Random triangulations in high genus

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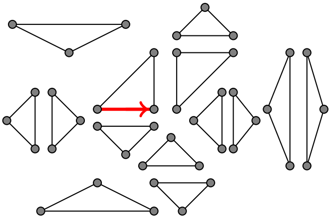
Outline

- Random triangulations: discrete models of random two-dimensional geometry.
- Planar case: very active in the last 20 years, motivated by 2-dimensional quantum gravity, exact enumeration helps a lot.
- Higher genus: more recent, enumeration is much more difficult!
- "Random planar maps are fractals, random high genus maps are expanders".

Finite triangulations



- A *triangulation* with $2n$ faces is a set of $2n$ triangles whose sides have been glued two by two, in such a way that we obtain a connected, orientable surface.
- The *genus* g of the triangulation is the number of holes of this surface ($g = 0$ on the figure).
- Our triangulations are *of type I* (we may glue two sides of the same triangle), and *rooted* (oriented root edge).

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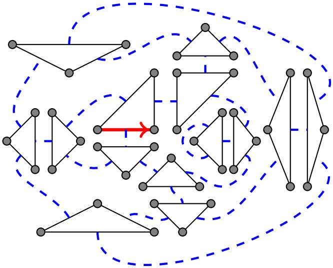

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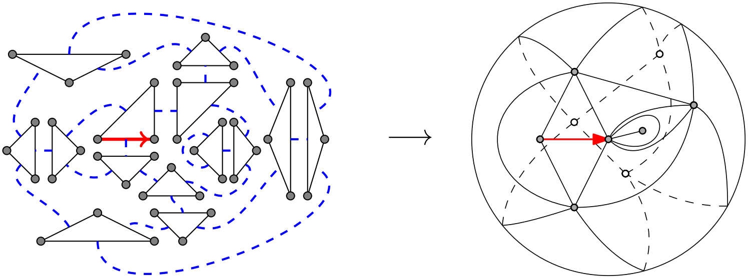

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Some combinatorics

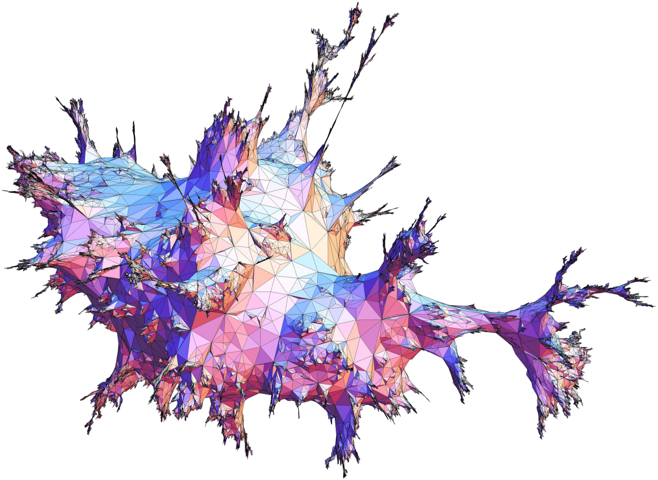
- Let $\mathcal{T}_{n,g}$ be the set of triangulations of genus g with $2n$ faces, and $\tau(n,g)$ its cardinal.
- Let also $\tau_p(n,g)$ be the number of triangulations of size n and genus g, where the face on the right of the root has perimeter p. Can we compute those numbers?
- In the planar case, exact formulas **[Tutte, 60s]**:

$$\tau(n,0) = 2\frac{4^n(3n)!!}{(n+1)!(n+2)!!} \underset{n\to+\infty}{\sim} \sqrt{\frac{6}{\pi}}(12\sqrt{3})^n n^{-5/2},$$

  where $n!! = n(n-2)(n-4)\ldots$. We also know $\tau_p(n,0)$ explicitely.
- In general, double recurrence relations **[Goulden–Jackson, 2008]**, but no close formula.
- Known asymptotics when $n \to +\infty$ with g fixed, but not when both $n, g \to +\infty$.

- Let $T_{n,g}$ be a uniform triangulation in $\mathcal{T}_{n,g}$. Equip $T_{n,g}$ with its graph metric.
- Distances in $T_{n,0}$ are typically of order $n^{1/4}$ [$\approx$ Chassaing–Schaeffer 2004].
- If the distances are renormalized by $n^{1/4}$, Gromov–Hausdorff convergence to the *Brownian sphere*[Le Gall 2011, $\approx$ Miermont 2011]: random compact metric space, homeomorphic to the sphere and with Hausdorff dimension 4.
- Similar results hold for $T_{n,g}$ with g fixed and $n \to +\infty$ (at least for quadrangulations). [Bettinelli 2010, Bettinelli–Miermont 2022].
- Main tool: bijections with labelled trees or labelled tree-like structures.

