Random maps with unconstrained genus

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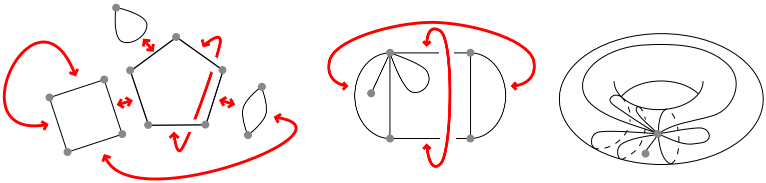
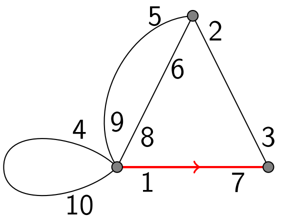
- A *rooted map* is a gluing of polygons which yields a connected, orientable surface. It is equipped with a distinguished oriented edge.



- We *do not* assume that this surface is the sphere.
- There are finitely many maps with n edges.
- Question: let M_n be a uniform map with n edges. What does it look like?

Encoding maps via permutations

- Add labels on the half-edges (1 is the label of the root edge).



$$\tau = (17)(23)(68)(59)(4\,10)$$
$$\sigma = (1\,8\,9\,4\,10)(256)(37)$$

- A map m with n edges can be encoded by a pair (τ, σ) of permutations of $\{1, 2, \ldots, 2n\}$.
- τ is an involution with no fixed point that matches the two halves of each edges.
- σ describes how to turn around a vertex, and may be any permutation.

Connectedness

- Small issue: a pair (τ, σ) does not always give a map (a map must be connected).
- However, the proportion of pairs (τ, σ) that correspond to a map goes to 1 as $n \to +\infty$.
- Consequence: the number of maps with n edges is asymptotically

$$\frac{(2n-1)!! \times (2n)!}{(2n-1)!} = 2n \times (2n-1)!!.$$

- Other consequence: a uniform map M_n is the map given by a uniform matching τ_n and an independent uniform permutation σ_n on $\{1, 2, \ldots, 2n\}$, conditioned on an event of probability $1 - o(1)$.

Basic properties of M_n

- The vertices of M_n correspond to the disjoint cycles of the uniform permutation σ_n. The number of cycles has the distribution of

$$\sum_{i=1}^{2n} \mathrm{Ber}\left(\frac{1}{i}\right) = (1+o(1)) \log n.$$

- By duality we also have $\#F(M_n) = (1+o(1)) \log n$.
- By the Euler formula $(f+v-e=2-2g)$, the genus of M_n is

$$\frac{n}{2} - \log n + o(\log n).$$

This is close to the maximal possible value $\frac{n}{2}$.

Vertex degrees of M_n

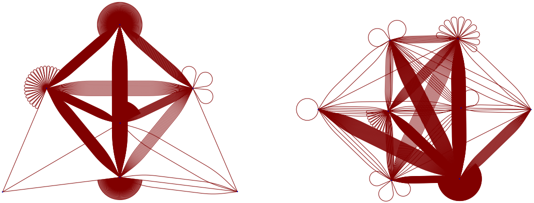
- The degrees of the vertices are the lengths of the cycles of σ_n.
- The length of the cycle containing 1 in σ_n has uniform distribution in $\{1, 2, \dots, 2n\}$. Conditionally on this cycle C_1^n, the rest of σ_n is a uniform permutation of the complementary of C_1^n.
- Poisson–Dirichlet process: let $(X_i)_{i\geq 1}$ decreasing re-ordering of the variables

$$U_1, (1 - U_1)U_2, (1 - U_1)(1 - U_2)U_3, \dots,$$

  where the U_i are i.i.d. uniform variables on $[0, 1]$.
- Let $v_1^n, v_2^n, \dots$ be the vertices of M_n, ranked by decreasing degrees. Then

$$\left(\frac{1}{2n}\deg(v_i^n)\right)_{i\geq 1} \xrightarrow[n\to+\infty]{(d)} (X_i)_{i\geq 1}.$$

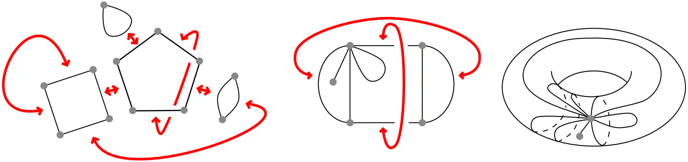
- Let $\mathcal{G}(m)$ be the graph associated to a map m.
- Conditionally on σ_n, the graph $\mathcal{G}(M_n)$ is given by the uniform matching τ_n: it is a *configuration model*.



