ENS de Lyon — Mathematic department — Master 1 — Spring 2025
Stochastic processes — V. Issa & E. Jacob

TD1: Continuous-time stochastic processes

Exercice 1 — *Modification and indistinguishability.*

(1) Show that two functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are rightcontinuous and coincide on a (possibly countable) dense subset of $\mathbb{R}_+$, are the same.

Let f and g be two such functions that coincide on some dense set $D \subset \mathbb{R}_+$. Let $t \geq 0$, for every $n \geq 1$ there exists $t_n \in D \cap [t+1/n, t+2/n]$. We have $f(t_n) = g(t_n)$ and $t_n \downarrow t$, letting $n \to \infty$, we obtain $f(t) = g(t)$.

(2) Deduce that two continuous-time stochastic processes with real values $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ that are modifications of each other and have rightcontinuous trajectories, are actually indistinguishable.

If Y is a modification of X then for every $t \geq 0$, $\mathbb{P}(X_t = Y_t) = 1$. Since $\mathbb{Q}$ is countable, $\mathbb{P}(\forall t \in \mathbb{Q} \cap [0, +\infty), X_t = Y_t) = 1$. Since the trajectories are right continuous, the event $\{\forall t \in \mathbb{Q} \cap [0, +\infty), X_t = Y_t\}$ and $\{\forall t \in [0, +\infty), X_t = Y_t\}$, coincide so $\mathbb{P}(\forall t \in [0, +\infty), X_t = Y_t) = 1$ and X is indistinguishable from Y.

(3) Construct a process $(X_t)_{t\geq 0}$ that is a modification of the trivial process $(Y_t)_{t\geq 0}$ defined by $Y_t = 0$, but whose trajectories are almost surely discontinuous everywhere. (Thus clearly X is distinguishable from Y).

(*Hint*: Construct first a process X that is a modification of Y but whose trajectories differ at one (necessarily random) time.)

Consider a uniform random variable $U \in [0, 1]$, the process $X_t = \mathbf{1}_U(t)$ is a modification of Y but its trajectories only have one point of discontinuity (at $t = U$). To circumvent this, we define $X_t = \mathbf{1}_{U+\mathbb{Q}}(t)$. Since $U + \mathbb{Q}$ and its complement are dense, the trajectories of X are almost surely discontinuous everywhere. We have,

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(X_t = 0) = \mathbb{P}(U \notin -t + \mathbb{Q}) = 1,$$

so X is a modification of Y.

Exercice 2 — *Scaling Property.*

Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Show that for every $c > 0$, $(N_{ct})_{t\geq 0}$ is a Poisson process with intensity $c\lambda$.

We call N' the new process. The trajectories of N are càdlàg so clearly the trajectories of N' are also càdlàg. We have $N'_0 = N_0 = 0$ almost surely. For every $t \geq 0$, $N'_t - N'_{t-} = N_{ct} - N_{(ct)-} \in \{0, 1\}$ almost surely. Finally, let $(J_n)_n$ denote the sequence of jump times of N, that is

$$J_n = \inf\{t \geq 0 \big| N_t \geq n\}.$$

Similarly, we define $(J'_n)_n$ for N' and we have $J'_n = J_n/c$. If $S_n = J_n - J_{n-1}$, then $S'_n = S_n/c$ and since $S_n \sim \exp(\lambda)$ we have $S'_n \sim \exp(c\lambda)$. So N' is a Poisson process with intensity $c\lambda$.

Exercice 3 — *Measurability.*
Let I be a nonempty set, we consider the space $\mathbb{R}^I$ equipped with the σ-algebra $\mathcal{B}(\mathbb{R})^{\otimes I}$. Recall that $\mathcal{B}(\mathbb{R})^{\otimes I}$ is the smallest σ–algebra such that each projection $\pi_i : \mathbb{R}^I \to \mathbb{R}$ is measurable.

(1) Build a π-system that generates $\mathcal{B}(\mathbb{R})^{\otimes I}$. Deduce that a probability measure on $(\mathbb{R}^I, \mathcal{B}(\mathbb{R})^{\otimes I})$ is characterized by its finite dimensional marginals.

