# A Hamilton-Jacobi Approach to Disordered Systems

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- Attribute to student $i$ a value $\sigma_i \in \{-1, 1\}$ dictating if they are in dormitory A or B.
- Denote by $J_{ij} \in \mathbb{R}$ the quality of the interaction between student $i$ and $j$. When $J_{ij} > 0$, student $i$ and $j$ prefer to be in the same dormitory, otherwise they prefer to be in different dormitories.

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- Frustration : if $J_{12} > 0$, $J_{23} > 0$ and $J_{13} < 0$ no configuration fully satisfies the preferences of student 1, 2 and 3 simultaneously.

# What are spin glass models?

Let $N \geqslant 1$, and $(J_{ij})_{1 \leqslant i,j \leqslant N}$ be i.i.d standard Gaussian random variables.

- For $\sigma \in \{-1, +1\}^N$ define,

$$H_N^{SK}(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j. \quad (1)$$

- $H_N^{SK}$ is a Gaussian process which satisfies,

$$\mathbb{E} H_N^{SK}(\sigma) H_N^{SK}(\tau) = N\xi^{SK}\left(\frac{\sigma \cdot \tau}{N}\right), \quad (2)$$

where $\xi^{SK}(x) = x^2$.

# The free energy

- One can associate a (random) probability measure to the process $H_N^{SK}$,

$$G_{N,t}^{SK}(\sigma) \propto e^{\sqrt{2t}H_N^{SK}(\sigma)-Nt}\frac{1}{2^N}. \tag{3}$$

- We are interested in computing the free energy of this probability measure

$$F_N^{SK}(t) = -\frac{1}{N}\mathbb{E}\log\left(\frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N} e^{\sqrt{2t}H_N^{SK}(\sigma)-Nt}\right). \tag{4}$$

- We have

$$\frac{1}{N}\max_{\sigma\in\{-1,+1\}^N} H_N^{SK}(\sigma) = \lim_{t\to\infty}\frac{t - F_N^{SK}(t)}{\sqrt{2t}}. \tag{5}$$

# The Parisi formula

Let $\mathcal{Q}$ be the space of square integrable increasing functions $q : [0,1) \to \mathbb{R}_+$.

**Theorem (Parisi 1979, Guerra & Talagrand 2006)**

*There exists an explicit functional, $\mathcal{P}_{t,\xi^{SK}} : \mathcal{Q} \to \mathbb{R}$, that satisfies*

$$\lim_{N\to\infty} F_N^{SK}(t) = \sup_{q\in\mathcal{Q}} \mathcal{P}_{t,\xi^{SK}}(q). \tag{6}$$

If we replace $H_N^{SK}$ by a Gaussian process $H_N^{\xi}$ such that

$$\mathbb{E} H_N^{\xi}(\sigma) H_N^{\xi}(\tau) = N\xi\left(\frac{\sigma\cdot\tau}{N}\right),$$

where $\xi(x) = \sum_{p=1}^{P} a_p x^p$ with $a_p \geqslant 0$, then the theorem above remains valid. (Panchenko 2011)

# General formulation of the Hamilton-Jacobi equation

The set $\mathcal{Q}^D = \mathcal{Q} \times \cdots \times \mathcal{Q}$ is the set of square integrable increasing functions $q : [0,1) \to \mathbb{R}_+^D$. Let $\xi : \mathbb{R}^D \to \mathbb{R}$ be a polynomial with nonnegative coefficients and $\Phi : \mathcal{Q}^D \to \mathbb{R}$ a Lipschitz function. We consider,

$$\begin{cases} \partial_t U - \int \xi(\nabla_q U) = 0 \text{ on } (0,\infty) \times \mathcal{Q}^D \\ U(0,\cdot) = \Phi \text{ on } \mathcal{Q}^D. \end{cases} \quad \text{(HJ)}$$

## Definition

Fréchet derivative Let $\mathcal{C}$ be a closed convex cone in a Hilbert space $(\mathcal{H}, \langle\cdot\rangle_{\mathcal{H}})$, we say that $h : \mathcal{C} \to \mathbb{R}$ is Fréchet differentiable at $x \in \mathcal{C}$ when there exists a unique $p \in \mathcal{H}$ such that as $x' \to x$ in $\mathcal{C}$,

$$h(x') = h(x) + \langle p, x' - x \rangle_{\mathcal{H}} + o(|x - x'|). \quad (7)$$

The Fréchet derivative of $h$ at $x$ is denoted $\nabla h(x)$ (or $\nabla_x h(x)$).

