HIGHER HIDA AND COLEMAN THEORY ON THE MODULAR CURVE

GEORGE BOXER AND VINCENT PILLONI

Abstract. We construct Hida and Coleman families for the degree 0 and degree 1 cohomology of automorphic line bundles on the modular curve and we define a p-adic duality pairing between the theories in degree 0 and degree 1.

1. Introduction

In the 80's, Hida introduced an ordinary projector on the space of modular forms and he constructed p-adic families of ordinary modular forms ([Hid86]). In the 90's, Coleman developed the finite slope theory ([Col97]) and Coleman and Mazur constructed the eigencurve ([CM98]). These theories have now been extended to higher dimensional Shimura varieties.

Hida and Coleman theories combine two ideas. The first one is to restrict modular forms from the full modular curve to the ordinary locus and its neighborhoods. The additional structure on the universal p-divisible group on and near the ordinary locus (the canonical subgroups) is used to p-adically interpolate the sheaves of modular forms, and their cohomology. The second idea is to use the Hecke operators at p to detect when a section of the sheaf of modular forms defined on (a neighborhood of) the ordinary locus comes from a classical modular form.

Until recently, Hida and Coleman theories had only been used for degree 0 coherent cohomology groups. In the recent works [Pil17], [BCGP18] we began to develop them further in order to study higher coherent cohomology of vector bundle on the Shimura varieties for the group GSp_4, and we are now convinced that Hida and Coleman theories should exist in all cohomological degrees for any Shimura variety.

The purpose of the current work is to confirm this prediction in the simple setting of modular curves and to construct Hida and Coleman theories for the degree 1 cohomology groups. We actually construct in parallel the theories for degree 0 and degree 1 cohomology, as this sheds some new light on the usual degree 0 theory. We also prove a p-adic Serre duality, which gives a perfect pairing between the theories in cohomological degree 0 and 1, but our constructions are independent of this pairing.

Let us describe the results we prove. Let $X \to \operatorname{Spec} \mathbb{Z}_p$ be the compactified modular curve of level $\Gamma_1(N)$, where $N \geq 3$ is an integer prime to p, and let D be the boundary divisor. Let $X_1 \to \operatorname{Spec} \mathbb{F}_p$ be the special fiber and X_1^{ord} be the ordinary locus.

Theorem 1.1 (Hida's control theorem)**.** *There is a Hecke operator T_p acting on the cohomology groups $\mathrm{R}\Gamma(X_1,\omega^k)$, $\mathrm{R}\Gamma_c(X_1^{ord},\omega^k)$ and $\mathrm{R}\Gamma(X_1^{ord},\omega^k)$, and an associated ordinary projector $e(T_p)$. Moreover, we have quasi-isomorphisms*

$$e(T_p)\mathrm{R}\Gamma(X_1,\omega^k) \to e(T_p)\mathrm{R}\Gamma(X_1^{ord},\omega^k) \text{ if } k \geq 3$$

and

$$e(T_p)\mathrm{R}\Gamma_c(X_1^{ord},\omega^k) \to e(T_p)\mathrm{R}\Gamma(X_1,\omega^k) \text{ if } k \leq -1.$$

The proof of this theorem relies on a local analysis of the cohomological correspondence T_p at supersingular points. The above theorem implies also a vanishing theorem: $e(T_p)\mathrm{R}\Gamma(X_1,\omega^k)$ is concentrated in degree 0 if $k \geq 3$, and degree 1 if $k \leq -1$, because the ordinary locus is affine. Of course, this vanishing theorem holds true for the entire cohomology from the Riemann-Roch theorem and the Kodaira-Spencer isomorphism, but the argument above is independent and generalizes.

Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwaswa algebra. Each integer $k \in \mathbb{Z}$ defines a character of $\mathbb{Z}_p^\times$, and a morphism $k : \Lambda \to \mathbb{Z}_p$.

Theorem 1.2. *There are two projective Λ-modules M and N carrying an action of the Hecke algebra of level prime to p, and there are canonical, Hecke-equivariant isomorphisms for all $k \geq 3$:*

1. $M \otimes_{\Lambda,k} \mathbb{Z}_p = e(T_p)\mathrm{H}^0(X,\omega^k)$,
2. $N \otimes_{\Lambda,k} \mathbb{Z}_p = e(T_p)\mathrm{H}^1(X,\omega^{2-k}(-D))$,

Moreover, there is a perfect pairing $M \times N \to \Lambda$ which interpolates the classical Serre duality pairing.

The modules M and N are obtained by considering the ordinary factor of the cohomology and cohomology with compact support of the ordinary locus with value in an interpolation sheaf of Λ-modules.

Let $X_0(p) \to \mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ be the adic modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$. We have two quasi-compact opens $X_0(p)^m$ and $X_0(p)^{et}$ inside $X_0(p)$ which are respectively the loci where the universal subgroup of order p has multiplicative and étale reduction. We let $X_0(p)^{m,\dagger}$ and $X_0(p)^{et,\dagger}$ be the corresponding dagger spaces (these are the inductive limit over all strict neighborhoods of $X_0(p)^m$ and $X_0(p)^{et}$).