- Let $[i,j]_n$ be the number of edges joining v_i^n and v_j^n, with the convention that $[i,i]_n$ is twice the number of loops on v_i^n. Then

$$\left(\frac{1}{2n}[i,j]_n\right)_{i,j\geq 1} \xrightarrow[n\to+\infty]{(d)} (X_i X_j)_{i,j\geq 1}.$$

- We want to deal with more general models like triangulations, maps with one face...
- We start with a family $\mathcal{P}_n$ of polygons with $2n$ edges in total, and glue the edges two by two in a uniform way to obtain a map $M_{\mathcal{P}_n}$.



- How many vertices in $M_{\mathcal{P}_n}$? Law of the largest degrees?

Universality

- General idea: the graph $\mathcal{G}\left(M_{\mathcal{P}_n}\right)$ looks a lot like $\mathcal{G}(M_n)$.

Conjecture

Assume that $\mathcal{P}_n$ has $o(\sqrt{n})$ one-gons and $o(n)$ two-gons.
Then the total variation distance between the laws of $\mathcal{G}\left(M_{\mathcal{P}_n}\right)$ and $\mathcal{G}(M_n)$ goes to 0 as $n \to +\infty$.

- The assumption on the number of 1-gons and 2-gons is here to ensure connectedness with high probability.
- Goal of the rest of the talk: present some results supporting this conjecture.

Algebraic methods

- The conjecture is true if we only look at the sequence of degrees instead of the graph:
  - for triangulations **[Gamburd 2006]**,
  - for any $(\mathcal{P}_n)$ as long as there is no 1-gon or 2-gon **[Chmutov–Pittel 2013]**.
- This implies $\#V\left(M_{\mathcal{P}_n}\right) = (1+o(1))\log n$, and convergence of the largest degrees to a Poisson–Dirichlet process.
- In both papers, the proof relies on algebraic methods (representations of the symetric group).
- Does not allow to say anything more precise about the graph.
- The assumption that there is no 1-gon or 2-gon does not look optimal.

Probabilistic methods?

Theorem (B.–Curien–Petri 2019)

We assume that $\mathcal{P}_n$ has $o(\sqrt{n})$ one-gons and $o(n)$ two-gons. Let $v_1^n, v_2^n, \dots$ be the vertices of $M_{\mathcal{P}_n}$, ranked by decreasing degrees. Let $[i,j]_n$ be the number of edges of $M_{\mathcal{P}_n}$ between v_i^n and v_j^n. Then

$$\left(\frac{1}{2n}\deg(v_i^n)\right)_{i\geq 1} \xrightarrow[n\to+\infty]{(d)} (X_i)_{i\geq 1}.$$

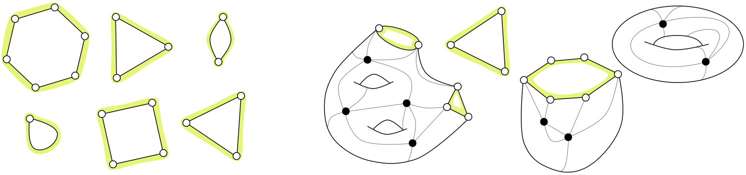
$$\left(\frac{1}{2n}[i,j]_n\right)_{i,j\geq 1} \xrightarrow[n\to+\infty]{(d)} (X_iX_j)_{i,j\geq 1},$$

where X is a Poisson–Dirichlet process.

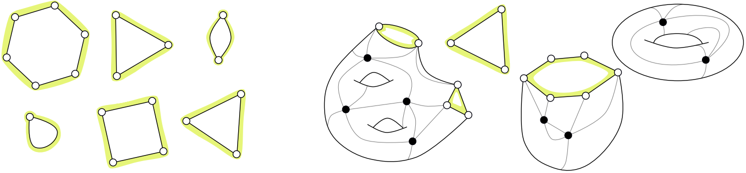
- Probabilistic proofs!

Peeling!

- Let ρ be the root vertex. Goal of the rest of the talk: explain why $\frac{1}{2n}\deg(\rho)$ converges to a uniform variable on $[0, 1]$.
- Peeling arguments! For us, a *peeling exploration* of $M_{\mathcal{P}_n}$ will be an increasing sequence of "partial gluings" $(S_i)_{0\leq i\leq n}$ of the polygons of $\mathcal{P}_n$.