# Viscosity solution to Hamilton-Jacobi equation

Let $U : \mathbb{R}_+ \times \mathcal{Q}^D \to \mathbb{R}$ be a continuous function.

1. The function $U$ is a *viscosity subsolution* of (HJ) when for every $(t, q) \in (0, +\infty)$ and every $h \in \mathcal{C}^1((0, +\infty) \times \mathcal{Q}^D)$ such that $U - h$ has a minimum at $(t, q)$ we have

$$\partial_t h(t, q) - \int \xi(\nabla h(t, q)) \geqslant 0.$$

2. The function $U$ is a *viscosity supersolution* of (HJ) when for every $(t, q) \in (0, +\infty)$ and every $h \in \mathcal{C}^1((0, +\infty) \times \mathcal{Q}^D)$ such that $U - h$ has a maximum at $(t, q)$ we have

$$\partial_t h(t, q) - \int \xi(\nabla h(t, q)) \leqslant 0.$$

3. The function $U : \mathbb{R}_+ \times \mathcal{Q}^D \to \mathbb{R}$ is a *viscosity solution* when it is both a viscosity subsolution and a a viscosity supersolution.

# Well-posedness of the Cauchy problem

Let $\mathcal{C}$ be a closed convex cone in a Hilbert space $(\mathcal{H}, \langle \cdot \rangle_{\mathcal{H}})$. We say that $h : \mathcal{C} \to \mathbb{R}$ is $\mathcal{C}^*$-increasing when for every $x, x' \in \mathcal{C}$,

$$\Big(\forall y \in \mathcal{C}, \langle x' - x, y \rangle_{\mathcal{H}} \geqslant 0\Big) \implies h(x') - h(x) \geqslant 0.$$

### Theorem (Chen 2022)

*When the initial condition is $(\mathcal{Q}^D)^*$-increasing, the Cauchy problem* (HJ) *admits a unique Lipschitz viscosity solution in the set of functions $U$ satisfying, for every $t \geqslant 0$, $U(t, \cdot)$ is $(\mathcal{Q}^D)^*$-increasing.*

### Remark

For the Cauchy problem arising in the context of spin glasses, the initial condition is $(\mathcal{Q}^D)^*$-increasing.

# Hopf-Lax representation for the viscosity solution

Theorem (Chen 2022)

*If $\xi$ is convex and the initial conditon is $(\mathcal{Q}^D)^*$-increasing, the unique viscosity solution of* (HJ) *satisfies*

$$U(t,q) = \sup_{p \in \mathcal{Q}^D} \inf_{r \in \mathcal{Q}^D} \left\{ \Phi(q+p) - \langle p, r \rangle_{L^2} + t \int \xi(r) \right\}.$$

# Hamilton-Jacobi equations and the free energy

Recall that $\mathcal{Q}$ is the space of square integrable increasing functions $q : [0,1) \to \mathbb{R}_+$. Let $V : \mathbb{R}_+ \times \mathcal{Q} \to \mathbb{R}$ be the unique viscosity solution to

$$\begin{cases} \partial_t V - \int (\nabla_q V)^2 = 0 \text{ on } (0,\infty) \times \mathcal{Q} \\ V(0,\cdot) = \Psi^{SK} \text{ on } \mathcal{Q}. \end{cases} \tag{HJ1}$$

Here $\Psi^{SK}$ denotes a functional transformation of the uniform measure on $\{-1,1\}$ called the cascade transform.

Theorem (Mourrat 2020)

$$\lim_{N\to+\infty} F_N^{SK}(t) = V(t,0). \tag{8}$$

Proof : Use the Hopf-Lax representation for $V$ and observe that $V(t,0) = \sup_q \mathcal{P}_{t,\xi^{SK}}(q)$.

# Multi-species models

- It is possible to define more generals models that allow for several types. For models with two types, we encode spin configurations with $\sigma = (\sigma_1, \sigma_2) \in \{-1, 1\}^N \times \{-1, 1\}^N$.
- For example, we can consider the bipartite model,

$$H_N^{\mathrm{BP}}(\sigma_1, \sigma_2) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} J_{ij} \sigma_{1i} \sigma_{2j}. \tag{9}$$

- $H_N^{\mathrm{BP}}$ is a Gaussian process which satisfies,

$$\mathbb{E} H_N^{\mathrm{BP}}(\sigma) H_N^{\mathrm{BP}}(\tau) = N \xi^{BP} \left( \frac{\sigma_1 \cdot \tau_1}{N}, \frac{\sigma_2 \cdot \tau_2}{N} \right), \tag{10}$$

where $\xi^{BP}(x, y) = xy$.