Recall a π-system is a subset of $\mathcal{P}(\mathbb{R})$ that is stable under finite intersection. Given a family $A = (A_i)_{i\in I}$ of $\mathcal{B}(\mathbb{R})$, we say that A is finitely supported if for every index $i \in I$, except a finite number, $A_i = \mathbb{R}$. By definition, the product σ-algebra $\mathcal{B}(\mathbb{R})^{\otimes I}$ is the σ-algebra generated by $\Pi = \{\prod_{i\in I} A_i,\ (A_i)_{i\in I} \text{ is finitely supported}\}$. If A and B are finitely supported, setting $C_i = A_i \cap B_i$ we have $\prod_{i\in I} C_i = (\prod_{i\in I} A_i) \cap (\prod_{\in I} B_i)$ so Π is a π-system.
Let P, Q be two probability measures on $(\mathbb{R}^I, B(\mathbb{R})^{\otimes I})$ with the same finite dimensional marginals. Then, P, Q are equal on the π-system Π. By the monotone class lemma we have $P = Q$ on $\sigma(\Pi) = \mathcal{B}(\mathbb{R})^I$.

(2) Show that $(X_t)_{t\in I}$ is a random variable on $(\mathbb{R}^I, \mathcal{B}(\mathbb{R})^{\otimes I})$ if and only if for every $t \in I$, X_t is a random variable on I.

If X is measurable, then $X_t = \pi_t \circ X$ is measurable as the composition of two measurable functions. If for every $t \in I$, X_t is measurable. Then the π-system Π is contained in

$$\mathcal{M} = \{A \in \mathcal{B}(\mathbb{R})^{\otimes I} \big| \, X^{-1}(A) \in F\}.$$

Since $\mathcal{M}$ is a monotone class (stable by difference and increasing union), $\mathcal{B}(\mathbb{R})^{\otimes I} = \sigma(\Pi) \subset \mathcal{M}$. Thus, $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^I, \mathcal{B}(\mathbb{R})^{\otimes I})$ is measurable.

(3) We equip $\mathbb{R}^I$ with the product topology, recall that the product topology is the coarsest topology (i.e the topology with the fewest open sets) that make the projections $\pi_i : \mathbb{R}^I \to \mathbb{R}$ continuous.

(a) Give a basis of open sets for the product topology on $\mathbb{R}^I$ and show that the σ–algebra generated by this basis is $\mathcal{B}(\mathbb{R}^I)$. $\mathcal{T} = \{(q, q'),\ q, q' \in \mathbb{Q}\}$ is a countable basis of open set of $\mathbb{R}$. So, $\mathcal{T}_I = \{\pi_i^{-1}(U),\ (i, U) \in I \times \mathcal{T}\}$ is a countable basis of $\mathbb{R}^I$. In particular $\mathcal{B}(\mathbb{R}^I) = \sigma(\mathcal{T}_I)$,

(b) Assume that I is countable and show that the Borel σ-algebra on $\mathbb{R}^I$ is $\mathcal{B}(\mathbb{R})^{\otimes I}$. When $\mathbb{R}^I$ is equipped with the product topology, the projections $\pi_i : \mathbb{R}^I \to \mathbb{R}$ are continuous so they are measurable and $\mathcal{B}(\mathbb{R})^{\otimes I} \subset \mathcal{B}(\mathbb{R}^I)$. This holds even when I is not countable. Conversely $\mathcal{T}_I \subset \mathcal{B}(\mathbb{R})^{\otimes I}$, and since $\mathcal{T}_I$ is countable the open sets of $\mathbb{R}^I$ are contained in $\mathcal{B}(\mathbb{R})^{\otimes I}$, so $\mathcal{B}(\mathbb{R}^I) \subset \mathcal{B}(\mathbb{R})^{\otimes I}$.