Theorem 1.3 (Coleman's classicality theorem)**.** *For all $k \in \mathbb{Z}$, there is a well defined Hecke operator U_p which is compact and has non negative slopes on $\mathrm{H}^i(X_0(p),\omega^k)$, $\mathrm{H}^i(X_0(p)^{m,\dagger},\omega^k)$ and $\mathrm{H}^i_c(X_0(p)^{et,\dagger},\omega^k)$. Moreover, the natural maps (where the superscript $< \star$ means slope less than $\star$ for U_p) :*

1. $\mathrm{H}^i(X_0(p),\omega^k)^{<k-1} \to \mathrm{H}^i(X_0(p)^{m,\dagger},\omega^k)^{<k-1}$,
2. $\mathrm{H}^i_c(X_0(p)^{et,\dagger},\omega^k)^{<1-k} \to \mathrm{H}^i(X_0(p),\omega^k)^{<1-k}$

are isomorphisms.

The proof of this theorem is based on some simple estimates for the operator U_p on the ordinary locus, reminiscent of [Kas06]. We can again derive from this theorem a vanishing theorem for the small slope classical cohomology (without appealing to Riemann-Roch theorem).

Coleman and Mazur constructed the eigencurve $\mathcal{C}$ of tame level $\Gamma_1(N)$. It carries a weight morphism $w : \mathcal{C} \to \mathcal{W}$ where $\mathcal{W}$ is the analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ associated with the Iwasawa algebra Λ.

Theorem 1.4. *The eigencurve carries two coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ interpolating the degree 0 and 1 finite slope cohomology. For any $k \in \mathbb{Z}$, we have*

(1) $\mathcal{M}_k^{<k-1} = \mathrm{H}^0(X_0(p), \omega^k)^{<k-1}$,
(2) $\mathcal{N}_k^{<k-1} = \mathrm{H}^1(X_0(p), \omega^{2-k}(-D))^{<k-1}$,

and a there is a perfect pairing between $\mathcal{M}$ and $\mathcal{N}$, interpolating the usual Serre duality pairing.

1.1. **Aknowledgements.** We would like to thank David Loeffler for his comments. This research is supported by the ERC-2018-COG-818856-HiCoShiVa.

2. Preliminaries

2.1. Finite flat cohomological correspondences.

2.1.1. *The functors $f_\star$, $f^\star$ and $f^!$.* Let $f : X \to Y$ be a finite flat morphism of schemes. We have a functor $f_\star : Coh(X) \to Coh(Y)$. It induces an equivalence of categories between $Coh(X)$ and the category of coherent sheaves of $f_\star\mathscr{O}_X$-modules over Y.

The functor $f_\star$ has a left adjoint $f^\star : Coh(Y) \to Coh(X)$, given by $\mathscr{F} \mapsto \mathscr{F} \otimes_{\mathscr{O}_Y} f_\star\mathscr{O}_X$, as well as a right adjoint $f^! : Coh(Y) \to Coh(X)$ defined by $f^!\mathscr{F} = \underline{\mathrm{Hom}}_{\mathscr{O}_Y}(f_\star\mathscr{O}_X, \mathscr{F})$. For any $\mathscr{F} \in Coh(X)$, we have an isomorphism $f^!\mathscr{F} = f^!\mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{F}$.

A finite flat morphism $f : X \to Y$ has a trace map $\mathrm{tr}_f : f_\star\mathscr{O}_X \to \mathscr{O}_Y$. This trace is by definition a global section of $f^!\mathscr{O}_Y$ or equivalently a morphism $f^\star\mathscr{O}_Y \to f^!\mathscr{O}_X$. It follows that the trace map provides a natural transformation $f^\star \Rightarrow f^!$.

A finite flat morphism $f : X \to Y$ is called Gorenstein if $f^!\mathscr{O}_X$ is an invertible sheaf. If $f : X \to Y$ if a local complete intersection morphism, then it is Gorenstein (see [Eis95], coro. 21.19).

2.1.2. *Cohomological correspondences.*

Definition 2.1. *A finite flat correspondence over a scheme X is a scheme C equipped with two finite flat morphisms $X \overset{p_2}{\leftarrow} C \overset{p_1}{\rightarrow} X$.*

Definition 2.2. *Let $\mathscr{F}$ be a coherent sheaf on X. A finite flat cohomological correspondence for $\mathscr{F}$ is the data of a pair (C, T) consisting of a finite flat correspondence C and a map $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$.*

Given a finite flat cohomological correspondence (C, T) on $\mathscr{F}$, we get a morphism in cohomology that we also denote by T,

$$T : \mathrm{R}\Gamma(X, \mathscr{F}) \to \mathrm{R}\Gamma(C, p_2^\star\mathscr{F}) \xrightarrow{T} \mathrm{R}\Gamma(C, p_1^!\mathscr{F}) \to \mathrm{R}\Gamma(X, \mathscr{F}).$$

2.1.3. *Composition.* It is possible to compose finite flat cohomological correspondences. Assume we have a diagram of finite flat correspondences C and D:

$$\begin{tikzcd} & C \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & & D \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ X & & X & & X \end{tikzcd}$$

We can form $E = C \times_{p_1, X, q_2} D$:

$$\begin{tikzcd} & & E \arrow[dl, "t_2"'] \arrow[dr, "t_1"] & & \\ & C \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & & D \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ X & & X & & X \end{tikzcd}$$