A sample of $T_{32400,0}$



The planar case: local properties

- *Local convergence*: two triangulations t and t' are close if there is a large r such that $B_r(t) = B_r(t')$, where B_r denotes the ball for the graph distance.

Theorem (Angel–Schramm 2003)

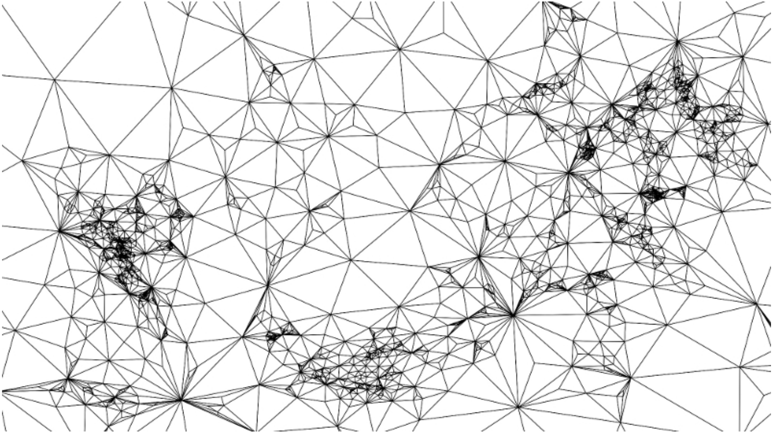
We have the convergence in distribution

$$T_{n,0} \xrightarrow[n\to+\infty]{(d)} \mathbb{T}$$

for the local topology, where $\mathbb{T}$ is an infinite triangulation of the plane called the *UIPT* (Uniform Infinite Planar Triangulation).

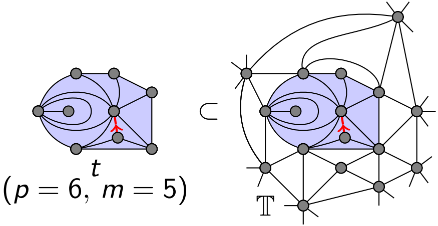
- Moreover, the UIPT has volume growth or order r^4 [Angel 2004].

The UIPT



Convergence to the UIPT $\mathbb{T}$: sketch of the proof

- Let t be a small triangulation with perimeter p and $2m$ internal faces.



- Then $P(t \subset T_{n,0}) = \frac{\tau_p(n-m,0)}{\tau(n,0)}$, and the limit is given by the results of Tutte.

Exact enumeration $\Rightarrow$ Probability

- In particular $\mathbb{P}(t \subset \mathbb{T})$ is explicit, which allows to explore $\mathbb{T}$ "face by face" in a Markovian way ("peeling").

The non-planar case: what is going on?

- Euler formula: $T_{n,g}$ has $\#E = 3n$ edges and $\#V = n + 2 - 2g$ vertices. In particular $g \leq \frac{n}{2}$.
- Hence, the *average degree* in $T_{n,g}$ is

$$\frac{2\#E}{\#V} = \frac{6n}{n+2-2g} \approx \frac{6}{1-2g/n}.$$

- Interesting regime: $\frac{g}{n} \to \theta \in \left(0, \frac{1}{2}\right)$. The average degree in the limit is strictly between 6 and $+\infty$.
- The d-regular infinite triangulation for $d > 6$ is hyperbolic, so we expect a *hyperbolic* behaviour.

The high genus case: local limits

Theorem (B.–Louf 2019)

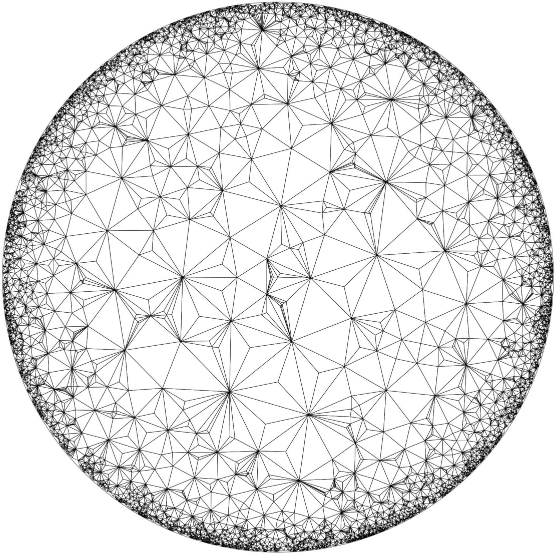
Let $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right)$. Then we have the convergence

$$T_{n,g_n} \xrightarrow[n\to+\infty]{(d)} \mathbb{T}_\theta$$

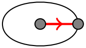
in distribution for the local topology, where $\mathbb{T}_\theta$ is a random infinite triangulation of the plane called *PSHT*.

