TD12 : Harmonic Functions and Miscellaneous

Exercise 1 — *Point transience of Brownian motion.*
Let $d \geq 2$, and let B be an $\mathbb{R}^d$-valued Brownian motion.

(1) Assume that $d = 2$, show that almost surely $\{B_t, t \in [0,1]\}$ is negligible with respect to the Lebesgue measure.
(2) Show that for every $x, y \in \mathbb{R}^d$, we have $\mathbb{P}_x(y \in \{B_t, t > 0\}) = 0$.

Exercise 2 — *Counterexample.*
Let $U = \{x \in \mathbb{R}^2, 0 < |x| < 1\} \subset \mathbb{R}^2$ be the punctured unit disk and let $\varphi : \partial U \to \mathbb{R}$ be the function defined by $\varphi(x) = \mathbf{1}_{x \neq 0}$. Consider the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = \varphi & \text{on } \partial U. \end{cases}$$

(1) Show that the Brownian expectation does not define a continuous solution of the equation above.
(2) Show that in fact this Laplace equation doesn't have any solution. (*Hint:* any continuous solution is of the form $u(x) = g(|x|)$)

Exercise 3 — *Gambler's ruin in several dimensions.*
Let $r, R \in (0, \infty)$ such that $r < R$ and $d \geq 1$, consider the annulus

$$U = \{x \in \mathbb{R}^d, r < |x| < R\}$$

Let B be a $\mathbb{R}^d$-valued Brownian motion, we let T_r (resp. T_R) denote the hitting time of the ball of radius r (resp. R) centered at 0 by B. The hitting time of ∂U by B is given by $T_{\partial U} = T_r \wedge T_R$. Recall the definition of the Laplacian operator,

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial^2 x_i}$$

(1) Let $c, C \in \mathbb{R}$ and let $\varphi : \partial U \to \mathbb{R}$ be the function defined by $\varphi(x) = c\mathbf{1}_{|x|=r} + C\mathbf{1}_{|x|=R}$. Give a solution of the Laplace equation

$$\begin{cases} \Delta u = 0 \text{ on } U \\ u = \varphi \text{ on } \partial U. \end{cases}$$

(2) Using u compute $\mathbb{P}_x(T_r < T_R)$.
(3) for $|x| > r$, compute $\mathbb{P}(T_r < \infty)$.
(4) Assume that $d = 2$, Show that almost surely, for every $x \in \mathbb{R}^2$ and $\varepsilon > 0$ there exists an increasing sequence $(t_n)_n \in (\mathbb{R}_+)^{\mathbb{N}}$ such that $t_n \to \infty$ and $|B(t_n) - x| \leq \varepsilon$.

(5) Assume that $d \geq 3$, show that $\mathbb{P}_0$-almost surely $\lim_{t\to\infty} |B_t| = \infty$. (*Hint:* Consider the events $A_n = \{ \text{ for every } t > T_{n^3},\ |B_t| > n\}$.)

Exercise 4 — *Law of iterated logarithms for random walks.*
Let B be a Brownian motion, define $\psi(t) = \sqrt{2t \log\log t}$.

(1) Let $(T_n)_n$ be a sequence of stopping times such that $T_n \to \infty$ almost surely and $T_n/T_{n+1} \to 1$ almost surely. For every $q > 4$, we define

$$D_k = \left\{ B(q^k) - B(q^{k-1}) \geq \psi(q^k - q^{k-1}) \right\}$$
$$\Omega_k = \left\{ \min_{q^k \leq t \leq q^{k+1}} B(t) - B(q^k) - \sqrt{q^k} \right\}$$

In what follows we admit the existence of $c > 0$ such that $\mathbb{P}(D_k) \geq c/(k \log k)$.

(a) Show that $\limsup \frac{B(T_n)}{\psi(T_n)} \leq 1$ almost surely.

(b) Show that $\mathbb{P}(\limsup D_{2k} \cap \Omega_{2k}) = 1$.

