

TD9 : Regularity of Brownian Trajectories and Stopping Times

Exercise 1 — *Warm-up.*

Let B be a Brownian motion, show that B has the simple Markov property. That is for every $s \geq 0$ show that $(B_{t+s} - B_s)_{t \geq 0}$ is a Brownian motion independent from $(B_t)_{t \leq s}$.

Exercise 2 — *Counter-example.*

Let B be a Brownian motion and let

$$T = \inf\{t \geq 0, B_t = \max_{s \in [0,1]} B_s\},$$

be the first hitting time by B of the maximum of B on $[0, 1]$, is T a stopping time ?

Exercise 3 — *Hölder regularity of Brownian trajectories.*

Let B be a Brownian motion, recall from Exercise 3 of TD8 that the process $X = (tB_{1/t})_{t \geq 0}$ is also a Brownian motion.

- (1) Show that $\lim_{t \rightarrow \infty} \frac{B_t}{t} = 0$ a.s. and that for every $\varepsilon > 0$,

$$\lim_{t \rightarrow 0} \frac{B_t}{t^{1/2-\varepsilon}} = 0 \text{ and } \lim_{t \rightarrow \infty} \frac{B_t}{t^{1/2+\varepsilon}} = 0 \text{ a.s..}$$

- (2) Let $(\xi_n)_{n \geq 1}$ be a sequence of standard random variables.
 (a) For every $K \in \mathbb{R}$, show that $\mathbb{P}(\limsup_{n \rightarrow \infty} \{\sum_{k=1}^n \xi_k \geq K\sqrt{n}\}) > 0$.
 (b) Show that almost surely

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k = +\infty.$$

- (3) Deduce from the previous question that,

$$\limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t \rightarrow \infty} \frac{B_t}{\sqrt{t}} = -\infty,$$

$$\limsup_{t \rightarrow 0} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t \rightarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

- (4) Recall that given a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $t \geq 0$, we say that f is locally α -Hölder at t if there exists a neighborhood V of t in \mathbb{R}_+ and a constant $c \geq 0$ such that for every $s, s' \in V$, we have

$$|f(s) - f(s')| \leq c|s - s'|^\alpha.$$

- (a) Show that almost surely B is not locally 1/2-Hölder at 0.

- (b) For every $t \geq 0$, define $L_t = \limsup_{s \rightarrow 0} \left| \frac{B_{t+s} - B_t}{\sqrt{s}} \right|$ and $A = \{t \in \mathbb{R}_+, L_t < +\infty\}$. Show that almost surely the set A is negligible with respect to the Lebesgue measure.
- (c) Show that the set of points at which B is locally $1/2$ -Hölder is almost surely negligible with respect to the Lebesgue measure.

Exercise 4 — *The set of zeros of B is perfect.*

Recall that for every $S \subset \mathbb{R}$ we say that $x \in S$ is an isolated point of S if there exists a neighborhood V of x in \mathbb{R} such that $V \cap S = \{x\}$. Let B be a Brownian motion, and

$$Z = \{t \geq 0, B_t = 0\}.$$

- (1) Show that the following events are almost sure,
- Z is infinite and is a closed set.
 - Z has Lebesgue measure 0.
 - Z has no isolated points.
- (2) (★) Make the notion of random closed set rigorous by defining a σ -algebra on the set of closed subsets of \mathbb{R}_+ .

Exercise 5 — *Another counter-example.*

Let $X_t = AB_t$ where B is a Brownian motion started from 1 and A an independent balanced Bernoulli.

- Show that X is a Markov process and give its transition kernel.
- Show that it does not verify the strong Markov property.

Exercise 6 — *Brownian motion on the circle.*

Define a Brownian motion on the circle \mathbb{S}^1 by setting $X_t = e^{iB_t}$ for $t \geq 0$. What is the distribution of the last point hit by X in \mathbb{S}^1 ?