- In particular $\mathbb{T}_0$ is the UIPT, so if $g_n = o(n)$, the limit is the same as for $g_n = 0$.
- For $\theta > 0$, the triangulation $\mathbb{T}_\theta$ is "hyperbolic": exponential volume growth, transience of the simple random walk... [Curien 2016]
- The case $\theta = \frac{1}{2}$ is degenerate (vertices with "infinite degrees").
- The limit is planar, although T_{n,g_n} has a high genus! Also true in other well-known contexts (e.g. random regular graphs).

A sample of a PSHT



Local limits: back to combinatorics

- Natural idea to prove the theorem: as in the planar case, use asymptotic results on the number $\tau_p(n, g_n)$ of triangulations of size n with genus g_n and a boundary of length p.
- Unfortunately, it seems very hard to obtain accurate asymptotics.
- On the other hand, let $t_0 =$  . Then

$$\mathbb{P}\left(t_0 \subset T_{n,g}\right) = \frac{\tau_1(n-1,g)}{\tau(n,g)} = \frac{\tau(n-1,g)}{\tau(n,g)}$$

by a simple root transformation (erase the boundary edge and glue two edges together).

- But we also know that if $\frac{g}{n} \to \theta$, then

$$\mathbb{P}\left(t_0 \subset T_{n,g}\right) \xrightarrow[n\to+\infty]{} \mathbb{P}\left(t_0 \subset \mathbb{T}_\theta\right) = \lambda(\theta),$$

where $\lambda(\theta)$ satisfies an explicit equation.

Local limits: back to combinatorics

- We know that $\frac{\tau(n-1,g)}{\tau(n,g)} \approx \lambda\left(\frac{g}{n}\right)$ and $\tau(2g-1,g)$ is explicit, so with a telescopic product we can estimate $\tau(n,g)$.

Theorem (B.–Louf, 2019)

When $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right]$, we have

$$\tau(n, g_n) = n^{2g_n} \exp\left(f(\theta)n + o(n)\right),$$

where $f(\theta) = 2\theta \log \frac{12\theta}{e} + \theta \int_{2\theta}^{1} \log \frac{1}{\lambda(\theta/t)} \mathrm{d}t$, and $\lambda(\theta)$ is as above.

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- Summary of the proof:

Very crude enumeration	$\Rightarrow$	Probability	$\Rightarrow$	Not so crude enumeration

- Tightness from crude combinatorial estimates (the ratio $\frac{\tau(n,g)}{\tau(n-1,g)}$ is bounded).
- Planarity and one-endedness of the limits come from the Goulden–Jackson recurrence:

$$\tau(n,g) = n^2\tau(n-2,g-1) + \dots$$

- Any subsequential limit T is *weakly Markovian*: for any finite triangulation t, the probability $\mathbb{P}(t \subset T)$ is a function of the perimeter and the volume of t.
- Any weakly Markovian random triangulation of the plane is a mixture of PSHT (i.e. $\mathbb{T}_\Theta$ for some random Θ).
- Ergodicity: Θ is deterministic, characterized by the fact that the average degree must be $\frac{6}{1-2\theta}$.

High genus triangulations: global properties

- The local behaviour of $T_{n,g}$ for $\frac{g}{n} \to \theta$ is now well understood, but what about the global one?
- At large scale, we expect $T_{n,g}$ to behave roughly like a random graph (e.g supercritical Erdös-Rényi, or uniform 3-regular):
  - hyperbolic local limits,
  - comparison with unicellular maps **[Ray 2014]** or with models of random hyperbolic surfaces **[Mirzakhani 2013]**,
  - the dual of a triangulation picked in $\bigcup_g \mathcal{T}_{n,g}$ is a uniform (connected) 3-regular graph (in this model $g \approx \frac{n}{2}$ w.h.p.).
- Many nice questions about the global structure of random graphs: distances? Isoperimetric inequalities? Spectral gap? Cut-off for the random walk? Percolation?

