

TD8 : Construction of the Brownian Motion

Exercise 1 — *Constructing a Brownian motion indexed by \mathbb{R}_+ .*

Let $(B^{(n)})_n$ be a sequence of independent Brownian motions defined on $[0, 1]$. For every $t \geq 0$, define

$$B_t = B_{t - [t]}^{([t])} + \sum_{i=0}^{[t]-1} B_1^{(i)}.$$

Show that $(B_t)_{t \geq 0}$ is a Brownian motion.

Exercise 2 — *Lévy's construction of the Brownian motion.*

Let $H = L^2([0, 1])$ with the usual inner product. For $t \geq 0$ let $I_t = \mathbb{1}_{[0,t]} \in H$. We also set $(e_i)_{i \in \mathbb{N}}$ to be an orthonormal basis of H .

- (1) Check that $\langle I_s, I_t \rangle = s \wedge t$.
- (2) Assume that there exists a H -valued standard Gaussian random variable. That is, a random variable $\xi \in H$, such that for every $x \in H$, $\langle x, \xi \rangle \sim \mathcal{N}(0, |x|^2)$.
 - (a) Using the random variable ξ and the functions $(I_t)_{t \geq 0}$, build a Gaussian process $(B_t)_{t \in [0,1]}$ such that $\text{Cov}(B_s, B_t) = s \wedge t$.
 - (b) Let $Z_i = \langle \xi, e_i \rangle$, so that $\xi = \sum_{i \in \mathbb{N}} Z_i e_i$. Show that the (Z_i) are independent standard Gaussians (*Hint*: Compute the characteristic function of finite subvectors.). Deduce that the process of the previous question would satisfy,

$$(\dagger) \quad B_t = \sum_{n=0}^{\infty} Z_n \int_0^t e_n(s) ds.$$

- (c) By computing $|\xi|^2$, show that ξ cannot exist.
- (3) Define $h_0 = 0$ and for $n \geq 0$ and $0 \leq k < 2^n$,

$$h_{k,n} := 2^{n/2} \left(\mathbb{1}_{\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right]} - \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]} \right),$$

We admit (or recall) that $(h_{k,n})_{k,n}$ is an orthonormal basis of H called the Haar wavelet basis. Let $(Z_{n,k})_{n,k}$ be a family of independent standard Gaussian random variables. For every $t \geq 0$ set

$$(\dagger\dagger) \quad B_t = tZ + \sum_{n=0}^{\infty} F_n(t),$$

where $F_n(t) = \sum_{k=0}^{2^n-1} Z_{n,k} f_{n,k}(t)$ and $f_{n,k}(t) = \int_0^t h_{n,k}(s) ds$.

- (a) Using the inequality $\mathbb{P}(|Y| \geq \lambda) \leq \frac{\sqrt{2/\pi}}{\lambda} e^{-\lambda^2}$ for $\lambda > 0$ and $Y \sim \mathcal{N}(0, 1)$, show that

$$\mathbb{P}\left(2^{-\frac{n+2}{2}} \max_{0 \leq k < 2^n} |Z_{n,k}| > \frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right).$$

- (b) Show that $\mathbb{P}(\|F_n\|_\infty \leq \frac{1}{n^2} \text{ for } n \text{ large enough}) = 1$.
 (c) Show that almost surely, the sum of functions in $(\dagger\dagger)$ converges uniformly on $[0, 1]$ to a (random) continuous function.
 (4) (\star) Prove the same result than in the previous question when we use the Fourier basis $e_0 = 1$, and $e_m(t) = \sqrt{2} \cos(\pi mt)$ in (\dagger) rather than the Haar wavelet basis.

Exercise 3 — *Time inversion.*

Let $(B_t)_{t \geq 0}$ be a Brownian motion. Set $X_t = tB_{1/t}$ for $t > 0$ and $X_0 = 0$.

- (1) Show that X has the finite-dimensional marginals of a Brownian motion.
- (2) Show that the set $U = \{f \in \mathbb{R}^{\mathbb{Q}^+}, \lim_{t \rightarrow 0, t \in \mathbb{Q}} f_t = 0\} \subset \mathbb{R}^{\mathbb{Q}^+}$ is measurable.
- (3) Deduce that $(X_t)_t$ is continuous almost surely, hence may be modified on a negligible event to form a Brownian motion.

Exercise 4 — *Brownian bridges.*

For $x, y \in \mathbb{R}$, we define the Brownian bridge of length one between x and y as follows: let B be a standard Brownian motion and set $\beta_t^{x,y} = x + B_t - tB_1 + t(y - x)$ for $t \in [0, 1]$.

- (1) Show that if X is a Brownian motion started from x , then the conditional distribution of the process $X_{|[0,1]}$ given $X_1 \in dy$ is $\beta^{x,y}$.
- (2) For $0 < a < 1$, what expression does the Markov property applied at a give for the joint distribution of $(X_{|[0,a]}, X_1)$? Deduce an expression for the conditional distribution of $X_{|[0,a]}$ given $X_1 \in dy$. Deduce a second expression from the previous question.
- (3) Show that the distribution of $\beta_{|[0,a]}^{x,y}$ is absolutely continuous with regard to that of $X_{|[0,a]}$ where $a < 1$.

Hint: use the fact that the conditional distributions are a.e. uniquely defined, along with some continuity argument.