If we are given $p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ and $q_2^\star \mathscr{F} \to q_1^! \mathscr{F}$, we can consider the following maps :

$$t_2^\star p_2^\star \mathscr{F} \to t_2^\star p_1^! \mathscr{F} \text{ and } t_1^! q_2^\star \mathscr{F} \to t_1^! q_1^! \mathscr{F}$$

moreover we claim that there is a natural transformation $t_2^\star p_1^! \Rightarrow t_1^! q_2^\star$ so that we can compose and get the cohomological correspondence :

$$t_2^\star p_2^\star \mathscr{F} \to t_1^! q_1^! \mathscr{F}.$$

It remains to check the claim. Equivalently (by adjunction) it suffices to find a natural transformation $(t_1)_\star t_2^\star p_1^! \Rightarrow t_1^! q_2^\star$. We have a base change isomorphism $(t_1)_\star t_2^\star = (q_2)^\star (p_1)_\star$, and by adjunction $(p_1)_\star p_1^! \Rightarrow \mathrm{Id}$, there is a natural transformation $(q_2)^\star (p_1)_\star p_1^! \Rightarrow q_2^\star$.

2.1.4. *Restriction.* Let X is a scheme and $Y \hookrightarrow X$ is a closed subscheme defined by an ideal $\mathscr{I} \hookrightarrow \mathscr{O}_X$. For any quasi-coherent sheaf $\mathscr{F}$ over X, we let

$$\underline{\Gamma}_Y(\mathscr{F}) = \mathrm{Ker}(\mathscr{F} \to \underline{\mathrm{Hom}}(\mathscr{I}, \mathscr{F}))$$

be the subsheaf of sections with support in Y.

Proposition 2.1. *Consider a commutative diagram of schemes :*

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Z \end{tikzcd}$$

where i is a closed immersion and g, f are finite flat morphisms. Let $\mathscr{F}$ be a coherent sheaf on Z. Then $i_\star g^! \mathscr{F} = \underline{\Gamma}_Y f^! \mathscr{F}$.

Proof. Let us denote by $\mathscr{I}$ the ideal sheaf of Y in X. This follows from the exact sequence :

$$0 \to \underline{\mathrm{Hom}}(g_\star \mathscr{O}_Y, \mathscr{F}) \to \underline{\mathrm{Hom}}(f_\star \mathscr{O}_X, \mathscr{F}) \to \underline{\mathrm{Hom}}(f_\star \mathscr{I}, \mathscr{F}).$$

□

2.2. **Local finiteness.** Let R be an artinian local ring with finite residue field.

Lemma 2.1. *Suppose M is a finite type R-module and $T \in \mathrm{End}_R(M)$ is an endomorphism of M. The sequence $(T^{n!})_{n\in\mathbb{N}} \in (\mathrm{End}_R(M))^{\mathbb{N}}$ is eventually constant and converges to an idempotent denoted $e(T) \in \mathrm{End}_R(M)$.*

Proof. We have a decreasing sequence of sub-modules $T^{n!}(M)$ of M which is eventually stationary since M is artinian. Let M_0 be the limit. For all n large enough $T^{n!}$ induces a permutation of the finite set M_0, and therefore, for all N large enough, this permutation is trivial. It follows that for all n large enough, $T^{n!}$ is a projector from M to M_0. □

Definition 2.3. *Let M be an R-module and let $T \in \mathrm{End}_R(M)$. We say that T is locally finite on M if M is a union of finite type R-modules which are stable under T.*

Remark 2.1. This condition is equivalent to the claim that for any finite type submodule $V \subseteq M$, $\sum_{n\geq 0} T^n V$ is a finite type submodule. It is also equivalent to the claim that for some $n \geq 1$, $T^n V \subseteq \sum_{i=0}^{n-1} T^i V$.

Proposition 2.2. *If M is an R-module and T is locally finite on M, there is an idempotent $e(T)$ attached to T such that T is an isomorphism on $e(T)M$ and T is locally nilpotent on $(1-e(T))M$.*

By locally nilpotent, we mean that for each $m \in (1-e(T))M$, there exists N such that $T^N m = 0$.

Proof. This follows from lemma 2.1. □

Lemma 2.2. *If $f : M \to N$ is an R-linear morphism and T is an R-linear operator acting equivariantly and locally finitely on M and N, then $f(e(T)M) \subseteq e(T)N$ and $f((1-e(T))M) \subseteq (1-e(T))N$.*

Proposition 2.3. *Let $0 \to M \to N \to L \to 0$ be a short exact sequence of R-modules and let T be an R-linear operator acting equivariantly on M, N and L.*

1. *T is locally finite on N if and only if T is locally finite on M and L.*
2. *If T is locally finite on M, N and L, the sequence : $0 \to e(T)M \to e(T)N \to e(T)L \to 0$ is exact.*

Proof. We prove the first point. The direct implication is trivial. Let us prove the reverse implication. Let $V \subset N$ be a finite submodule. Since T is locally finite on L, we deduce that there exists $n \geq 1$ and a finite submodule $W \subset M$ such that :

$$T^n(V) \subseteq \sum_{i=0}^{n-1} T^i V + W.$$

Since T is locally finite on M, there exists $m \geq 1$ such that :

$$T^m(W) \subseteq \sum_{i=0}^{m-1} T^i W.$$

We finally conclude that $T^{n+m}(V+W) \subseteq \sum_{i=0}^{n+m-1} T^i(V+W)$, so that $\sum_{i=0}^{n+m-1} T^i(V+W)$ is a finite R-module, stable by T, containing V. The second point follows easily. □

We will actually need to work with certain topological R-modules that we call profinite R-modules and are in a certain sense duals of (discrete) R-modules. A profinite R-module M is a topological R-module which is homeomorphic to a projective limit $\lim_{i\in I} M_i$ of finite R-modules (I is a filtered ordered set).

