TD2: Poisson Processes

Exercice 1 — *Jump times of a Poisson process.*
Let $\lambda > 0$ consider a Poisson process X with intensity λ, let $(J_n)_n$ be the jump times of X. Let $n \geq 1$ and $t > 0$, let $U_1, \dots, U_n$ be independent uniform random variables in $[0, t]$. Let σ be the (random) permutation of $\{1, \dots, n\}$ such that,

$$U_{\sigma(1)} \leq U_{\sigma(2)} \leq \dots \leq U_{\sigma(n)}.$$

(1) Prove that σ is well-defined on a set of measure 1.
(2) Show that the random variable $(U_{\sigma(i)})_{1 \leq i \leq n}$ has density

$$d_n(u_1, \dots, u_n) = n!/t^n \mathbf{1}\{u_1 < \dots < u_n\}.$$

(3) Show that the density of $(J_1, \dots, J_n)$ conditionally on $\{X_t = n\}$ is d_n, that is for any non-negative measurable function $f : \mathbb{R}^n \to \mathbb{R}_+$, we have

$$\mathbb{E}\left[f(J_1, \dots, J_n) | X_t = n\right] = \int_{[0,t]^n} f(s_1, \dots, s_n) \frac{n!}{t^n} \prod_{i=1}^{n} ds_i.$$

(4) Deduce from the previous questions a way to sample the trajectories on $[0, t]$ of a Poisson process using only uniform random variables and Poisson random variables.

Exercice 2 — *M/GI/∞ queue.*

Let $X = (X_t)_{t \geq 0}$ be a Poisson process of intensity $\lambda > 0$, we denote $(J_n)_n$ the jump times of X. Let $(Z_n)_n$ be iid random variables, we denote G the cdf of Z_1 and $1/\mu$ the mean of Z_1. Consider the following model, you operate a restaurant in which the n^{th} customer arrives at time J_n and leaves at time $J_n + Z_n$. You want to estimate the number N_t of customers in the shop at time t. Note that for every $t \geq 0$, we have

$$N_t = \sum_n \mathbf{1}\{J_n \leq t \leq J_n + Z_n\}.$$

(1) Let X a Poisson random variable with parameter $\alpha > 0$ and $(B_n)_n$ be iid Bernoulli random variables with parameter p independent from X, show that $Y = \sum_{n=1}^{X} B_n$ is a Poisson random variable with parameter αp.
(2) Let $t \geq 0$, $n \geq 0$ and let U denote a uniform random variable in $[0, t]$, define $p = \mathbb{P}(Z_1 > U)$. Show that conditionally on $X_t = n$, the random variable N_t is Binomial random variable with parameter (n, p).
(3) Let $t > 0$ and $\alpha(t) = \lambda \int_0^t \mathbb{P}(Z_1 > x) dx$, show that N_t is a Poisson random variable with parameter $\alpha(t)$.
(4) Show that as $t \to \infty$, N_t converges in law toward a Poisson law with parameter $\rho = \lambda/\mu$.

In France approximately, 1903896 new cars have been bought each year between 1967 and 2023 (source : CCFA, Comité des Constructeurs Français d'Automobiles). Assume that the French people buy cars according to a Poisson Process with intensity $\lambda = 1903896$ per year and that there was no car bought before 1967.

(5) Assume that each car owner keeps its car for a duration uniform between 0 and 20 years. What is the expected number of cars in the French fleet in the year 1977 ? what about in the year 1987 ? and Afterward ?

(6) Answer the previous question now assuming that each owner keeps its car for an exponential duration with parameter 1/10.