(c) Show that almost surely for infinitely many $k \geq 1$, we have

$$\min_{q^k \leq t \leq q^{k+1}} B_t \geq \psi(q^k)\left(1 - \frac{1}{q} - \frac{2}{\sqrt{q}}\right) - \sqrt{q^k}$$

(*Hint:* You may use the inequality $\psi(q^k - q^{k-1}) \geq \psi(q^k)(1 - 1/q)$.)

(d) By considering the sequence $n(k) = \inf\{n \geq 1, T_n \geq q^k\}$, show that almost surely,

$$\limsup_{n\to\infty} \frac{B(T_n)}{\psi(T_n)} = 1.$$

(2) Define a sequence of stopping times recursively by $T_0 = 0$ and

$$T_{n+1} = \inf\{t > T_n, |B(t) - B(T_n)| = 1\},$$

show that $T_n/n \to 1$ almost surely.

(3) Let $(X_k)_k$ be a sequence of independent uniform random variables in $\{-1, 1\}$ and $S_n = \sum_{k=1}^n X_k$. Show that,

$$\limsup \frac{S_n}{\psi(n)} = 1.$$

Exercise 5 — *Quadratic and absolute variation.*
Let $t \geq 0$, a partition $\underline{t}$ of $[0,t]$ is a finite sequence $0 = t_0 \leq t_1 \leq \ldots \leq t_n = t$, given a partition we define its length $\#\underline{t} = n$ and its mesh-size $|\underline{t}| = \max_{1\leq i\leq \#\underline{t}} |t_i - t_{i-1}|$. Let $f : [0,t] \to \mathbb{R}$ be a measurable function, we define the total variation of f on $[0,t]$ by

$$TV_t(f) = \lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} |f(t_i) - f(t_{i-1})|.$$

Where $\sup_{|\underline{t}|\leq\epsilon}$ should be understood as the supremum over all partitions of $[0,t]$ with mesh-size $\leq \epsilon$. Similarly, we define the quadratic variation of f on $[0,t]$ by

$$QV_t(f) = (\lim_{\epsilon\to 0}\sup_{|\underline{t}|\leq\epsilon}\sum_{i=1}^{\#\underline{t}}(f(t_i)-f(t_{i-1}))^2.$$

(1) Let $(\underline{t}^{(k)})_k$ be a sequence of partitions with $|\underline{t}^{(k)}|\to 0$. For every $k\geq 1$, let

$$X_k = \sum_{i=1}^{\#\underline{t}^{(k)}}(B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2.$$

  (a) Assume that the sequence $(X_k)_k$ converges in $L^2(\Omega)$ to some constant random variable X, show that $X=t$ almost surely.
  (b) Show that $(X_k)_k$ converges in $L^2(\Omega)$ toward the constant random variable taking only the value t.
  (c) Show that if $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty}\sum_{j=1}^{\#\underline{t}^{(k)}}(t_i^{(k)} - t_{i-1}^{(k)})^2 < \infty$, then $(X_k)_k$ converges almost surely.
  (d) What can you say about the random variable $QV_t(B)$?

(2) Show that almost surely the trajectories of the Brownian do not have bounded total variation, that is $\mathbb{P}(TV_t(B)=\infty)=1$. (*Hint*: what can you say about the quadratic variation of a continuous function with finite total variation ?).

Exercise 6 — *A weaker condition for the first Wald's lemma.*
We wish to show that when T is a stopping time with $\mathbb{E}[T^{1/2}]<\infty$, Wald's lemma still applies and $\mathbb{E}[B_T]=0$

(1) Define $\tau := \min\{k : 4^k \geq T\}$. Set $M(t) := \max_{[0,t]} B$ and $X_k := M(4^k) - 2^{k+2}$. Show that (X_k) is a supermartingale for the filtration $(\mathcal{F}_{4^k})_k$, and that τ is a stopping time.
(2) Show that $\mathbb{E}[M(4^\tau)]<\infty$ and conclude.
(3) Show that when T is the hitting time of 1, then $\mathbb{E}[T^\alpha]<\infty$ for all $\alpha<1/2$, yielding that our result is in some sense optimal.