In other words, a topological R-module M is a profinite R-module if it is separated and complete and if there is a basis of neighborhoods of $0 \in M$, $\{N_i\}_{i\in I}$ consisting of cofinite submodules. In such a case, we have $M = \lim_i M/N_i$.

Definition 2.4. *Let M be a profinite R-module, and let $T \in \mathrm{End}_R(M)$ be a continuous endomorphism. We say that T is locally finite on M if M has a basis of neighborhoods of 0 consisting of submodules $\{N_i\}$ such that $T(N_i) \subseteq N_i$.*

Remark 2.2. We observe that the first condition in the definition is equivalent to the condition that for any open submodule $V \subseteq M$, the submodule $\cap_{n\geq 0} T^{-n}V$ is open. Since $M/\cap_{n\geq 0}T^{-n}V$ if a finite module and therefore an artinian module, we deduce that the condition is also equivalent to the condition that $\cap_{i=0}^{n-1} T^{-i}V \subseteq T^{-n}V$ for some $N \geq 1$.

Proposition 2.4. *If M is a profinite R-module and T is locally finite on M, there is an idempotent $e(T)$ attached to T such that T is an isomorphism on $e(T)M$ and T is topologically nilpotent on $(1-e(T))M$.*

By topologically nilpotent, we mean that for each $m \in (1-e(T))M$, $T^N m \to 0$ when $N \to +\infty$.

Proof. This follows from lemma 2.1. □

Lemma 2.3. *If $f : M \to N$ is a continuous R-linear morphism and T is a locally finite operator acting equivariantly on M and N, then $f(e(T)M) \subseteq e(T)N$ and $f((1-e(T))M) \subseteq (1-e(T))N$.*

We say that an exact sequence of profinite R-modules $0 \to M \stackrel{d}{\to} N \stackrel{s}{\to} L \to 0$ is strict if s is is open. We remark that this forces s to be continuous, and M to be a closed subspace of N (so that d is continuous and closed).

Proposition 2.5. *Let $0 \to M \to N \to L \to 0$ be a strict short exact sequence of profinite R-modules and let T be a continuous R-linear operator acting equivariantly on M, N and L.*

1. *T is locally finite on N if and only if T is locally finite on M and L.*
2. *If T is locally finite on M, N and L, the sequence : $0 \to e(T)M \to e(T)N \to e(T)L \to 0$ is exact.*

Proof. We prove the first point. The direct implication is trivial. Let us prove the reverse implication. Let $V \subset N$ be an open submodule. Since T is locally finite on M, we deduce that there is $n \geq 1$ such that $\cap_{i=0}^{n-1} T^{-i}(V \cap M) \subseteq T^{-n}(V \cap M)$. It follows that :

$$(T^{-n}V + M) \bigcap \cap_{i=0}^{n-1} T^{-i}(V) \subseteq T^{-n}(V).$$

To shorten notations, let us denote by $W = T^{-n}V + M$. Since W is open in N, its image $\overline{W}$ in L is open. Since T is locally finite on L, there is $m \geq 0$ such that $\cap_{i=0}^{m-1} T^{-i}(\overline{W}) \subset T^{-m}(\overline{W})$. We deduce that $\cap_{i=0}^{m-1} T^{-i}(W) \subset T^{-m}(W)$. It follows that :

$$\begin{aligned}
T^{-(n+m)}(V \cap W) &= T^{-m}(T^{-n}V) \cap T^{-(n+m)}(W) \\
&\supseteq T^{-m}(W \bigcap \cap_{i=0}^{n-1} T^{-i}(V)) \bigcap \cap_{i=0}^{m-1} T^{-i}W \\
&\supseteq T^{-m}(W) \bigcap \cap_{i=0}^{m+n-1} T^{-i}(V) \bigcap \cap_{i=0}^{m-1} T^{-i}W \\
&\supseteq \cap_{i=0}^{m+n-1} T^{-i}(V) \bigcap \cap_{i=0}^{m-1} T^{-i}W \\
&\supseteq \cap_{i=0}^{m+n-1} T^{-i}(V \cap W)
\end{aligned}$$

Therefore, $\cap_{i=0}^{m+n-1} T^{-i}(V \cap W)$ is an open submodule of V which is stable under T. The rest of the properties are easy. □

3. The cohomological correspondence T_p

Let N be an integer $N \geq 3$. Let p be a prime number. Let $X \to \operatorname{Spec} \mathbb{Z}_p$ be the compactified modular curve of level $\Gamma_1(N)$ ([DR73]). This is a proper smooth relative curve. Denote by D the boundary divisor, and by $E \to X$ the semi-abelian scheme which extends the universal elliptic curve and denote by e the unit section. We let $\omega_E = e^\star \Omega^1_{E/X}$. For any $k \in \mathbb{Z}$, we denote by $\omega^k = \omega_E^{\otimes k}$.