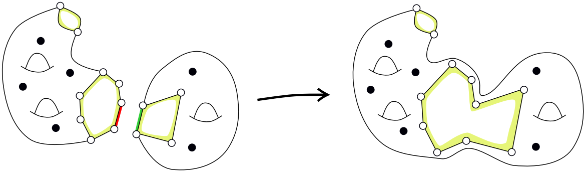
- S_0 is just the collection of polygons $\mathcal{P}_n$ (as on the left) with a root, and $S_n = M_{\mathcal{P}_n}$.
- The non-glued edges of S_i form the *boundary* of S_i.



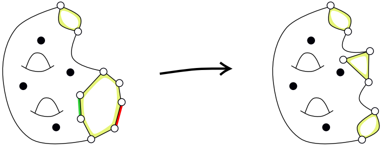
- To move from S_i to S_{i+1}, we choose a boundary edge $\mathcal{A}(S_i)$ according to a *peeling algorithm*, and glue it where it has to be glued in $M_{\mathcal{P}_n}$.
- Important: at each step i, conditionally on S_i, the gluing of the non-glued edges is uniform, so the peeled edge is glued uniformly to one of the $2(n-i)-1$ boundary edges.
- At each step, there are "true" vertices (in black) and "false" vertices (in white).

Peeling cases (Episode 1)

- Case 1: gluing two edges on two different components.

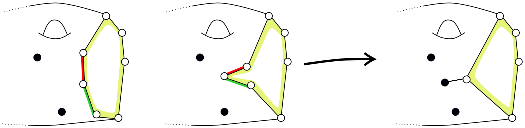


- Case 2: gluing two edges on the same hole of the same component

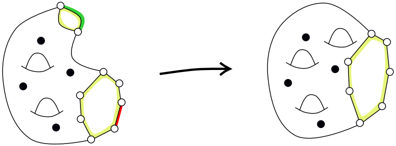


Peeling cases (Episode 2)

- Case 3: gluing two *neighbour* edges on the same hole.



- Case 4: gluing two edges on two different holes of the same component.



- Case 3 creates vertices, and Case 4 creates genus.

Peeling algorithm and closure time

- Peeling algorithm: always peel the edge on the right of the root vertex.
- At the time τ where ρ becomes a true vertex, stop the exploration.
- The closure of the root occurs when the peeled edge is glued to the edge on its left, so $\mathbb{P}\left(\tau = i+1 | S_0, \ldots, S_i\right) = \frac{1}{2(n-i)-1}$.
- Consequence: convergence in distribution $\frac{1}{n}\tau \to T$, where T has density $\frac{1}{2\sqrt{1-t}}\mathrm{d}t$.

First and second moment estimates

- Let us fix a "blue" vertex v on a polygon at the beginning, and let θ be the time at which it is glued to the peeled vertex.
- For the same reason as τ, we have $\frac{1}{n}\theta \to \Theta \sim \frac{1}{2\sqrt{1-t}}\mathrm{d}t$, with Θ independent of T.
- Hence $\mathbb{P}\left(v \text{ glued to } \rho|\tau\right) = 1 - \sqrt{1-\tau}$. By summing over all vertices

$$\frac{1}{2n}\mathbb{E}\left[\deg(\rho)|\tau\right] \xrightarrow[n\to+\infty]{(P)} 1 - \sqrt{1-\tau}.$$

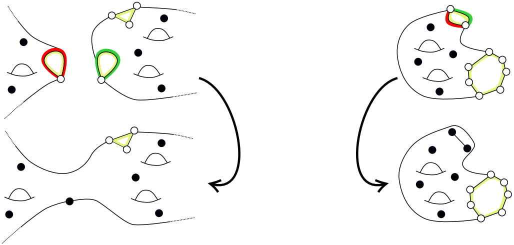
- Same thing with two blue vertices gives

$$\mathrm{Var}\left(\frac{1}{2n}\deg(\rho)\middle|\tau\right) \xrightarrow[n\to+\infty]{(P)} 0.$$

- Finally $\frac{1}{2n}\deg(\rho)$ converges in distribution to $1 - \sqrt{1-\tau} \sim \mathrm{Unif}\left([0,1]\right)$.

Uncheating

- Peeling cases 5 and 6:



- We need show that peeling case 5 does not occur before time $1 - o(1)$.
- For this, we needed to understand how 1-gons appear, and why they disappear quickly when we peel them.

Geometric motivations

- By the uniformization theorem, this provides a way to build random hyperbolic Riemannian surfaces (i.e. constant curvature -1) [Brooks–Makover 2004].
- Peeling techniques allow to control the diameter of these surfaces [B.–Curien–Petri 2019+].
- The surface we obtain has genus $g \sim \frac{n}{2}$ and diameter $(2+o(1)) \log g$.
- Question: are there hyperbolic surfaces with genus g and diameter $(1+o(1)) \log g$?

THANK YOU !