The local limit of uniform triangulations in high genus

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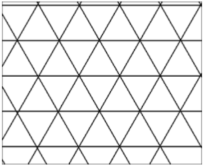
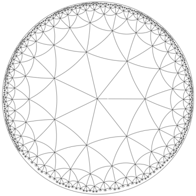
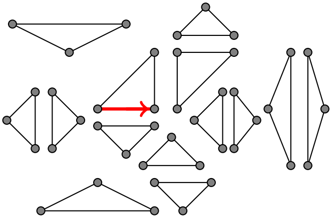
Motivations

- We would like to define a discrete "random two-dimensional geometry", in a way that is as uniform as possible.
- Regular prototypes: the 6-regular and the 7-regular triangular lattices.



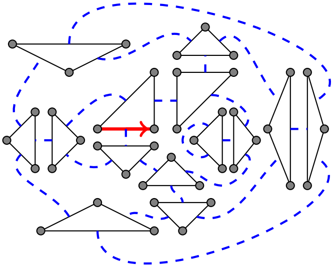


- For physicists, discrete model of "2d quantum gravity".
- Basic idea: use finite random objects, and take the limit.



- A *triangulation* with $2n$ faces is a set of $2n$ triangles whose sides have been glued two by two, in such a way that we obtain a connected, orientable surface.
- The *genus* g of the triangulation is the number of holes of this surface ($g = 0$ on the figure).
- Our triangulations are *of type I* (we may glue two sides of the same triangle), and *rooted* (oriented root edge).

Finite triangulations



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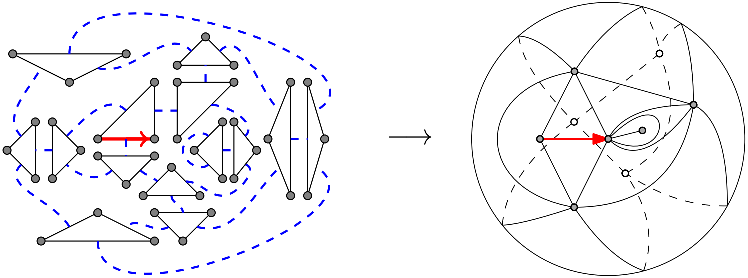

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Some combinatorics

- Let $\mathscr{T}_{n,g}$ be the set of triangulations of genus g with $2n$ faces, and $\tau(n,g)$ its size.
- Let also $\tau_p(n,g)$ be the number of triangulations of size n and genus g, where the face on the right of the root has perimeter p. Can we compute those numbers?
- In the planar case, exact formulas [Tutte, 60s]:

$$\tau(n,0) = 2\frac{4^n(3n)!!}{(n+1)!(n+2)!!} \underset{n\to+\infty}{\sim} \sqrt{\frac{6}{\pi}}(12\sqrt{3})^n n^{-5/2},$$

  where $n!! = n(n-2)(n-4)\ldots$. We also know $\tau_p(n,0)$ explicitely.
- In general, double recurrence relations [Goulden–Jackson, 2008], but no close formula.
- Known asymptotics when $n \to +\infty$ with g fixed, but not when both $n, g \to +\infty$.

Local convergence

- For any (finite or infinite) rooted triangulation t and $r \geq 0$, let $B_r(t)$ be the ball of radius r around the root vertex in t.
- For any two triangulations t and t', set

$$d_{loc}(t,t') = \left(1 + \max\{r \geq 0 | B_r(t) = B_r(t')\}\right)^{-1}.$$

  This is the *local distance*: we focus on the neighbourhood of the root.
- Let $T_{n,g}$ be uniform in $\mathscr{T}_{n,g}$. We want to study local limits of $T_{n,g}$ when both n and g go to infinity.

The planar case

Theorem (Angel–Schramm, 2003)

We have the convergence

$$T_{n,0} \xrightarrow[n\to+\infty]{(d)} \mathbb{T}$$

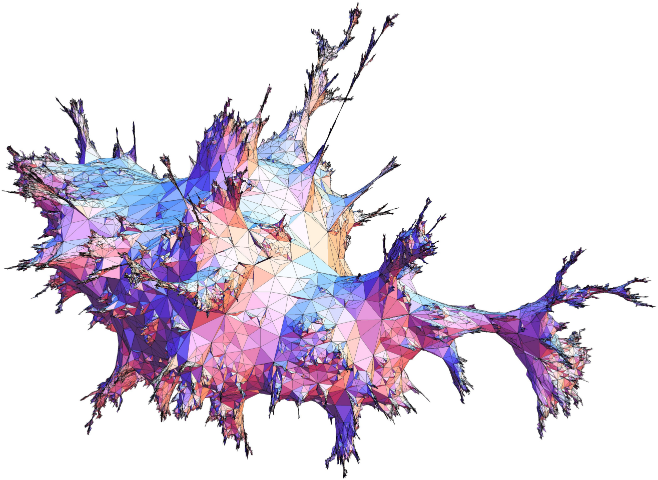
in distribution for the local topology, where $\mathbb{T}$ is an infinite triangulation of the plane called the *UIPT* (Uniform Infinite Planar Triangulation).