3.1. **The cohomological correspondences T_p.** We denote by $p_1, p_2 : X_0(p) \to X$ the Hecke correspondence which parametrizes an isogeny $\pi : p_1^\star E \to p_2^\star E$ of degree p. We denote by $D_0(p)$ the boundary divisor in $X_0(p)$ (which is reduced, so that $D_0(p) = (p_i^{-1}D)^{red}$). We let $\pi_k : p_2^\star \omega^k \dashrightarrow p_1^\star \omega^k$ the rational map which is deduced from the pull-back map on differentials $p_2^\star \omega_E \to p_1^\star \omega_E$ (this map is well defined if $k \geq 0$, and is an isomorphism over $\mathbb{Q}_p$ for all k). We also denote by π_k^{-1} : $p_1^\star \omega^k \dashrightarrow p_2^\star \omega^k$ the inverse of π_k. We also have a dual isogeny $\pi^\vee : p_2^\star E \to p_1^\star E$ and we denote by $\pi_k^\vee : p_1^\star \omega^k \dashrightarrow p_2^\star \omega^k$ the rational map which is deduced from the pull-back map on differentials $p_1^\star \omega_E \to p_2^\star \omega_E$. We also denote by $(\pi_k^\vee)^{-1} : p_1^\star \omega^k \dashrightarrow p_2^\star \omega^k$ the inverse of $\pi_k^\vee$. We have the following formula relating π_k and $\pi_k^\vee$:

$$\pi_k \circ \pi_k^\vee = p^k \mathrm{Id}.$$

We have natural trace maps $\mathrm{tr}_{p_1} : \mathscr{O}_{X_0(p)} \to p_1^! \mathscr{O}_X$ and $\mathrm{tr}_{p_2} : \mathscr{O}_{X_0(p)} \to p_2^! \mathscr{O}_X$. Note that since p_1 and p_2 are local complete intersection morphisms, $p_1^! \mathscr{O}_X$ and $p_2^! \mathscr{O}_X$ are invertible sheaves. We can restrict the map tr_{p_i} to $\mathscr{O}_{X_0(\ell)}(-D_0(p))$ and :

Lemma 3.1. *We have factorizations :* $\mathrm{tr}_{p_1} : \mathscr{O}_{X_0(p)}(-D_0(p)) \to p_1^!(\mathscr{O}_X(-D))$ *and* $\mathrm{tr}_{p_2} : \mathscr{O}_{X_0(p)}(-D_0(p)) \to p_2^!(\mathscr{O}_X(-D))$.

Proof. The boundary divisors in X and $X_0(p)$ are reduced. The lemma boils down to the statement that the trace of function vanishing along the boundary on $X_0(p)$ vanishes on the boundary on X. □

We have a "naive" cohomological correspondence :

$$T_{p,k}^{naive} : p_2^\star \omega^k \dashrightarrow p_1^! \omega^k$$

which is defined by taking the tensor product of the map $p_2^\star \omega^k \dashrightarrow p_1^\star \omega^k$ and the map $\mathrm{tr}_{p_1} : \mathscr{O}_{X_0(\ell)} \to p_1^! \mathscr{O}_X$, and similarly a map $T_{p,k}^{naive} : p_2^\star(\omega^k(-D)) \dashrightarrow p_1^!(\omega^k(-D))$. We finally denote by $T_{p,k} = p^{-\inf\{1,k\}} T_{p,k}^{naive}$.

Proposition 3.1. *$T_{p,k}$ is a cohomological correspondence : $p_2^\star \omega^k \to p_1^! \omega^k$.*

Proof. The map $T_{p,k}$ is a rational map between invertible sheaves over the regular scheme $X_0(p)$. To check that it is well defined, it is enough to work in codimension 1. Since the map is well defined over $\mathbb{Q}_p$, we can thus localize at a generic points ξ of the special fiber and we are left to prove that it is well defined locally at these points. There are two types of generic points corresponding for the possibility of the isogeny $p_1^\star E \to p_2^\star E$ to be either multiplicative or étale. Let us first assume that ξ is étale. The differential map $(p_2^\star \omega)_\xi \tilde{\to} (p_1^\star \omega)_\xi$ is an isomorphism and the map $(tr_{p_1})_\xi : (p_1^\star \mathscr{O}_X)_\xi \to (p_1^! \mathscr{O}_X)_\xi$ factors into an isomorphism: $(tr_{p_1})_\xi : (p_1^\star \mathscr{O}_X)_\xi \tilde{\to} p(p_1^! \mathscr{O}_X)_\xi$. It follows that $(T_{p,k}^{naive})_\xi : (p_2^\star \omega^k)_\xi \tilde{\to} p(p_1^! \omega^k)_\xi$. Let us next assume that ξ is multiplicative. The differential map $(p_2^\star \omega)_\xi \tilde{\to} (p_1^\star \omega)_\xi$ factors into an isomorphism $(p_2^\star \omega)_\xi \tilde{\to} p(p_1^! \omega)_\xi$ and the map $(tr_{p_1})_\xi : (p_1^\star \mathscr{O}_X)_\xi \tilde{\to} (p_1^! \mathscr{O}_X)_\xi$ is an isomorphism. It follows that $(T^{naive}_{p,k})_\xi : (p_2^\star \omega^k)_\xi \tilde{\to} p^k (p_1^! \omega^k)_\xi$. We deduce that $T_{p,k}$ is indeed a well defined map and that it is optimally integral. □

When the weight is clear, we often write T_p. The map T_p induces a map on cohomology :

$$T_p \in \mathrm{End}\mathrm{R}\Gamma(X, \omega^k) \text{ and } \mathrm{End}\mathrm{R}\Gamma(X, \omega^k(-D))$$

obtained by composing the maps :

$$\mathrm{R}\Gamma(X, \omega^k) \overset{p_2^\star}{\to} \mathrm{R}\Gamma(X_0(p), p_2^\star\omega^k) \overset{T_p}{\to} \mathrm{R}\Gamma(X_0(p), p_1^!\omega^k) \overset{\mathrm{tr}_{p_1}}{\to} \mathrm{R}\Gamma(X, \omega^k)$$

and similarly for cuspidal cohomology.

