TD4 : Stopping times

Exercice 1 — *Elementary results on stopping times.*
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. We denote by $\mathbb{F}_+ = (\mathcal{F}_{t+})_{t\geq 0}$ the right-continuous completion of the filtration $\mathbb{F}$, where for every $t \geq 0$

$$\mathcal{F}_{t+} = \bigcap_{t<s} \mathcal{F}_s.$$

(1) Let T be an $\mathbb{F}$-stopping time, let $t \geq 0$ show that $\{T < t\}$ and $\{T = t\}$ are $\mathcal{F}_t$-measurable.
(2) Let T be a random variable valued in $\mathbb{R}_+ \cup \{+\infty\}$, show that T is a $\mathbb{F}_+$-stopping time if and only if for every $t \geq 0$, $\{T < t\} \in \mathcal{F}_t$.
(3) Let T, S be two $\mathbb{F}$-stopping times, assume that $T \leq S$ almost surely and show that $\mathcal{F}_T \subset \mathcal{F}_S$.
(4) Let T be an $\mathbb{F}$-stopping time, for every $n \geq 1$ show that $T_n = \frac{\lceil Tn \rceil}{n}$ is a $\mathbb{F}$-stopping time and that almost surely $(T_n)_n$ is a decreasing sequence that converges to T.

Let E be a metric space equipped with its Borel σ-algebra $\mathcal{E}$. Let $A \in \mathcal{E}$ be a measurable set and let X be an E valued $\mathbb{F}$-adapted process. Define,

$$T_A = \inf\{t \geq 0, X_t \in A\}.$$

(5) Assume that X has right continuous trajectories. Let $O \subset E$ be an open set, show that T_O is a $\mathbb{F}_+$-stopping time.
(6) Assume that X has continuous trajectories. Let $F \subset E$ be a closed set, show that T_F is a $\mathbb{F}$-stopping time.
(7) Give an example where T_O is not a $\mathbb{F}$-stopping time.

Exercice 2 — *Transition Matrix Computations.*
Let X be a continuous time Markov chain valued in a discrete set I. For every $i, j \in I$ and $t \geq 0$ we define $p_{i,j}(t) = \mathbb{P}(X_t = j | X_0 = i)$ and we let $P(t) = (p_{i,j}(t))_{i,j\in I}$.

(1) Recall the relation between the intensity matrix Q of the process X and $(P(t))_{t\geq 0}$.
(2) Let $n \geq 1$ and $Q \in M_n(\mathbb{R})$ be a diagonalizable matrix, let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ be the eigenvalues of Q. Show that there exists a unique d-uplet $A_1, \ldots, A_d \in M_n(\mathbb{R})$ such that for every $t \geq 0$,

$$e^{tQ} = \sum_{i=1}^{d} e^{\lambda_i t} A_i.$$

Show that this result becomes false when Q is not assumed to be diagonalizable. (*Hint :* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.)

(3) Compute $P(t)$ for every $t \geq 0$ assuming that $I = \{1, 2\}$ and that,

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

(4) Compute $P(t)$ for every $t \geq 0$ assuming that $I = \{1, 2, 3\}$ and that,

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

Exercice 3 — *Explosion time.*

Let X be a continuous time Markov chain, with intensity matrix Q such that for every $i \in I$, $q(i) \neq 0$. Let $(Y_n)_n$ denote the jump process of X and ζ be the explosion time of X. Let $(E_n)_{n\geq 1}$ be independent exponential random variables with parameters $\lambda_n \in (0, +\infty)$. We let E denote the value in $\mathbb{R}_+ \cup \{+\infty\}$ of $\sum_{n\geq 1} E_n$.

(1) Assume that $\sum_{n\geq 1} \frac{1}{\lambda_n} < +\infty$, show that $\mathbb{P}(E < +\infty) = 1$.

(2) Assume that $\sum_{n\geq 1} \frac{1}{\lambda_n} = +\infty$, show that $\mathbb{P}(E = +\infty) = 1$.

(3) We say that X is a Yule process when, X is $\mathbb{N}$-valued and for every $n \in \mathbb{N}$, $Y_n = Y_0 + n$. Let λ be a probability measure on $\mathbb{N}$, assume that X is a Yule process and compute the probability of explosion of X under $\mathbb{P}_\lambda$ in terms of $(q_n)_n$.

(4) Show that X almost surely doesn't blow up if and only if $\mathbb{P}\left(\sum_n \frac{1}{q(Y_n)} = +\infty\right) = 1$.

(5) Show that in the following cases the condition of question (4) is satisfied and so X almost surely doesn't blow up.

   (a) The state space is finite.

   (b) $\sup_{i\in I} q(i) < +\infty$.

   (c) The jump process Y admits a recurrent state. That is, there exists $i \in I$ such that almost surely the set $\{n \in \mathbb{N}, Y_n = i\}$ is infinite.