TD1: Continuous-time stochastic processes

Exercice 1 — *Modification and indistinguishability.*

(1) Show that two functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are rightcontinuous and coincide on a (possibly countable) dense subset of $\mathbb{R}_+$, are the same.
(2) Deduce that two continuous-time stochastic processes with real values $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ that are modifications of each other and have rightcontinuous trajectories, are actually indistinguishable.
(3) Construct a process $(X_t)_{t\geq 0}$ that is a modification of the trivial process $(Y_t)_{t\geq 0}$ defined by $Y_t = 0$, but whose trajectories are almost surely discontinuous everywhere. (Thus clearly X is distinguishable from Y).
(*Hint*: Construct first a process X that is a modification of Y but whose trajectories differ at one (necessarily random) time.)

Exercice 2 — *Scaling Property.*
Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Show that for every $c > 0$, $(N_{ct})_{t\geq 0}$ is a Poisson process with intensity $c\lambda$.

Exercice 3 — *Measurability.*
Let I be a nonempty set, we consider the space $\mathbb{R}^I$ equipped with the σ-algebra $\mathcal{B}(\mathbb{R})^{\otimes I}$. Recall that $\mathcal{B}(\mathbb{R})^{\otimes I}$ is the smallest σ–algebra such that each projection $\pi_i : \mathbb{R}^I \to \mathbb{R}$ is measurable.

(1) Build a π-system that generates $\mathcal{B}(\mathbb{R})^{\otimes I}$. Deduce that a probability measure on $(\mathbb{R}^I, B(\mathbb{R})^{\otimes I})$ is characterized by its finite dimensional marginals.
(2) Show that $(X_t)_{t\in I}$ is a random variable on $(\mathbb{R}^I, B(\mathbb{R})^{\otimes I})$ if and only if for every $t \in I$, X_t is a random variable on I.
(3) We equip $\mathbb{R}^I$ with the product topology, recall that the product topology is the coarsest topology (i.e the topology with the fewest open sets) that make the projections $\pi_i : \mathbb{R}^I \to \mathbb{R}$ continuous.
  (a) Give a basis of open sets for the product topology on $\mathbb{R}^I$ and show that the σ–algebra generated by this basis is $\mathcal{B}(\mathbb{R}^I)$.
  (b) Assume that I is countable and show that the Borel σ-algebra on $\mathbb{R}^I$ is $\mathcal{B}(\mathbb{R})^{\otimes I}$.
(4) We equip $\mathcal{C}([0, 1])$ with the topology of uniform convergence on compact sets, and we denote $\mathcal{E}$ the Borel σ-algebra associated to this topology. Show that the restriction of $\mathcal{B}(\mathbb{R})^{\otimes[0,1]}$ to $\mathcal{C}([0, 1])$ is $\mathcal{E}$.

Exercice 4 — *Composed Poisson Processes.*

Let ν be a probability measure on $\mathbb{R}^n$ and $\lambda > 0$. Let $X = (X_t)_{t\geq 0}$ be a Poisson process of intensity λ and let $(M_k)_{1\leq k\leq n}$ be a sequence of iid random variables with law ν taken independent of X. We define, the composed Poisson process of parameter $\lambda\nu$ by

$$Z_t = \sum_{k=1}^{X_t} M_k.$$

(1) Show that the increments of the process Z are stationary and independent,
(2) Let $t \geq 0$ and $\xi \in \mathbb{R}$, compute $\mathbb{E}[e^{i\xi Z_t}]$. What is the expectation and the variance of Z_t ?
(3) Let $p \in [0,1]$, assume that $\nu = (1-p)\delta_0 + p\delta_1$. Show that the processes Z and $X - Z$ are independent Poisson process of respective intensity $p\lambda$ and $(1-p)\lambda$.
(4) Deduce from the previous question that the sum of two Poisson process is also a Poisson process.