Remark 3.1. The proposition 3.1 is a particular instance of constructions performed in [FP19], where the problem of constructing Hecke operators on the integral coherent cohomology of more general Shimura varieties is considered.

Remark 3.2. One can check that our map $T_{p,k}$ has the following effect on q-expansions (of given Nebentypus $\chi : \mathbb{Z}/N\mathbb{Z}^\times \to \overline{\mathbb{Z}}_p^\times$) : it maps $\sum a_n q^n$ to $\sum a_{np} q^n + p^{k-1}\chi(p) \sum a_n q^{np}$ if $k \geq 1$ and to $p^{1-k}\sum a_{np}q^n + \chi(p)\sum a_n q^{np}$ if $k \leq 1$.

Remark 3.3. Our normalization is consistant with standard conjectures on the existence and properties of Galois representations associated to automorphic forms ([BG14]). The cohomology groups $\mathrm{H}^i(X, \omega^k) \otimes \mathbb{C}$ can be computed using automorphic forms and for any $\pi = \pi_\infty \otimes \pi_f$ contributing to the cohomology, we find that the infinitesimal character of π_∞ is given by $(t_1, t_2) \mapsto t_1^{\frac{1}{2}} t_2^{k-\frac{1}{2}}$ and is indeed C-algebraic. By the Satake isomorphism, we know that $T_p^{naive}|\pi_p = p^{1/2}\mathrm{Trace}(\mathrm{Frob}_p^{-1}|\mathrm{rec}(\pi_p))$ (because T_p^{naive} corresponds to the co-character $t \mapsto (1, t^{-1})$ [FP19], rem. 5.6 !). It is convenient to introduce the twist $\pi \otimes |.|^{-\frac{1}{2}}$ which is L-algebraic, for which we find that the infinitesimal character of $\pi_\infty \otimes |.|^{-\frac{1}{2}}$ is $(t_1, t_2) \mapsto t_2^{k-1}$. We make the following normalizations : the Hodge-Tate weight of the cyclotomic character is -1, and we normalize the reciprocity law by using geometric Frobenii. With these conventions, the Hodge cocharacter is $t \mapsto (1, t^{1-k})$ and the corresponding Hodge polygon has slopes $1-k$ and 0. We find that $T_p^{naive}|\pi_p = p\mathrm{Trace}\big(\mathrm{Frob}_p^{-1}|\mathrm{rec}(\pi_p \otimes |.|^{-\frac{1}{2}})\big)$. The Katz-Mazur inequality predicts that the Newton polygon (which has slopes the p-adic valuations of two eigenvalues of Frob_p) is above the Hodge polygon with same ending and inital point, from which we find that $v(T_p^{naive}|\pi_p) \geq \inf\{1, k\}$ and that T_p is indeed optimally integral.

3.2. **Duality.** We let $D_{\mathbb{Z}_p} = \mathrm{RHom}(\cdot, \mathbb{Z}_p)$ be the dualizing functor on the category of bounded complexes of finite type $\mathbb{Z}_p$-modules ([Har66], chap V). We denote by $f : X \to \mathbb{Z}_p$, and by $f^!\mathbb{Z}_p = \omega_{X/\mathbb{Z}_p}$. This is a dualizing complex on X ([Har66], chap V, thm. 8.3) . We recall that $\omega_{X/\mathbb{Z}_p} = \Omega^1_{X/\mathbb{Z}_p}[1]$ (see [Har66], chap. III, sect. 8). We denote by $g : X_0(p) \to \mathbb{Z}_p$, and by $\omega_{X_0(p)/\mathbb{Z}_p} = g^!\mathbb{Z}_p = p_1^!\omega_{X/\mathbb{Z}_p}$. This is also a dualizing complex on $X_0(p)$. We also recall that $\omega_{X_0(p)/\mathbb{Z}_p} = \Omega^1_{X_0(p)/\mathbb{Z}_p}(\log(SS))[1]$ where SS is the reduced closed subscheme of supersingular points in $X_0(p)$. We let $D_X = \mathrm{RHom}(\cdot, \omega_{X/\mathbb{Z}_p})$ and $D_{X_0(p)} = \mathrm{RHom}(\cdot, \omega_{X_0(p)/\mathbb{Z}_p})$ the corresponding dualizing functors on the derived category of (say) bounded complexes of coherent sheaves on X and $X_0(p)$. When the context is clear we only write D for the dualizing functor.

We have the following Serre duality isomorphism ([Har66], chap III, thm. 11.1):

$$D(\mathrm{R}f_\star\omega^k) = \mathrm{R}f_\star D(\omega^k)$$

and similarly for cuspidal cohomology.