# The free energy for nonconvex models: a conjecture

For the bipartite model, Parisi formula breaks down and it is not even proven that the free energy converges as $N \to +\infty$. But, we can still consider $W$ the unique viscosity solution of

$$\begin{cases} \partial_t W - \int \nabla_{q_1} W \nabla_{q_2} W = 0 \text{ on } (0,\infty) \times \mathcal{Q}^2 \\ W(0,\cdot) = \Psi^{BP} \text{ on } \mathcal{Q}^2, \end{cases} \tag{HJ2}$$

where $\Psi^{BP}(q_1, q_2) = \Psi^{SK}(q_1) + \Psi^{SK}(q_2)$.

Conjecture (Mourrat 2020)

$$\lim_{N \to +\infty} -\frac{1}{N} \mathbb{E} \log \left( \frac{1}{2^{2N}} \sum_{\sigma_1, \sigma_2 \in \{-1,1\}^N} e^{\sqrt{2t} H_N^{BP}(\sigma_1,\sigma_2) - Nt} \right) = W(t,0).$$

# The enriched free energy

- Define $H_N^{t,q} = \sqrt{2t} H_N^{BP}(\sigma) - Nt + H_N^q(\sigma, \alpha)$ for some well-chosen Gaussian process $H_N^q$.
- Consider the associated free energy

$$F_N(t,q) = -\frac{1}{N} \mathbb{E} \log \left( \frac{1}{2^{2N}} \sum_{\sigma \in (\{-1,1\}^N)^2} \int e^{H_N^{t,q}(\sigma,\alpha)} \mathrm{d}\mathcal{R}(\alpha) \right).$$

- In addition, when $q = 0$ we have $H_N^q = 0$ and thus $F_N(t,0) = F_N^{BP}(t)$.
- Any subsequential limit of $(F_N)_{N \geqslant 1}$ satisfies a Hamilton-Jacobi equation in a weak sense in the limit $N \to +\infty$. (see next slide)
- To prove the conjecture, it suffices to prove that any weak solution of the Hamilton-Jacobi equation is equal to the viscosity solution.

# The enriched free energy

The family $(F_N)_{N\geqslant 1}$ is uniformly Lipschitz on $\mathbb{R}_+ \times \mathcal{Q}^2$.

Theorem (Chen and Mourrat 2023)

*Any subsequential limit of $F$ of $F_N$ is a semi-concave weak solution of* (HJ2)*. More precisely $F$ satisfies,*

1. $F(0,\cdot) = \Psi^{BP}$*. (in fact $\Psi^{BP} = F_1(0,\cdot)$)*
2. $\partial_t F - \int \nabla_{q_1} F \nabla_{q_2} F = 0$ *"almost everywhere" on $(0,+\infty) \times \mathcal{Q}^2$ (outside a Gaussian null set).*
3. *$F$ is locally semi-concave on $(0,+\infty) \times \mathcal{Q}^2$.*

Theorem (Mourrat 2020)

*Let $W$ be the viscosity solution of* (HJ2)*. For any subsequential limit $F$ of $F_N$, we have $F \geqslant W$. In particular,*

$$\liminf_{N\to+\infty} F_N^{BP}(t) \geqslant W(t,0). \tag{11}$$

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# Some selection principles on $\mathbb{R}^d$

Let $H : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function and let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a differentiable Lipschitz function with Lipschitz gradient. Consider $u$ the viscosity solution of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t u - H(\nabla u) = 0 \text{ on } (0, +\infty) \times \mathbb{R}^d \\ u(0, \cdot) = \phi. \end{cases} \tag{12}$$

**Theorem (I. 2024)**

*Assume that $H$ is convex, then (12) admits at most one semi-concave weak solution and if it exists it is equal to $u$.*

**Theorem (I. 2024)**

*Assume that the viscosity solution $u$ is $\mathcal{C}^{1,1}$, then (12) admits exactly one semi-concave weak solution, and it is equal to $u$.*

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# Hamilton-Jacobi equation with concave initial condition

Assume that $\phi : \mathbb{R}^d \to \mathbb{R}$ be is concave and Lipschitz function, we have

$$\phi(x) = \inf_{y \in \mathbb{R}^d} \{x \cdot y - \phi_*(y)\},$$

where $\phi_*(y) = \inf_{x \in \mathbb{R}^d} \{x \cdot y - \phi(x)\}$.

