TD5 : Kolmogorov's Equations

Exercice 1 — *Birth and death processes.*
We consider the continuous time Markov process X with values in $\mathbb{N}$ and intensity matrix Q given by:

$$q_{i,j} = \begin{cases} \beta_i & \text{si } j = i+1 \\ \delta_i & \text{si } j = i-1 \\ -\beta_i - \delta_i & \text{si } j = i \neq 0 \\ -\beta_i & \text{si } j = i = 0 \\ 0 & \text{sinon,} \end{cases}$$

where the β_i et δ_i are assumed to be nonnegative.

(1) Let $i \in \mathbb{N}$, using Kolmogorov's equation, write down a system of differential equations satisfied by $p_{ij}(t) = \mathbb{P}_i(X_t = j)$.
(2) Show that under the hypothesis that for all $i \in \mathbb{N}$, $\delta_i = 0$ or for all $i \in \mathbb{N}$, $\beta_i = 0$, the system of differential equations of the previous question admits at most one solution on $\mathbb{R}_+$.
(3) Assume that for all $i \in I$, $\beta_i = \beta$ and $\delta_i = 0$, show that when the process is started at $X_0 = 0$, the law of X_t is Poisson with parameter βt.
(4) Assume that for all $i \in I$, $\beta_i = 0$ and $\delta_i = i\delta$, show that when the process is started at $X_0 = C > 0$, the law of X_t is binomial with parameters $(C, e^{-\delta t})$.
(5) Give an interpretation of the Markov chain in (4), give another way to compute the value of the extinction probability $\mathbb{P}_C(X_t = 0)$.

Exercice 2 — *Kolmogorov's equations makes your life easier.*
Let I be a set and X a continuous time Markov chain with intensity matrix Q. Let λ be a signed measure on I and $f : I \to \mathbb{R}$, we let $g_\lambda(t) = \mathbb{E}_\lambda[f(X_t)] := \sum \lambda(\{i\})\, \mathbb{E}_i[f(X_t)]$. In this exercise, we assume that the integrals/sums are well-defined and that we can derive under the integral/sum. Note this is in particular always the case when I is finite, but could be false in general.

(1) Identifying the measure λ with the lign vector $(\lambda(\{i\}))_{i\in I}$ and the function f with the column vector $(f(j))_{j\in I}$, show we have

$$g_\lambda(t) = \lambda P(t) f.$$

(2) Show that g_λ is differentiable and that for every $t \geq 0$,

$$g'_\lambda(t) = \mathbb{E}_{\lambda Q}[f(X_t)] = \mathbb{E}_\lambda[Qf(X_t)].$$

Consider the model of exercise 3 from TD3. A population of independent bacterias. Each bacteria splits into two bacterias after an exponential time of parameter λ. Let X_t denote the number of bacterias in the population at time t. We have shown previously that X is a Markov chain with intensity matrix,

$$q_{i,j} = \begin{cases} -\lambda i & \text{when } j = i \\ \lambda i & \text{when } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

We also recall that the law of X_t started from $X_0 = 2$ is the equal to the law of the sum of two independent copies of X_t started from 1.

(3) Let $r \in (-1, 1)$, use the previous questions to find a differential equation satisfied by $g_1(t) = \mathbb{E}_1[r^{X_t}]$.

(4) *Bonus :* Using $f(x) = \mathbf{1}_{\mathbb{R}_+^*}(x)$ find some special cases of the birth and death process from exercise 1 where you can derive some bounds on the probability of extinction $\mathbb{P}_\lambda(X_t = 0)$.

Exercice 3 — *Intensity matrix and transition matrices.*

Let I be a finite set, we set that a matrix P on I is stochastic when all of its entries are nonnegative and for every $i \in I$,

$$\sum_{j \in J} P_{i,j} = 1.$$

Let $Q = (q_{i,j})_{i,j \in I}$ be a matrix on I, for $t \geq 0$, let $P(t) = e^{tQ}$. We aim to show the equivalence of the following three statements:

(i) Q is an intensity matrix

(ii) $P(t)$ is a stochastic matrix for all t in a neighbourhood of 0.

(iii) $P(t)$ is a stochastic matrix for all $t \geq 0$.

(1) Show (ii) and (iii) are equivalent.

(2) Show (ii) implies (i).

(3) We now suppose (i) is satisfied.

   (a) Show that for all i and all t, we have $\sum_j P(t)_{i,j} = 1$. (*Hint: Use the ODE satisfied by* P)

   (b) Show the entries of the matrix $P(t)$ are nonnegative.