We now want to understand how this duality isomorphism behaves with respect to Hecke operators. The Hecke operator

$$\mathrm{R}\Gamma(X,\omega^k) \stackrel{p_2^\star}{\to} \mathrm{R}\Gamma(X_0(p), p_2^\star\omega^k) \stackrel{T_p}{\to} \mathrm{R}\Gamma(X_0(p), p_1^!\omega^k) \stackrel{\mathrm{tr}_{p_1}}{\to} \mathrm{R}\Gamma(X,\omega^k).$$

dualizes to an operator :

$$D(\mathrm{R}\Gamma(X,\omega^k)) \stackrel{p_1^\star}{\to} D(\mathrm{R}\Gamma(X_0(p), p_1^!\omega^k)) \stackrel{D(T_p)}{\to} D(\mathrm{R}\Gamma(X_0(p), p_2^\star\omega^k)) \stackrel{\mathrm{tr}_{p_2}}{\to} D(\mathrm{R}\Gamma(X,\omega^k)).$$

We have

$$D(\mathrm{R}\Gamma(X_0(p), p_1^!\omega^k)) = \mathrm{R}\Gamma((X_0(p), p_1^\star D(\omega^k)),$$

$$D(\mathrm{R}\Gamma(X_0(p), p_2^\star\omega^k)) = \mathrm{R}\Gamma((X_0(p), p_2^! D(\omega^k)),$$

by [Har66], chap. III, thm 11.1 and chap V, prop. 8.5, and it remains to understand what is $D(T_p) : p_1^\star D(\omega^k) \to p_2^! D(\omega^k)$. We first recall that we have the Kodaira-Spencer isomorphism over X ([Kat73], A.1.3.17) :

$$KS : \omega^2(-D) \to \Omega^1_{X/\mathbb{Z}_p}.$$

We consider the correspondence $T_p : p_2^\star\omega^k \to p_1^!\omega^k$. Applying the functor $D_{X_0(p)}$ yields a map : $D(T_p) : p_1^\star(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p}) \to p_2^!(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p})$. If we use the Kodaira-Spencer isomorphism on both sides and make a $[-1]$-shift, we obtain a map : $D(T_p) : p_1^\star(\omega^{-k+2}(-D)) \to p_2^!(\omega^{-k+2}(-D))$. The correspondence $X_0(p)$ is isomorphic to its transpose, by the automorphism sending the isogeny $p_1^\star E \to p_2^\star E$ to the dual isogeny. We can therefore think of $D(T_p)$ as a map $p_2^\star(\omega^{-k+2}(-D)) \to p_1^!(\omega^{-k+2}(-D))$ by applying this isomorphism. The main result of this section is the following :

Proposition 3.2. $D(T_p) = T_p$.

Lemma 3.2. *The following diagram is commutative :*

$$\begin{tikzcd}
& \Omega^1_{X_0(p)/\mathbb{Z}_p}(\log(SS+D)) & \\
p_2^\star\Omega^1_{X/\mathbb{Z}_p}(logD) \arrow[ur, "\mathrm{tr}_{p_2}"] & & p_2^\star\Omega^1_{X/\mathbb{Z}_p}(\log D) \arrow[ul, "\mathrm{tr}_{p_1}"'] \\
p_2^\star\omega^2 \arrow[u, "p_2^\star KS"] \arrow[rr, "p^{-1}\pi_2"] & & p_1^\star\omega^2 \arrow[u, "p_1^\star KS"']
\end{tikzcd}$$

Proof. Over $X_0(p)$ we have a map $\pi^\star : p_2^\star(\mathcal{H}^1_{dR}(E/X),\nabla) \to p_1^\star(\mathcal{H}^1_{dR}(E/X),\nabla)$ which induces a commutative diagram :

$$\begin{tikzcd}
\Omega^1_{X_0(p)/\mathbb{Z}_p}(\log(SS+D)) \otimes p_2^\star\omega^{-1} \arrow[r, "1\otimes(\pi_{-1}^\vee)^{-1}"] & \Omega^1_{X_0(p)/\mathbb{Z}_p}(\log(SS+D)) \otimes p_1^\star\omega^{-1} \\
p_2^\star\Omega^1_{X/\mathbb{Z}_p}(\log D) \otimes p_2^\star\omega^{-1} \arrow[u, "\mathrm{tr}_{p_2}"] & p_1^\star\Omega^1_{X/\mathbb{Z}_p}(\log D) \otimes p_1^\star\omega^{-1} \arrow[u, "\mathrm{tr}_{p_1}"'] \\
p_2^\star\omega \arrow[u, "p_2^\star KS"] \arrow[r, "\pi_1"] & p_1^\star\omega^2 \arrow[u, "p_1^\star KS"']
\end{tikzcd}$$

or equivalently

$$\begin{tikzcd}
& \Omega^1_{X_0(p)/\mathbb{Z}_p}(\log(SS+D)) & \\
p_2^\star\Omega^1_{X/\mathbb{Z}_p}(logD) \arrow[ur, "\mathrm{tr}_{p_2}"] & & p_2^\star\Omega^1_{X/\mathbb{Z}_p}(\log D) \arrow[ul, "\mathrm{tr}_{p_1}"'] \\
p_2^\star\omega^2 \arrow[u, "p_2^\star KS"] \arrow[rr, "\pi_1\otimes(\pi_1^\vee)^{-1}"] & & p_1^\star\omega^2 \arrow[u, "p_1^\star KS"']
\end{tikzcd}$$

It remains to observe that $\pi_1^\vee\pi_1 = p$.