Global distances

- Distances are logarithmic and "almost all the same":

Theorem (B.–Chapuy–Louf 2023)

For $\theta \in \left(0, \frac{1}{2}\right)$, there are constants $c_\theta, C_\theta > 0$ such that if $\frac{g_n}{n} \to \theta$, then

$$\mathbb{P}\left(c_\theta \log n \leq \mathrm{diam}(T_{n,g_n}) \leq C_\theta \log n\right) \xrightarrow[n \to +\infty]{} 1.$$

Theorem (B.–Chapuy–Louf 2023)

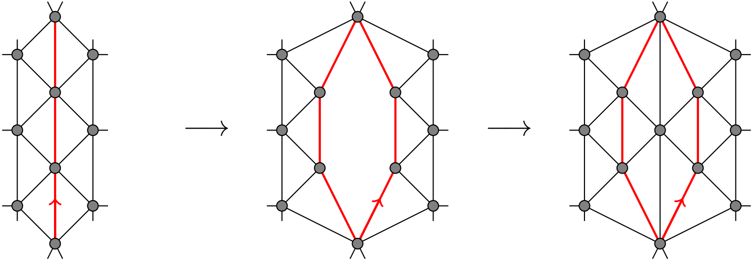
Let $\frac{g_n}{n} \to \theta \in \left(0, \frac{1}{2}\right)$ and let x_n, y_n, u_n, v_n be four independent uniform vertices of T_{n,g_n}. Then

$$d_{T_{n,g_n}}(x_n, y_n) - d_{T_{n,g_n}}(u_n, v_n)$$

is tight.

Distances: logarithmic lower bound

- Idea: first moment computation on paths of length ℓ.
- The size of the ball is bounded by the number of simple paths of length ℓ from the root.
- Cut along a path: the expected number of such paths is $\frac{\tau_{2\ell}(n,g)}{\tau(n,g)}$.
- Fill the hole: $\frac{\tau_{2\ell}(n,g)}{\tau(n,g)} \leq \frac{\tau(n+\ell,g)}{\tau(n,g)} \leq C_\theta^\ell = o(n)$ if $\ell \leq c_\theta \log n$.

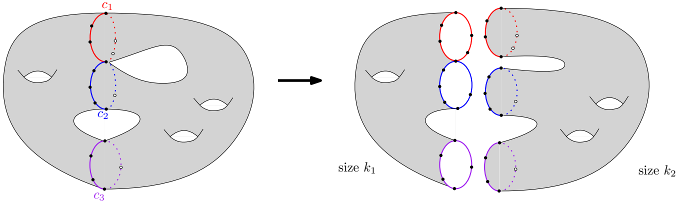


Isoperimetric inequality

- Two natural ways to tackle the upper bound: bijections with labelled trees, or isoperimetric inequalities.
- Ideally: if $A \subset T_{n,g}$, then $|\partial A| \geq c \min(|A|, |T_{n,g} \setminus A|)$.
- But random local limit, so there are "small defects".

Theorem (B.–Chapuy–Louf 2023)

For $\theta \in \left(0, \frac{1}{2}\right)$, there are $\delta_\theta, K_\theta > 0$ such that if $\frac{g_n}{n} \to \theta$, the following holds with high probability:
For any multicurve η which splits T_{n,g_n} into two connected components with n_1 and n_2 faces, if $n_2 \geq n_1 \geq K_\theta \log n$, then the length of η is at least $\delta_\theta n_1$.



From isoperimetric inequalities to distance estimates

- Let $x \in T_{n,g}$. For $r \geq K_\theta \log n$, we have

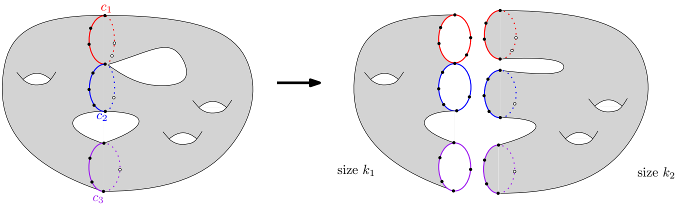
$$|B_{r+1}(x) \backslash B_r(x)| \approx |\partial B_r(x)| \geq \delta_\theta |B_r(x)|,$$

  so $|B_{r+1}(x)| \geq (1+\delta)|B_r(x)$ and $|B_r(x)| \geq (1+\delta)^r$, so $B_r(x) = T_{n,g}$ for $r = C \log n$, up to the small defects.
- Defects are actually not a problem: their volume is $O(\log n)$, and so is their diameter.
- Why are distances almost all the same?
  - let r be the first radius such that $|B_r(x)| \geq \varepsilon n$,
  - then $|B_{r+M}(x)| \geq (1+\delta)^M \varepsilon n \geq (1-\varepsilon)n$ for some finite M,
  - so most vertices y satisfy

$$r \leq d_{T_{n,gn}}(x,y) \leq r + M.$$

Proof of the isoperimetric inequality

- Again, first moment computation on separating multicurves:



