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TD10 : Continuous Time Martingales

Exercise 1 — *Hitting time of a line.*
Let $a \geq 0$ and $b \in \mathbb{R}$, define $T = \inf\{t \geq 0, B_t = at + b\}$. Compute $\mathbb{P}(T < \infty)$.

Exercise 2 — *All hypotheses matter.*
Let B be a Brownian motion and S, T two stopping times such that $S \leq T < \infty$ almost surely.

(1) Show that if $\mathbb{E}[S] < \infty$ and $\mathbb{E}[T] < \infty$, then $\mathbb{E}[B_S^2] \leq \mathbb{E}[B_T^2]$.
(2) Find two stopping times S and T with $\mathbb{E}[S] < \infty$, such that $\mathbb{E}[B_S^2] > \mathbb{E}[B_T^2]$.

Exercise 3 — *Brownian gambler's ruin.*
For any $c \in \mathbb{R}$, we let

$$T_c := \inf\{t \geq 0 \ : \ B_t = c\}$$

be the hitting time of c by $(B_t)_{t\geq 0}$. Let $a, b > 0$, we let $T := T_{-a} \wedge T_b$ be the hitting time of $\{-a, b\}$ by $(B_t)_{t\geq 0}$.

(1) What is the law of B_T?
(2) Compute $\mathbb{E}[T]$.

Exercise 4 — *Exponential martingale and computations.*
Let B be a Brownian motion, we recall that for every $\lambda \in \mathbb{R}$, the process $(e^{\lambda B_t - t\lambda^2/2})_{t\geq 0}$ is a martingale, called the exponential martingale. We let for any $a > 0$,

$$T_{a+} := \inf\{t \geq 0 \ : \ B_t > a\}$$

(1) For every $a > 0$ and $\mu \geq 0$, compute the Laplace transform $\mathbb{E}[e^{-\mu T_{a+}}]$. (*Hint:* use the exponential martingale).
(2) Let $(B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion. For every $a \geq 0$, we let

$$C_a := B^{(2)}_{T^{(1)}_{a+}}.$$

   (a) Show that for any $b > 0$, the process $C^{(b)} = (C_{b+a} - C_b)_{a\geq 0}$ is independent of $\mathcal{F}_{T_{b+}}$ and has the same law as $(C_a)_{a\geq 0}$. Deduce that $(C_a)_{a\geq 0}$ is a Markov process and give its transition kernel.
   (b) Show that $(e^{\lambda(B_t^{(1)} + iB_t^{(2)})})_{t\geq 0}$ is a complex martingale, and deduce the characteristic function of C_a for $a > 0$ fixed.
   (c) Compute the distribution of C_a.

Exercise 5 — *Exponential Martingale.*
Show that if $(X_t)_{t\geq 0}$ is a process such that for any $\lambda \in \mathbb{R}$, $(e^{\lambda X_t - t\lambda^2/2})_{t\geq 0}$ is a continuous martingale, then $(X_t)_{t\geq 0}$ has the law of a Brownian motion.

Exercise 6 — *Martingales derived from B.*
Let B be a Brownian motion. For $n \geq 0$, we define the n-th Hermite polynomial H_n by,

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

We equip the vector space $\mathbb{R}[X]$ of real polynomials with the scalar product,

$$P \cdot Q = \int_{\mathbb{R}} P(x)Q(x) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx.$$

(1) Show that $(H_n)_{n\geq 0}$ is an orthogonal family in $\mathbb{R}[X]$.
(2) Show that for every $\lambda, b \in \mathbb{R}$, $e^{\lambda b - \frac{\lambda^2}{2}} = \sum_{n\geq 0} \frac{H_n(b)}{n!} \lambda^n$.
(3) Show that for every $n \geq 0$, the process $(t^{n/2} H_n\left(\frac{B_t}{\sqrt{t}}\right))_{t\geq 0}$ is a martingale.