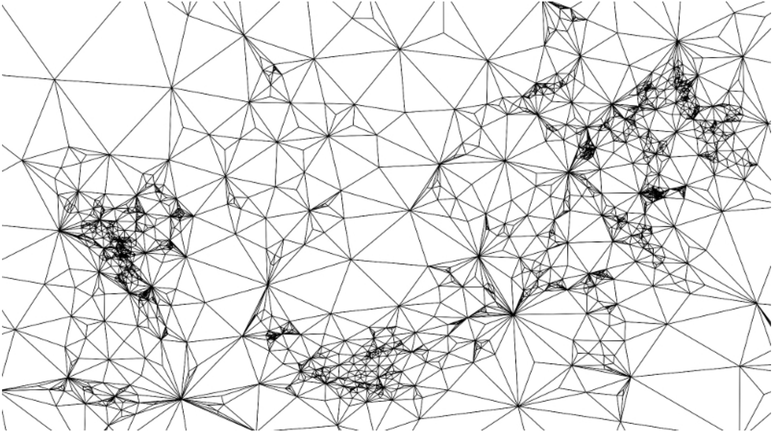
- Quick sketch of the proof: if t has size v and perimeter p, then

$$\mathbb{P}\left(t \subset T_{n,0}\right) = \frac{\tau_p(n-v,0)}{\tau(n,0)},$$

and the limit is given by the results of Tutte.

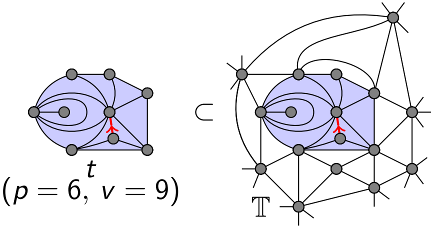
A sample of $T_{32400,0}$





The spatial Markov property of $\mathbb{T}$

- Let t be a small triangulation with perimeter p and v vertices in total.



- Then $\mathbb{P}(t \subset \mathbb{T}) = C_p \times \lambda_c^v$, where $\lambda_c = \frac{1}{12\sqrt{3}}$ and the C_p are explicit.
- Consequence: conditionally on $t \subset \mathbb{T}$, the law of $\mathbb{T} \backslash t$ only depends on p.
- Allows to explore $\mathbb{T}$ in a Markovian way: *peeling explorations* are one of the most important tools in the study of $\mathbb{T}$ [Angel, 2004...].

The non-planar case: what is going on?

- Euler formula: $T_{n,g}$ has $\#E = 3n$ edges and $\#V = n + 2 - 2g$ vertices. In particular $g \leq \frac{n}{2}$.
- Hence, the *average degree* in $T_{n,g}$ is

$$\frac{2\#E}{\#V} = \frac{6n}{n+2-2g} \approx \frac{6}{1-2g/n}.$$

- Interesting regime: $\frac{g}{n} \to \theta \in \left(0, \frac{1}{2}\right)$. The average degree in the limit is strictly between 6 and $+\infty$, so we expect a *hyperbolic* behaviour.

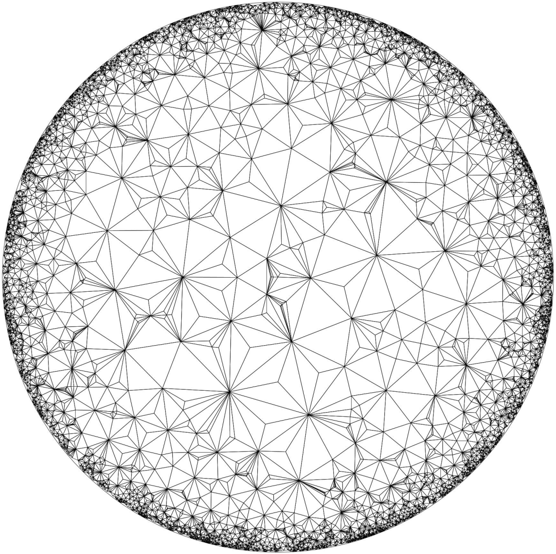
The Planar Stochastic Hyperbolic Triangulations