- We want to bound

$$\frac{\tau_{p_1,\ldots,p_k}(n_1,g_1)\tau_{p_1,\ldots,p_k}(n_2,g_2)}{\tau(n,g)}$$

$$\text{for } \begin{cases} n_1+n_2=n, \\ n_2 \geq n_1 \geq K \log n, \\ g_1+g_2=g-k+1, \\ p=p_1+\cdots+p_k \leq \delta n_1. \end{cases}$$

Proof of the isoperimetric inequality

- First, get rid of boundaries: $\tau_{p_1,\ldots,p_k}(n_1,g_1) \le n^k \tau(n_1+p,g_1) \le n^k C^p \tau(n_1,g_1)$.
- Reminder: $\tau(n,g) = n^{2g} \exp\left(nf\left(\frac{g}{n}\right)\right) e^{o(n)}$, so

$$\begin{aligned} &\frac{\tau_{p_1,\ldots,p_k}(n_1,g_1)\tau_{p_1,\ldots,p_k}(n_2,g_2)}{\tau(n,g)} \\ &\le C^p \frac{n^{2k} n_1^{2g_1} n_2^{2g_2}}{n^{2g}} e^{o(n)} \\ &\times \exp\left(n_1 f\left(\frac{g_1}{n_1}\right) + n_2 f\left(\frac{g_2}{n_2}\right) - nf\left(\frac{g}{n}\right)\right). \end{aligned}$$

- The function f is concave! (explicit computation)
- $\frac{n^{2k} n_1^{2g_1} n_2^{2g_2}}{n^{2g}} = n^2 \left(\frac{n_1}{n}\right)^{2g_1} \left(\frac{n_2}{n}\right)^{2g_2}$.
- If both pieces are macroscopic, this is e^{-cn}, so this beats C^p for $p \le \delta n$.

Proof of the isoperimetric inequality

- If $n_1 \ll n$, the error factor $e^{o(n)}$ may be too large!
- But we understand the ratios $\frac{\tau(n+1,g)}{\tau(n,g)}$, so we can estimate the quotient $\frac{\tau(n_2,g_2)}{\tau(n,g)}$ with an error $e^{o(n_1)}$ instead of $e^{o(n)}$.
- Everything is $\leq e^{-cn_1}$, so if $n_1 \geq K \log n$, we can union bound over $(p_1, \dots, p_k)$, n_1, n_2, g_1, g_2...

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Very crude enumeration $\Rightarrow$ Probability $\Rightarrow$ Not so crude enumeration $\Rightarrow$ More Probability!

Tentacles

- In the isoperimetric inequality, do we really need to assume $n_1, n_2 \geq K_\theta \log n$?
- Yes: with high probability, "tentacles" of length $c_\theta \log n$ bounded by only 2 edges appear in T_{n,g_n}.
- Reason:

$$\begin{aligned}\mathbb{E}\left[\#\text{tentacles of length } \ell\right] &\approx n \times \mathbb{P}\left(\text{tentacle around the root}\right)\\ &= n\frac{\tau(n-\ell,g)}{\tau(n,g)} \approx n\lambda(\theta)^\ell.\end{aligned}$$

- In particular, the Cheeger constant is of order $\frac{1}{\log n}$.
- Interpretation: T_{n,g_n} looks more like a supercritical Erdös–Rényi component than like a uniform 3-regular graph.

A few questions

Conjecture

There are two constants $D_\theta, D'_\theta > 0$ such that if $\frac{g_n}{n} \to \theta$ and x_n, y_n are two uniform vertices of T_{n,g_n}, then

$$\frac{1}{\log n} d_{T_{n,g_n}}(x_n, y_n) \xrightarrow[n\to+\infty]{P} D_\theta \quad \text{and} \quad \frac{1}{\log n} \mathrm{diam}(T_{n,g_n}) \xrightarrow[n\to+\infty]{P} D'_\theta,$$

with $D'_\theta = 3D_\theta$.

- Because of tentacles, if this is true, then $D'_\theta > D_\theta$ (as in supercritical Erdös–Rényi components).
- Intermediate regime $1 \ll g \ll n$?
- Transfer of any nice property of random graphs?
- Universality? Proved for local limits [B.–Louf 2021], but not for global properties (the function f is not explicit).

THANK YOU !