(4) We equip $\mathcal{C}([0, 1])$ with the topology of uniform convergence, and we denote $\mathcal{E}$ the Borel σ-algebra associated to this topology. Show that the restriction of $\mathcal{B}(\mathbb{R})^{\otimes[0,1]}$ to $\mathcal{C}([0, 1])$ is $\mathcal{E}$. For short, we let $\mathcal{C} = \mathcal{C}([0, 1])$, and we define

$$\mathcal{F} = \{A \cap \mathcal{C} \big| \, A \in \mathcal{B}(\mathbb{R}^I)\}.$$

The restriction to $\mathcal{C}$ of the projections $\pi_i : \mathbb{R}^I \to \mathbb{R}$ are continuous with respect to the sup-norm, so they are $\mathcal{E}$-measurable. Since $\mathcal{F}$ is generated by the sets of the

form $\pi_i^{-1}(A) \cap \mathcal{C}$ with $A \in \mathcal{B}(\mathbb{R})$, we have shown $\mathcal{F} \subset \mathcal{E}$.
Let D denote the set of polynomial functions on $[0,1]$, by the Stone-Weierstrass theorem, D is a dense subset of $\mathcal{C}$. In particular, $\mathbb{B} = \{B(f,r) \big| (f,r) \in D \times \mathbb{Q}_+^*\}$ is countable basis of open set for $\mathcal{C}$. In particular, every open set in $\mathcal{C}$ is a countable union of open set in $\mathbb{B}$, so $\mathcal{E} = \sigma(\mathbb{B})$. To conclude, let us show that $\mathbb{B} \subset \mathcal{F}$. We fix $(f,r) \in D \times \mathbb{Q}_+^*$, observe that $B(f,r)$ is the countable union of the closed balls $\bar{B}(f,r')$ with $r' \in \mathbb{Q} \cap (0,r)$. Finally,

$$\begin{aligned}\bar{B}(f,r') &= \{g \in \mathcal{C} \big| \forall x \in [0,1] \cap \mathbb{Q}, |f(x) - g(x)| \le r'\} \\ &= \bigcap_{i \in [0,1] \cap \mathbb{Q}} \pi_i^{-1}([f(x) - r', f(x) + r']) \cap \mathcal{C} \\ &\in \mathcal{F}.\end{aligned}$$

Exercice 4 — *Composed Poisson Processes.*

Let ν be a probability measure on $\mathbb{R}^n$ and $\lambda > 0$. Let $X = (X_t)_{t \ge 0}$ be a Poisson process of intensity λ and let $(M_k)_{1 \le k \le n}$ be a sequence of iid random variables with law ν taken independent of X. We define, the composed Poisson process of parameter $\lambda\nu$ by

$$Z_t = \sum_{k=1}^{X_t} M_k.$$

(1) Show that the increments of the process Z are stationary and independent, By definition, the process $(X_t)_t$ has stationary and independent increments such that $X_t - X_s$ has a Poisson law with parameter $\lambda(t-s)$. Since we have $Z_t - Z_s = \sum_{k=X_s}^{X_t} M_k$, the result follows.

(2) Let $t \ge 0$ and $\xi \in \mathbb{R}$, compute $\mathbb{E}[e^{i\xi Z_t}]$. What is the expectation and the variance of Z_t ? We can use Wald's formula to compute $\mathbb{E}[e^{i\xi Z_t}]$,

$$\begin{aligned}\mathbb{E}[e^{i\xi Z_t}] &= \mathbb{E}\left[\mathbb{E}\left[\prod_{k=1}^{X_t} e^{i\xi M_k} \big| X_t\right]\right] \\ &= \mathbb{E}\left[\prod_{k=1}^{X_t} \mathbb{E}\left[e^{i\xi M_k} \big| X_t\right]\right] \\ &= \mathbb{E}\left[\prod_{k=1}^{X_t} \mathbb{E}\left[e^{i\xi M_k}\right]\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}\left[e^{i\xi M_k}\right]\right)^{X_t}\right].\end{aligned}$$

If we let $\varphi(\xi)$ denote the characteristic function of M_k we have

$$\begin{aligned}\mathbb{E}[Z_t] &= \frac{d}{d\xi}\Big|_{\xi=0}\mathbb{E}[e^{i\xi Z_t}] \\ &= \mathbb{E}\left[X_t\varphi(0)^{X_t-1}\varphi'(0)\right] \\ &= \mathbb{E}\left[X_t\right]\mathbb{E}\left[M_1\right].\end{aligned}$$

(3) Let $p \in [0,1]$, assume that $\nu = (1-p)\delta_0 + p\delta_1$. Show that the processes Z and $X - Z$ are independent Poisson process of respective intensity $p\lambda$ and $(1-p)\lambda$.
(4) Deduce from the previous question that the sum of two Poisson process is also a Poisson process.