Master 1 — Spring 2024
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TD6 : Invariant Measures & Explosion

Exercice 1 — *Birth and death processes II.*
We consider the pure jump Markov process with values in $\mathbb{N}$ and intensity matrix Q given by:

$$q_{i,j} = \begin{cases} \beta_i & \text{si } j = i+1 \\ \delta_i & \text{si } j = i-1 \\ -\beta_i - \delta_i & \text{si } j = i \neq 0 \\ -\beta_i & \text{si } j = i = 0 \\ 0 & \text{sinon,} \end{cases}$$

where we assume $\beta_i > 0$. We let $r_n = \frac{1}{\beta_n} + \sum_{k=0}^{n-1} \frac{\delta_{k+1} \ldots \delta_n}{\beta_k \ldots \beta_n}$, note that the sequence $(r_n)_n$ satisfies

$$r_n = \frac{1 + \delta_n r_{n-1}}{\beta_n}.$$

We consider x a nonnegative solution to the equation $Qx = x$.

(1) Show that $x_0 = 0$ if and only if $x = 0$. Show that if $x \neq 0$, then the sequence (x_n) is increasing.
(2) Show that for every $i \geq 0$, we have $x_i + r_i x_0 \leq x_{i+1} \leq (1 + r_i) x_i$.
(3) We say that a process doesn't explode when for every $i \in E$, the probability of explosion starting from i is 0. Show that markov chain with intensity matrix Q doesn't explode if and only $\sum r_n = +\infty$.

Let $\lambda_i > 0$ and $p \in (0, 1)$ set $q = 1 - p$ and consider X the continuous time Markov chain with intensity matrix Q with parameters $\beta_i = p\lambda_i$ and $\delta_i = q\lambda_i$.

(4) Show that the equation $\mu Q = 0$ has a unique solution up to multiplicative constant, assuming that $\mu_0 = \frac{1}{\lambda_0}$, give an explicit expression of μ_i for every $i \in \mathbb{N}$.
(5) Assume that there exists $\lambda > 0$ such that for every $i \in \mathbb{N}$, $\lambda_i = \lambda$ and that $p < 1/2$. Show that X doesn't explode, that X admits a unique invariant probability measure and describe the set of invariant measures of X.
(6) Assume that there exists $\lambda > 0$ such that for every $i \in \mathbb{N}$, $\lambda_i = \lambda$ and that $p \geq 1/2$. Show that X doesn't explode and that X admits no invariant probability measure.

Exercice 2 — *M/M/1 queue invariant measure.*
Consider a shop where customers are served one at a time. Customers arrive at independent times and each arrival time follows an exponential law of parameter $\lambda > 0$. Customers are served one after the other, service times are independent and each service time follows an exponential law of parameter $\mu > 0$. We let X_t denote the number of customers in queue at time $t \geq 0$ (including the customer currently being served). We assume that the queue is empty at time 0 ($X_0 = 0$).

(1) Show that X is a continuous time Markov chain, give its intensity matrix and show that X doesn't blow up.
(2) Show that the process X admits a reversible measure.
(3) Using Exercise 1, give a necessary and sufficient condition for X to admit an invariant probability distribution π. Express π in terms of $\rho = \lambda/\mu$.
(4) Assume that the condition of question (3) is fulfilled, on average how much time do we have to wait until we see 0 customers in the queue for the first time (excluding $t = 0$)? In the large t limit, what is the probability that here is no customers left in the queue ? In the large t limit, what is the average number of customers in the queue?
(5) Bonus : Find again an invariant measure of X by using the following observation (that you will show). Let π be an invariant measure of X, and let $\Pi(s) = \sum_{n\geq 1} \pi(n)s^n$ be its generating function of, then for every $s \in (-1, 1)$,

$$\lambda s^2 \Pi(s) - (\lambda + \mu)s(\Pi(s) - \pi_0) + \mu(\Pi(s) - \pi_0) - \lambda \pi_0 s = 0.$$

Exercice 3 — *More hitting times.*
Let T^A be the hitting time of A and $h_A(i) = \mathbb{P}_i(T^A < +\infty)$,

(1) Show that the vector $(h_A(i))_{i\in I}$ is the minimal non-negative solution to

$$\begin{cases} h_A(i) = 1 & \text{if } i \in A \\ \sum_{j\in I} q_{i,j} h_A(j) = 0 & \text{otherwise.} \end{cases}$$

(2) Provide a similar interpretation to the minimal nonnegative solution of the system

$$\begin{cases} k(i) = 0, & \text{if } i \in A, \\ \sum_{j\in I} q_{i,j} k(j) = -h_A(i) & \text{otherwise.} \end{cases}$$

(3) Applications: Let Q be the intensity matrix on $I = \{1, 2, 3, 4\}$ given by:

$$Q = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For any given initial state, compute the probability of hitting state 3, as well as the expectation of the hitting time of state 4.