Master 1 — Spring 2024
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TD7 : Conditional probability distributions, first properties of the Brownian Motion

Exercise 1 — *Gaussian vectors.*
Let X be a random vector in $\mathbb{R}^n$. We say that it is a Gaussian vector if for every $t \in \mathbb{R}^n$, the random variable $\langle t, X\rangle \in \mathbb{R}$ has a Gaussian distribution (with possibly null variance).

(1) Recall the parameters, the characteristic function, and (when it exists) the p.d.f. of a Gaussian distribution on $\mathbb{R}$.
(2) Show that $t \mapsto \mathbb{E}[\langle t, X\rangle]$ is a linear form, and $(s,t) \mapsto \mathrm{Cov}[\langle s, X\rangle, \langle t, X\rangle]$ is a *positive semi-definite* bilinear form. Let them be represented by $\langle \cdot, m\rangle$ and $\langle \cdot, \Sigma \cdot\rangle$. Give an interpretation of m_i and Σ_{ij} for every $i, j \in \{1, \ldots, n\}$.
(3) Let X be a Gaussian vector, for every $t \in \mathbb{R}^n$, compute $\mathbb{E}[e^{i\langle t, X\rangle}]$. Briefly explain why the distribution of X is characterized by the parameters m and Σ.
(4) Let X be a Gaussian vector with parameters (m, Σ) and A be a $p \times n$ matrix, show that $AX \in \mathbb{R}^p$ is a Gaussian vector, and compute its parameters.
(5) We say that two processes A and B are uncorrelated when for every index t, s, $Cov(A_t, B_s) = 0$. Let V_1 and V_2 be two subspaces of $\mathbb{R}^n$ and X a Gaussian vector. Show that the σ-algebras $\sigma(\langle t, X\rangle, t \in V_1)$ and $\sigma(\langle t, X\rangle, t \in V_2)$ are independent if and only if $(\langle t, X\rangle)_{t\in V_1}$ and $(\langle s, X\rangle)_{s\in V_2}$ are uncorrelated.
(6) Build two standard Gaussian variables X and Y that are uncorrelated yet not independent (they obviously do not form a Gaussian vector !)
(7) Show that the vector $(X_1, \ldots X_n)$ with $X_1, \ldots, X_n$ independent standard Gaussian variables, is Gaussian. Use it to build a Gaussian vector with arbitrary parameters. Deduce its p.d.f. when it has one.

Exercise 2 — *Gaussian conditional distribution and Bayesian statistics 101.*
Let (X, Y) be a non-degenerate centered Gaussian vector in $\mathbb{R}^2$ with covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho \\ \rho & \sigma_y^2 \end{pmatrix}.$$

(1) For every $y \in \mathbb{R}$, compute the conditional distribution of X given $Y = y$.
(2) Let $\theta \sim \mathcal{N}(0, \tau^2)$ and $Y_1, \ldots, Y_n$ i.i.d. $\sim \mathcal{N}(0, \sigma^2)$ random variables, define $X_i = \theta + Y_i$. What is the conditional distribution of θ given $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i = \overline{x}$?
(3) Give an interpretation of the situation discribed in the previous question.
(4) Compute the limit of the distribution of θ given $\overline{X} = \overline{x}$ and give an interpretation in each of the following cases.
   (a) $\sigma \to +\infty$
   (b) $\sigma \to 0$

(c) $\tau \to +\infty$
(d) $\tau \to 0$

(5) ($\star$) What about the conditional distribution of θ given $(X_1, \dots X_n)$?

Exercise 3 — *Borel-Kolmogorov paradox.*
Let P denote a uniform point in the sphere $\mathbb{S}^2$, i.e. for every bounded measurable f,

$$\int f(p)\, \mathbb{P}_P(dp) = \frac{1}{\mathrm{Leb}_3(B_{\mathbb{R}^3}(0,1))} \int_{B_{\mathbb{R}^3}(0,1)} f\left(\frac{p}{|p|}\right) \mathrm{Leb}_3(dp).$$

Denote $\phi_P \in (-\pi/2, \pi/2]$ its latitude and $\theta_P \in (-\pi, \pi]$ its (almost surely defined) longitude.

(1) Compute the conditional distribution of P given $(\theta_P \mod \pi)$.
(2) Compute the conditional distribution of P given ϕ_P.
(3) Justify that there is only one "right way" of specializing those answers at $\theta \mod \pi = 0$ (resp. $\phi = 0$).
(4) What is the paradox ?

Exercise 4 — *Transformations.*
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(1) Show that for any $\lambda \in \mathbb{R}_+^\star$, the process $(\lambda^{-1/2} B_{\lambda t})_{t\geq 0}$ is a Brownian motion.
(2) Show that $B_1 - B_{1-t}$ is a Brownian motion on $[0, 1]$.

Exercise 5 — *A nowhere continuous version of the Brownian motion.*
Let (W_t) be a Brownian motion, the goal of this exercise is to build a process $(B_t)_{t\geq 0}$ such that for every ω, the function $t \mapsto B_t(\omega)$ is nowhere continuous and for every $t_1 < \ldots < t_n$, the vector $(B_{t_1}, \ldots, B_{t_n})$ has the same distribution as $(W_{t_1}, \ldots, W_{t_n})$.

(1) Let $(U_i)_{i\geq 1}$ be sequence of exponential random variables with parameter 1. Show that, almost surely, the set $\{U_i, i \geq 1\}$ is dense in $[0, +\infty)$.
(2) Build B as described.

Hint: change the value of W on a countable dense random subset of $\mathbb{R}$, so that the value at a fixed deterministic time is almost surely not changed.

Exercise 6 — *Brownian motion is nowhere monotonous.*
Let B be a Brownian motion. Show that almost surely, the function $t \mapsto B_t$ is not monotonous on any nonempty open interval.

Exercise 7 — *The stationary Ornstein-Uhlenbeck process.*
For $t \in \mathbb{R}$, set $X_t = e^{-t} B_{e^{2t}}$, where B is a Brownian motion. Show that X is a continuous Gaussian process, compute its covariance function. For any given t, what is the distribution of X_t ? Does it have independent increments ?