# Hamilton-Jacobi equation with concave initial condition

The solution to

$$\begin{cases} \partial_t u - H(\nabla u) = 0 \text{ on } (0, +\infty) \times \mathbb{R}^d \\ u(0, x) = x \cdot y - \phi_*(y), \end{cases}$$

is $u(t, x) = x \cdot y + tH(y) - \phi_*(y)$.

Theorem (Hopf representation)

*Assume that $\phi$ is concave, then the unique viscosity solution to*

$$\begin{cases} \partial_t u - H(\nabla u) = 0 \text{ on } (0, +\infty) \times \mathbb{R}^d \\ u(0, \cdot) = \phi, \end{cases}$$

*satisfies*

$$u(t, x) = \inf_{y \in \mathbb{R}^d} \left\{ x \cdot y + tH(y) - \phi_*(y) \right\}.$$

Sketch of proof : Write $\phi(x) = \inf_{y \in \mathbb{R}^d} \{x \cdot y - \phi_*(y)\}$ and permute the HJ semigroup with $\inf_{y \in \mathbb{R}^d}$.

# Hopf representation

**Theorem (Chen 2022)**

*Assume that* $\Phi : \mathcal{Q}^D \to \mathbb{R}$ *is concave, then* $U$ *the unique viscosity solution to* (HJ) *satisfies*

$$U(t,q) = \inf_p \left\{ \langle q, p \rangle_{L^2} + t \int \xi(p) - \Phi_\circ(p) \right\},$$

*where* $\Phi_\circ(p) = \inf_{q \in \mathcal{Q}} \{ \langle q, p \rangle_{L^2} - \Phi(q) \}$.

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*where* $\Phi_\circ(p) = \inf_{q \in \mathcal{Q}} \{ \langle q, p \rangle_{L^2} - \Phi(q) \}$.

- In the context of spin glasses, the initial condition $\Psi^{SK} : \mathcal{Q} \to \mathbb{R}$ and $\Psi^{BP} : \mathcal{Q}^2 \to \mathbb{R}$ are not concave (nor convex) in general. So this result cannot be used directly.

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- In the context of spin glasses, the initial condition $\Psi^{SK} : \mathcal{Q} \to \mathbb{R}$ and $\Psi^{BP} : \mathcal{Q}^2 \to \mathbb{R}$ are not concave (nor convex) in general. So this result cannot be used directly.
- But, the initial conditions $\Psi^{SK}$ and $\Psi^{BP}$ have some hidden concavity properties that can be revealed by altering the geometry of $\mathcal{Q}$.

# On the concavity of the initial condition

- Given $q \in \mathcal{Q}^D$, one can define a probability measure $\mu = \mathrm{Law}(q(U))$ where $U$ is a uniform random variable on $[0, 1)$.
- We denote by $\mathcal{P}^{\uparrow}(\mathbb{R}_+^D)$ the set of probability measure of the form $\mathrm{Law}(q(U))$ with $q \in \mathcal{Q}^D$.
- The map $q \mapsto \mathrm{Law}(q(U))$ is injective and given $\mu \in \mathcal{P}^{\uparrow}(\mathbb{R}_+^D)$ we denote by $q_\mu$ the unique element of $\mathcal{Q}^D$ such that $\mu = \mathrm{Law}(q_\mu(U))$.
- In general, we have

$$q_{\lambda\mu+(1-\lambda)\mu'} \neq \lambda q_\mu + (1-\lambda) q_{\mu'}.$$

- The set $\mathcal{P}^{\uparrow}(\mathbb{R}_+^2)$ is nonconvex.

# On the convexity of the initial condition

**Theorem (Auffinger and Chen[1] 2014)**
*The function $\mu \mapsto \Psi^{SK}(q_\mu)$ is concave on $\mathcal{P}^\uparrow(\mathbb{R}_+)$.*

**Remark**
Since $\Psi^{BP}(q_1, q_2) = \Psi^{SK}(q_1) + \Psi^{SK}(q_2)$ we also have that $\mu \mapsto \Psi^{BP}(q_\mu)$ is concave *on every convex subset of* $\mathcal{P}^\uparrow(\mathbb{R}_+^2)$.

**Watch out !**
The initial conditions $\Psi^{SK}$ and $\Psi^{BP}$ are concave in a *modified* geometry. This does not directly yield the Hopf representation for the viscosity solution of (HJ1) and (HJ2) since those are formulated using the Fréchet derivative (= derivative along straight lines in $\mathcal{Q}^D$).

[1] Wei-Kuo Chen.