- The PSHT $(\mathbb{T}_\lambda)_{0<\lambda\leq\lambda_c}$, where $\lambda_c = \frac{1}{12\sqrt{3}}$, have been introduced in **[Curien, 2014]**, following similar works on half-planar maps **[Angel–Ray, 2013]**.
- For every triangulation t with perimeter p and volume v, we have

$$\mathbb{P}\left(t \subset \mathbb{T}_\lambda\right) = C_p(\lambda)\lambda^v,$$

where the number $C_p(\lambda)$ are explicit **[B. 2016]**.
- $\mathbb{T}_{\lambda_c}$ is the UIPT. For $\lambda < \lambda_c$, they have a *hyperbolic behaviour*:
    - exponential volume growth **[Curien, 2014]**,
    - transience and positive speed of the simple random walk **[Curien, 2014]**,
    - existence of infinite geodesics in many different directions **[B., 2018]**...

A sample of a PSHT



The local limit of $T_{n,g}$

Theorem (B.–Louf, 2019)

Let $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right)$. Then we have the convergence

$$T_{n,g_n} \xrightarrow[n\to+\infty]{(d)} \mathbb{T}_{\lambda(\theta)}$$

in distribution for the local topology, where $\lambda(\theta)$ and θ are linked by an explicit equation.

- In particular, if $g_n = o(n)$, then the limit is the UIPT.
- It may seem surprising that highly non-planar objects become planar in the limit, but this is already the case in other contexts (ex: random regular graphs).
- The case $\theta = \frac{1}{2}$ is degenerate (vertices with "infinite degrees").

Back to combinatorics

- Natural idea to prove the theorem: as in the planar case, use asymptotic results on the number $\tau_p(n, g_n)$ of triangulations of size n with genus g and a boundary of length p.
- Unfortunately, this seems very hard to obtain directly asymptotics, so new ideas are needed.
- On the other hand, our local convergence result gives the limit value of the ratio $\frac{\tau(n+1,g_n)}{\tau(n,g_n)}$ when $\frac{g_n}{n} \to \theta$, and allows to obtain asymptotic enumeration results up to sub-exponential factors.

Theorem (B.–Louf, 2019)

When $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right]$, we have

$$\tau(n, g_n) = n^{2g_n} \exp\left(f(\theta)n + o(n)\right),$$

where $f(\theta) = 2\theta \log \frac{12\theta}{e} + \theta \int_{2\theta}^{1} \log \frac{1}{\lambda(\theta/t)} \mathrm{d}t$, and $\lambda(\theta)$ is the same as in the previous theorem.

Steps of the proof

- Tightness result, plus planarity and one-endedness of the limits.
- Any subsequential limit T is *weakly Markovian*: for any finite t, the probability $\mathbb{P}(t \subset T)$ only depends on the perimeter and volume of t.
- Any weakly Markovian random triangulation of the plane is a mixture of PSHT (i.e. $\mathbb{T}_\Lambda$ for some random Λ).
- Ergodicity: Λ is deterministic, characterized by the fact that the average degree must be $\frac{6}{1-2\theta}$.

Tightness: the bounded ratio lemma

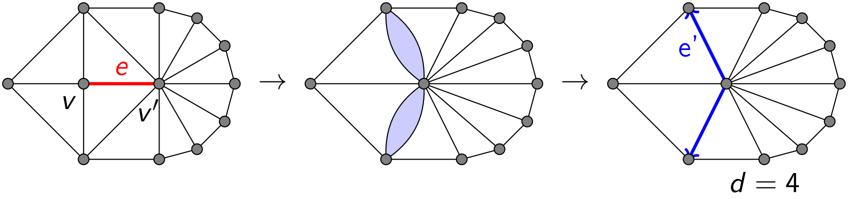
Lemma

Fix $\varepsilon > 0$. There is a constant C_ε such that, for every p, n and for every $g \leq \left(\frac{1}{2} - \varepsilon\right) n$, we have

$$\frac{\tau_p(n,g)}{\tau_p(n-1,g)} \leq C_\varepsilon.$$

- This is the "minimal combinatorial input" needed to adapt the Angel–Schramm argument for tightness.
- Proof: the average degree is $\frac{6n}{n+2-2g_n} \leq \frac{3}{\varepsilon}$, so there are εn *good* vertices with degree $\leq \frac{6}{\varepsilon}$.
- Consider a good vertex v and remember its degree $d \leq \frac{6}{\varepsilon}$. Choose an edge e joining v to another vertex v'. We will contract e.

