TD1: Continuous-time stochastic processes

Exercice 1 — *Modification and indistinguishability.*

(1) Show that two functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are rightcontinuous and coincide on a (possibly countable) dense subset of $\mathbb{R}_+$, are the same.
(2) Deduce that two continuous-time stochastic processes with real values $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ that are modifications of each other and have rightcontinuous trajectories, are actually indistinguishable.
(3) Construct a process $(X_t)_{t\geq 0}$ that is a modification of the trivial process $(Y_t)_{t\geq 0}$ defined by $Y_t = 0$, but whose trajectories are almost surely discontinuous everywhere. (Thus clearly X is distinguishable from Y).
Hint: Construct first a process X that is a modification of Y but whose trajectories differ at one (necessarily random) time.

Exercice 2 — *Stopping times and measurability.*

Let X be a minimal continuous-time stochastic process, with values in a discrete countable subset I of $\mathbb{R}$.

(1) Draw two possible trajectories, one with and one without explosion, and remind the main notations used in the course.
(2) Assume that If X is right continuous of left continuous and show the following
  (a) If T is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$, finite a.s., then X_T is F_T-measurable.
  (b) $J_0, J_1, \ldots$ are $\mathcal{F}_\infty$-measurable, and stopping times with respect to $\mathcal{F}$.
  (c) $S_1, S_2, \ldots$ are $\mathcal{F}_\infty$-measurable.
  (d) Y_n is $\mathcal{F}_\infty$-measurable and $\mathcal{F}_{J_n}$ -measurable.
Hint: For (a), you may first show that a function Z is $\mathcal{F}_T$-measurable if and only if for all $t \geq 0$, the function $Z\,\mathbb{1}_{T\leq t}$ is $\mathcal{F}_t$-measurable.

Exercice 3 — *Poisson process.*

Keeping the same notations, we say that X is a standard Poisson process if:

- The times S_n between consecutive jumps are iid exponential random variables with parameter 1.
- The jump process satisfies $Y_n = n$.

(1) Show that there is almost surely no explosion.
(2) Show that X has independent and stationary increments.

(3) Compute the expectation

$$\mathbb{E}[f(J_1, \dots, J_n)\, \mathbb{1}_{X_t=n}],$$

for $n \geq 1$ and f nonnegative measurable test function.

(4) Deduce the law of X_t and the law of the Poisson process.

(5) Deduce also the law of $(J_1, \dots, J_n)$ conditioned on $X_t = n$.

(6) (Feller's autobus paradox) Modelize the arrival times of buses at a stop by a Poisson process. Assuming you arrive at the stop at time t, what is the distribution of the time you have to wait for a bus? What is the distribution of the time between this bus and the subsequent one? Where is the paradox?

Exercice 4 — *Composed Poisson Processes.*

Let ν be a probability measure on $\mathbb{R}^n$ and $\lambda > 0$. Let $X = (X_t)_{t\geq 0}$ be a Poisson process of intensity λ and let $(M_k)_{1\leq k\leq n}$ be a sequence of iid random variables of law ν taken independent of X. We define, the composed Poisson process of parameter $\lambda\nu$ by

$$Z_t = \sum_{k=1}^{X_t} M_k.$$

(1) Show that the increments of the process Z are stationary and independent,

(2) Let $t \geq 0$ and $\xi \in \mathbb{R}$, compute $\mathbb{E}[e^{i\xi Z_t}]$, deduce the value of the expectation and of the variance of Z_t.

(3) Let $p \in [0, 1]$, assume that $\nu = (1 - p)\delta_0 + p\delta_1$. Show that the processes Z and $X - Z$ are independent Poisson process of respective intensity $p\lambda$ and $(1 - p)\lambda$.

(4) Deduce from the previous question that the sum of two Poisson process is also a Poisson process.

(5) Assume that ν is supported on $\mathbb{N}^*$, show that a composed Poisson process of intensity $\lambda\nu$ is a continuous time Markov chain, describe its intensity matrix.