# Viscosity solution with affine initial condition

Let $\Phi : \mathcal{Q} \to \mathbb{R}$ be a Lipschitz and $\mathcal{Q}^*$-increasing function such that $\mu \mapsto \Phi(q_\mu)$ is concave on $\mathcal{P}(\mathbb{R}_+)$. Then,

$$\Phi(q) = \inf_{\chi} \left\{ \int_0^1 \chi(q(u))\mathrm{d}u - \Phi_*(\chi) \right\},$$

where the infimum is taken over the set of functions $\chi : \mathbb{R}_+ \to \mathbb{R}$ that are 1-Lipschitz, increasing and convex, and where we have defined $\Phi_*(\chi) = \inf_q \left\{ \int_0^1 \chi(q(u))\mathrm{d}u - \Phi(q) \right\}$.

**Remark**

The function $\mu \mapsto \int_0^1 \chi(q_\mu(u))\mathrm{d}u = \int_0^\infty \chi(x)\mathrm{d}\mu(x)$ is affine on $\mathcal{P}^\uparrow(\mathbb{R}_+)$.

# Viscosity solution with affine initial condition

"If the initial condition is affine as a function of $\mu$, then the viscosity solution is also affine as a function of $\mu$"

**Theorem (I. 2024)**

*Let $b \in \mathbb{R}$ and let $\chi : \mathbb{R}_+ \to \mathbb{R}$ be a 1-Lipschitz, increasing and convex function. The unique viscosity solution of*

$$\begin{cases} \partial_t U - \int \xi(\nabla U) = 0 \text{ on } (0, +\infty) \times \mathcal{Q} \\ U(0, q) = \int_0^1 \chi(q(u))\mathrm{d}u - b, \end{cases}$$

*Is*

$$U(t, q) = \int_0^1 S_t\chi(q(u))\mathrm{d}u - b,$$

*where $S_t\chi$ is the unique viscosity solution of*

$$\begin{cases} \partial_t u - \xi(\nabla u) = 0 \text{ on } (0, +\infty) \times \mathbb{R}_+ \\ u(0, \cdot) = \chi. \end{cases}$$

# A Hopf-like representation for the SK model

Given a Lipschitz and increasing function $\chi : \mathbb{R}_+ \to \mathbb{R}$, we let $S_t \chi$ denote the unique viscosity solution of

$$\begin{cases} \partial_t v - (\nabla v)^2 = 0 \text{ on } (0, +\infty) \times \mathbb{R}_+ \\ v(0, \cdot) = \chi. \end{cases}$$

also define

$$(\Psi^{SK})_*(\chi) = \inf_{q \in \mathcal{Q}} \left\{ \int_0^1 \chi(q(u)) \mathrm{d}u - \Psi^{SK}(q) \right\}$$

**Theorem (I. 2024)**

*The unique viscosity solution $V$ of* (HJ1) *satisfies*

$$V(t, q) = \inf_{\chi} \left\{ \int_0^1 S_t \chi(q(u)) \mathrm{d}u - (\Psi^{SK})_*(\chi) \right\},$$

*where the infimum is taken over the set of* 1*-Lipschitz, increasing and convex functions* $\chi : \mathbb{R}_+ \to \mathbb{R}$.

# A conjectural variational formula

Given a Lipschitz and $\mathbb{R}^2_+$-increasing function $\chi : \mathbb{R}^2_+ \to \mathbb{R}$, we let $S_t\chi$ denote the unique viscosity solution of

$$\begin{cases} \partial_t w - \nabla_{x_1} w \nabla_{x_2} w = 0 \text{ on } (0, +\infty) \times \mathbb{R}^2_+ \\ w(0, \cdot) = \chi. \end{cases}$$

also define

$$(\Psi^{BP})_*(\chi) = \inf_{q \in \mathcal{Q}^2} \left\{ \int_0^1 \chi(q(u))\mathrm{d}u - \Psi^{BP}(q) \right\}.$$

## Conjecture (I. 2024)

*The unique viscosity solution $W$ of* (HJ2) *satisfies*

$$W(t, q) = \inf_{\chi} \left\{ \int_0^1 S_t\chi(q(u))\mathrm{d}u - (\Psi^{BP})_*(\chi) \right\},$$

*where the infimum is taken over the set of functions of the form $\chi(x_1, x_2) = \chi_1(x_1) + \chi_2(x_2)$, with $\chi_1$ and $\chi_2$ 1-Lipschitz, increasing and convex.*

# References

For more on this subject, a 20 page expository note is available on Jean-Christophe Mourrat's webpage.


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