Proof of the bounded ratio lemma



- From a triangulation with size n and a good vertex v, we obtain a triangulation with size $n-1$ with a marked (oriented) edge e', and a degree $d \leq \frac{6}{\varepsilon}$.
- Given d, we can find the other blue edge and reverse the operation, so the operation is injective.
- At least $\tau(n, g_n) \times \varepsilon n$ inputs, and at most $\tau(n-1, g_n) \times 6n \times \frac{6}{\varepsilon}$ outputs, so $\frac{\tau(n,g_n)}{\tau(n-1,g_n)} \leq \frac{36}{\varepsilon^2}$.

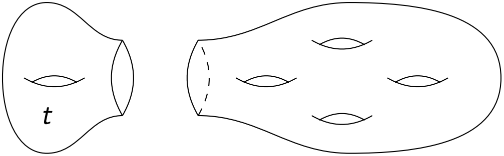
Tightness

- As in [Angel–Schramm, 2003], we first prove that the degree of the root in T_{n,g_n} is tight.
- We explore the neighbours of the root vertex ρ step by step.

- We have $\mathbb{P}\left(t^+ \subset T_{n,g_n} | t \subset T_{n,g_n}\right) = \frac{\tau_p(n-v-1,g_n)}{\tau_p(n-v,g_n)} \geq \frac{1}{C_\varepsilon}$.
- Hence, the number of steps needed to finish the exploration of the root has exponential tail uniformly in n, so the root degree is tight.
- The root vertex degree is tight and T_{n,g_n} is stationary for the simple random walk, so the degrees in all the neighbourhood of the root are tight, which is enough to ensure tightness for the local topology.

Planarity and the Goulden–Jackson formula

- Let T be a subsequential limit. If T is not planar, it contains a finite t which is not planar, say with genus 1.
- $\mathbb{P}(t \subset T) = \lim_{n\to+\infty} \mathbb{P}(t \subset T_{n,g_n}) = \lim_{n\to+\infty} \frac{\tau_p(n-v,g_n-1)}{\tau(n,g_n)}$.



- Goulden–Jackson formula (algebraic black box):

$$\tau(n,g) = \frac{4}{n+1}\Big(n(3n-2)(3n-4)\tau(n-2,g-1) + \sum_{\substack{n_1+n_2=n-2\\ g_1+g_2=g}} (3n_1+2)(3n_2+2)\tau(n_1,g_1)\tau(n_2,g_2)\Big).$$

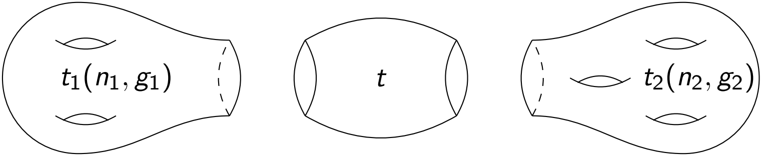
- Looking at the first term gives

$$\tau(n,g-1) \leq \frac{c}{n^2}\tau(n+2,g) \leq \frac{c'}{n^2}\tau(n+v,g),$$

and surgery operations allow to add a boundary, so $\mathbb{P}(t \subset T) = 0$.

One-endedness

- To be sure that T can be nicely embedded in the plane, we need it to be one-ended, i.e. for any finite $t \subset T$, the complementary $T \backslash t$ has only one infinite connected component.
- We want to show that this does not occur with n_1 and n_2 large:



- Use the second part of the Goulden–Jackson formula:

$$\tau(n,g) = \frac{4}{n+1}\Big(n(3n-2)(3n-4)\tau(n-2,g-1) + \sum_{\substack{n_1+n_2=n-2 \\ g_1+g_2=g}} (3n_1+2)(3n_2+2)\tau(n_1,g_1)\tau(n_2,g_2)\Big).$$

Weakly Markovian triangulation and mixture of PSHT

- Let T be a subsequential limit of (T_{n,g_n}), and let t be a finite triangulation with perimeter p and volume v.
- Then $\mathbb{P}(t \subset T) = a_v^p$. We say that T is *weakly Markovian*.
- The PSHT are weakly Markovian with $a_v^p = C_p(\lambda)\lambda^v$, so any PSHT with a random parameter Λ is weakly Markovian with $a_v^p = \mathbb{E}[C_p(\Lambda)\Lambda^v]$.

Theorem (B.–Louf, 2019)

Any weakly Markovian random triangulation of the plane is a PSHT with random parameter.

