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TD3 : Continuous time Markov chains

Exercice 1 — *Transition semigroups.*
In this exercise, questions (3) is independent from the rest.

(1) Let $d \geq 1$, let $(\mu_t)_{t\geq}$ be a family of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Assume that there exists a measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that for every $t \geq 0$, the characteristic function of μ_t is given by $\xi \mapsto e^{t\phi(\xi)}$. Let $(X_t)_{t\geq 0}$ be independent random variables such that for every $t \geq 0$, X_t has law μ_t. For every $t \geq 0$ and $x \in \mathbb{R}^d$, let $P_t(x, \cdot)$ be the law of $x + X_t$. Show that $(P_t)_{t\geq 0}$ is a Markov semigroup on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

(2) Using the previous question, show that the Poisson semigroup, the Gaussian semigroup, and the lognormal semigroup from lecture 2 are indeed Markov semigroups. Briefly explain why it's not possible to use question (1) to show that the Cauchy semigroup is indeed a Markov semigroup.

(3) Let $(E, \mathcal{E})$ be a measurable space, let $(P_t)_{t\geq 0}$ be a Markov semigroup on $(E, \mathcal{E})$ and let $h : \mathbb{R}_+ \times E \to \mathbb{R}_+^*$ be a measurable function. For every $t \geq 0$ and every $x \in \mathbb{R}^d$, we define a measure $\hat{P}_t(x, \cdot)$ on E via,
$$\frac{d\hat{P}_t(x, \cdot)}{dP_t(x, \cdot)} = \frac{1}{h(t, x)} h(t, \cdot).$$
Assume that for every $t \geq 0$ and every $x \in \mathbb{R}^d$, we have,
$$h(t, x) = \int_E h(t, x') dP_t(x, dx').$$
Show that $(\hat{P}_t)_{t\geq 0}$ is a Markov semigroup on $(E, \mathcal{E})$.

Exercice 2 — *Markov Property at the first jump time.*
Let X be a continuous Markov chain valued in a discrete set I with intensity matrix Q. Let ζ be the blow-up time of X and let T be a stopping time. Let η be a, possibly infinite, $\mathcal{F}_T$-measurable random variable. Consider a positive measurable function $f : I \to \mathbb{R}$.

(1) Let A be an $\mathcal{F}_T$-measurable and let $i \in I$, define $F_i(t) = \mathbb{E}_i[f(X_t)]$. Show,
$$\mathbb{E}[f(X_{T+\eta})\mathbf{1}_{A\cap\{T<\zeta\}\cap\{\eta<\infty\}\cap X_T=i}] = \mathbb{E}[F_i(\eta)\mathbf{1}_{A\cap\{T<\zeta\}\cap\{\eta<\infty\}\cap X_T=i}].$$

(2) Let $i_0 \in I$, deduce from the previous question that,
$$\mathbb{E}_{i_0}[f(X_t)\mathbf{1}_{J_1\leq t}] = \sum_{i\neq i_0} q_{i_0 i} \int_0^t e^{-q(i_0)s}\mathbb{E}_i[f(X_{t-s})]ds.$$

Exercice 3 — *Bacterial Proliferation.*
Consider a population of independent bacterias. Each bacteria splits into two bacterias

after an exponential time of parameter λ. Let X_t denote the number of bacterias in the population at time t and $m(t) = \mathbb{E}_1[X_t]$ the expected number of bacterias when there is only one bacteria at time $t = 0$.

(1) Show that X is a continuous time Markov chain and give its intensity matrix.
(2) Show that for every $t \geq 0$, we have

$$m(t) = e^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \mathbb{E}_2[X_{t-s}] ds.$$

(3) Deduce from the previous question that m satisfies an ordinary differential equation and use it to determine an explicit expression of m.
(4) Let $t \geq 0$, using a similar approach, compute the moment generating function of X_t. Give the law of X_t and for every $k \geq 1$, compute $\mathbb{P}_1(X_t = k)$.