most polynomial in $\log r$ [Gwynne–Miller 2017],
- Circle packing arguments for recurrence are expected to give lower bounds of order $\log \log r$,
- Conjectured order of magnitude: $\log r$. 
Recurrence of half-plane models

- Half-plane models: models of infinite maps with an infinite (simple) boundary.
- Recurrence of the Uniform Infinite *Half-Plane* Triangulation [Angel–Ray 2016].
- UIHPM: similar model for general maps.

**Theorem (B.–Lehéricy 2019)**

The UIHPM $M_\infty$ s a.s. recurrent. More precisely, there is a constant $c$ such that a.s.:

$$R_{M_\infty} (\rho \leftrightarrow \partial B_r(M_\infty)) \geq c \log r$$

for $r$ large enough.
First definition: local limit of critical Boltzmann maps with simple boundary. Let $M_p$ be such that

$$P(M_p = m) = \frac{1}{Z_p} \left( \frac{1}{12} \right)^{\#\text{Edges}(m)}$$

for all map $m$ with a simple boundary of length $p$, rooted on the boundary. Then $M_\infty$ is the local limit of $M_p$ as $p \to +\infty$.

Second construction: from the Uniform Infinite Half-Plane Quadrangulation, using the Tutte bijection.
Correspondence between quadrangulations and general maps:

- Start from the UIHPQ $Q_\infty$ and apply the Tutte correspondence. There is a unique infinite 2-connected component, which has the distribution of $M_\infty$.
- The finite 2-connected components have negligible contribution to distances and to resistances, so we can work on $\text{Tutte}(Q_\infty)$. 
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Central idea: use the self-duality of the model. We find a self-dual block in which the resistance is bounded from below with positive probability.

Using peeling estimates in the UIHPQ, we build a logarithmic number of these self-dual blocks between the root and infinity.

Similarities with RSW theory in percolation. Also used in the context of SRW on $\mathbb{Z}^2$ with weights given by the exponential of a Gaussian Free Field [Biskup–Ding–Goswami 2016].
Spatial Markov property: the quadrilateral incident to the root edge has one of the following shapes:

- Finite regions are filled with *critical Boltzmann quadrangulations*.
- The infinite region has the law of $Q_\infty$.
Consequence: we can explore $Q_\infty$ almost face by face. Explored region after $n$ steps:

\[ \partial_t E_n(Q_\infty) \]

\[ \partial_b E_n(Q_\infty) \]

The boundary length variation $X_n = |\partial_t E_n(Q_\infty)| - |\partial_b E_n(Q_\infty)|$ is a random walk with explicitly known transitions.

Scaling limit of $(X_n)$: stable Lévy process with index 3/2 and only negative jumps.

Flexibility: at each step, we can choose the next boundary edge to explore (peeling algorithm).
Let $M$ be a planar map with two vertices $a_1, a_3$ on the unbounded face. Let $(M^*, a_2^*, a_4^*)$ be the dual map.

Then $R_{M^*} (a_2^* \leftrightarrow a_4^*) = (R_M (a_1 \leftrightarrow a_3))^{-1}$. 
Consider $Q_p$ critical Boltzmann quadrangulation of the $2p$-gon, i.e. $\mathbb{P}(Q_p = q) = \frac{1}{Z_p} \left( \frac{1}{12} \right)^{\#\text{Faces}(q)}$.

Split the boundary into 4 parts $A_1$, $A_2^*$, $A_3$, $A_4^*$.

Let $M_p$ (resp. $M_p^*$) be the map obtained by the Tutte bijection on white (resp. black) vertices.
Then \( R_{M_p}(A_1 \leftrightarrow A_3) = \left( R_{M^*_p}(A_2^* \leftrightarrow A_4^*) \right)^{-1} \).

Symmetry black/white: if \( |A_1|, |A_3| \leq |A_2^*|, |A_4^*| \), then \( R_{M_p}(A_1 \leftrightarrow A_3) \) stochastically dominates \( R_{M^*_p}(A_2^* \leftrightarrow A_4^*) \).

Consequence: for all \( p \),

\[
P\left( R_{M_p}(A_1 \leftrightarrow A_3) \geq 1 \right) \geq \frac{1}{2}.
\]
Fix a red segment of length \( L \geq 1 \) on \( \partial Q_\infty \). We want to build a self-dual Boltzmann block \( B \) which:
- separates the red segment from infinity,
- has top and bottom boundaries smaller than the left and right boundaries.

Explore \( Q_\infty \) with the following peeling algorithm: always peel the edge at distance \( 2L \) on the left of the red segment.

Stop the exploration at the time \( \tau \) where the boundary length process makes a jump \( \leq -L \).
Stop the exploration at the time $\tau$ where the boundary length process makes a jump $\leq L$.

In particular, if a point at distance $< L$ to the red segment is hit, the exploration stops.
Construction of a self-dual block at a given scale

- Stop the exploration at the time $\tau$ where the boundary length process makes a jump $\leq L$.
- In particular, if a point at distance $< L$ to the red segment is hit, the exploration stops.

Assume the jump at time $\tau$ is $< -4L$ and is caused by swallowing a region $B$ on the right of the peeled edge.
Then the hole in green is filled by a Boltzmann quadrangulation with top and bottom boundaries $\leq L$ and left, right boundaries $\geq L$. 
Construction of a self-dual block at a given scale

- $P(\text{the first jump } < -L \text{ is } < -4L) \to \delta > 0 \text{ as } L \to +\infty$.
- So with probability $\geq \frac{\delta}{2}$, the green block has the right dimensions.
- Scaling limit results on the boundary length process: $\mathbb{E}\left[\frac{L'}{L}\right]$ is bounded.
- Iterate this construction in the unexplored region, with $L'$ playing the role of the new $L$. 
Construction of a logarithmic number of blocks

- Repeat this construction, starting from $L_0 = 1$:

At each scale, conditionally on the previous ones, probability to have a "nice" green block $\geq \frac{\delta}{2}$, so the probability to have a resistance at least 1 is $\geq \frac{\delta}{4}$.

Hence $R(\rho \leftrightarrow n$-th blue region) $\geq \frac{\delta}{4}n$

On the other hand $\mathbb{E}[L_n] \leq e^{cn}$, so the segment of length $L_n$ is at exponential distance from $\rho$. 
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Self-duality is crucial, so no hope to generalize the approach to other classes of maps (e.g. triangulations).

Full plane topology?

Matching lower bound? Natural first step: prove that the resistance in a large self-dual block is typically of order 1.

Self-dual random maps equipped with statistical physics models?
THANK YOU!