Weakly Markovian triangulation: sketch of the proof

- The numbers a_v^p are linked by the *peeling equations*:

$$a_v^p = a_{v+1}^{p+1} + 2\sum_{i=0}^{p-1}\sum_{j=0}^{+\infty} \tau_{i+1}(j,0)a_{v+j}^{p-i}.$$

- In particular, we can express a_{v+1}^{p+1} in terms of constants with smaller values of p, so everything is determined by $(a_v^1)_{v\geq 1}$.
- For the PSHT, we have $a_v^1 = C_1(\lambda)\lambda^v = \lambda^{v-1}$, so we are looking for a variable $\Lambda \in (0, \lambda_c]$ such that

$$\forall v \geq 1, a_v^1 = \mathbb{E}[\Lambda^{v-1}].$$

Weakly Markovian triangulation: sketch of the proof

- If we want $\Lambda \in [0,1]$, this is precisely the Hausdorff moment problem. It is enough to check that

$$\forall k \geq 0, \forall v \geq 1, (\Delta^k a^1)_v \geq 0,$$

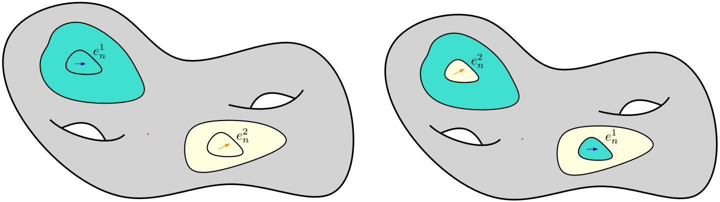
where Δ is the discrete derivative operator:

$$(\Delta u)_n = u_n - u_{n+1}.$$

- The numbers a_v^p are linear functions of the a_v^1 and are nonnegative. This proves $(\Delta^k a^1)_v \geq 0$ by doing th right algebraic manipulations.
- If $\Lambda > \lambda_c$, the sum in the peeling equations does not converge. If $\lambda = 0$, then T has vertices with infinite degrees, so $\Lambda \in (0, \lambda_c]$.

Ergodicity: the two holes argument

- We know that any subsequential limit of T_{n,g_n} is of the form $\mathbb{T}_\Lambda$, where Λ is random and we want Λ deterministic.
- In other words, T_{n,g_n} looks like $\mathbb{T}_\Lambda$ around the root edge e_n. We first prove that Λ does not depend on the choice of e_n on T_{n,g_n}, and then that it does not depend on T_{n,g_n}.
- Idea: pick two uniform root edges e_n^1 and e_n^2 on T_{n,g_n}. The neighbourhoods of e_n^1 and e_n^2 converge to $\mathbb{T}^1_{\Lambda_1}$ and $\mathbb{T}^2_{\Lambda_2}$.
- We consider two "balls" around e_n^1 and e_n^2 with the same perimeter and swap them.



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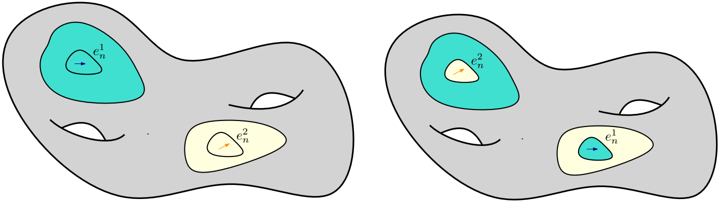

- The triangulation on the right is still uniform, so the neighbourhoods of e_n^1 on the right should look like a PSHT.
- On the other hand, the volume growth in $\mathbb{T}_\lambda$ is $f(\lambda)^r$, where $f(\lambda)$ is strictly monotone[Curien, 2014].
- Hence, if $\Lambda_1 \neq \Lambda_2$, the neighbourhood of e_1^n on the right grows like $f(\Lambda_1)^r$ up to a certain r and then like $f(\Lambda_2)^r$, which is very unlikely for a PSHT, so $\Lambda_1 = \Lambda_2$ a.s.

Ergodicity: end of the proof

- Since Λ only depends on T_{n,g_n} and not on the root, we can "group" the triangulation according to the corresponding Λ.
- For any T_{n,g_n}, the average root degree over all choices of the root is $\frac{6n}{n+2-g_n} \to \frac{6}{1-2\theta}$. Hence, conditionally on Λ, the average root degree is $\frac{6}{1-2\theta}$.
- On the other hand, the average degree $d(\lambda)$ in $\mathbb{T}_\lambda$ can be explicitely computed, and we must have

$$\frac{6}{1-2\theta} = d(\Lambda).$$

Since d is monotone, this fixes the value of Λ and we are done.

Further questions

- What about more general models? k-angulations? Boltzmann planar maps?
- Models with boundary? With both a high genus and a large boundary?
- Maps decorated with statistical physics models?
- Global structure of uniform triangulations with high genus? Interaction between local and scaling limits?

THANK YOU !