$\square$

Lemma 3.3. $D(T_p^{naive}) = p^{k-1}T_p^{naive}$.

Proof. The dual of the map $p_2^\star\omega^k \dashrightarrow p_1^\star\omega^k \to p_1^!\omega^k$ writes $p_1^\star(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p}) \to p_1^!(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p}) \dashrightarrow p_2^!(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p})$. The first map $p_1^\star\omega^{-k}\otimes p_1^\star\omega_{X/\mathbb{Z}_p} \to p_1^\star\omega^{-k}\otimes\omega_{X_0(p)/\mathbb{Z}_p}$ is just $1\otimes\mathrm{tr}_{p_1}$.

The second map $p_1^!(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p}) \to p_2^!(\omega^{-k}\otimes\omega_{X/\mathbb{Z}_p})$ writes also $p_1^\star\omega^{-k}\otimes\omega_{X_0(p)/\mathbb{Z}_p} \to p_2^\star\omega^{-k}\otimes\omega_{X_0(p)/\mathbb{Z}_p}$ and is $\pi_{-k}^{-1}\otimes 1$.

On the other hand, $\omega_{X_0(p)/\mathbb{Z}_p} = p_1^!\mathscr{O}_X\otimes p_1^\star\omega_{X/\mathbb{Z}_p} = p_1^!\mathscr{O}_X\otimes p_1^\star\omega_2(-D)[1]$ using Kodaira-Spencer and similarly $\omega_{X_0(p)/\mathbb{Z}_p} = p_2^!\mathscr{O}_X\otimes p_2^\star\omega_{X/\mathbb{Z}_p} = p_2^!\mathscr{O}_X\otimes p_2^\star\omega_2(-D)[1]$ using again Kodaira-Spencer. The identity map : $p_1^!\mathscr{O}_X\otimes p_1^\star\omega_2(-D) \to p_2^!\mathscr{O}_X\otimes p_2^\star\omega_2(-D)$ decomposes into :

$$\mathrm{tr}_{p_2}\mathrm{tr}_{p_1}^{-1}\otimes(\pi_1^\vee\otimes(\pi_1)^{-1})$$

according to lemma 3.2. We get that $D(T_\ell^{naive}) : p_1^\star(\omega^{-k+2}(-D)) \dashrightarrow p_2^!(\omega^{-k+2}(-D))$ is $\mathrm{tr}_{p_2}\mathrm{tr}_{p_1}^{-1}\circ\mathrm{tr}_{p_1}\otimes(\pi_1^\vee\otimes(\pi_1)^{-1})\otimes(\pi_{-k})^{-1}$. It remains to observe that $\pi_{1-k}\pi_{1-k}^\vee = p^{1-k}$. $\square$

Corollary 3.1. $D(T_p) = T_p$.

Proof. This follows from the identity :

$$-\inf\{1,k\}+k-1 = -\inf\{1,2-k\}.$$

$\square$

4. Higher Hida theory

4.1. **The mod p theory.** Let $\mathfrak{X}$ be the p-adic completion of X and $X_n \to \mathrm{Spec}\,\mathbb{Z}/p^n\mathbb{Z}$ be the scheme obtained by reduction modulo p^n. Let X_n^{ord} be the ordinary locus in X_n and $\mathfrak{X}^{ord}$ the ordinary locus in $\mathfrak{X}$.

We recall that $X_0(p)_1 = X_0(p)_1^F \cup X_0(p)_1^V$ is the union of the Frobenius and Vershiebung correspondences. We let p_i^F and p_i^V be the restrictions of the projections p_i to these components. The projection $p_2^V : X_0(p)_1^V \to X_1$ is an isomorphism (and $X_0(p)_1^V$ parametrizes the Vershiebung isogeny $(p_1^V)^\star E \simeq (p_2^V)^\star E^{(p)} \to (p_2^V)^\star E$). The projection $p_1^F : X_0(p)_1^F \to X_1$ is an isomorphism (and $X_0(p)_1^F$ parametrizes the Frobenius isogeny $(p_1^F)^\star E \to (p_2^F)^\star E \simeq (p_1^F)^\star E^{(p)}$). We denote by i^F and i^V the inclusions $X_0(p)_1^F \hookrightarrow X_0(p)_1$ and $X_0(p)_1^V \hookrightarrow X_0(p)_1$.

Lemma 4.1. *If $k \geq 2$, we have a factorization :*

$$\begin{tikzcd} p_2^\star\omega^k \arrow[r, "T_p"] \arrow[d] & p_1^!\omega^k \\ i_\star^V(p_2^V)^\star\omega^k \arrow[r] & i_\star^V(p_1^V)^!\omega^k \arrow[u] \end{tikzcd}$$

If $k \leq 0$, we have a factorization :

$$\begin{tikzcd} p_2^\star\omega^k \arrow[r, "T_p"] \arrow[d] & p_1^!\omega^k \\ i_\star^F(p_2^F)^\star\omega^k \arrow[r] & i_\star^F(p_1^F)^!\omega^k \arrow[u] \end{tikzcd}$$

Proof. By proposition 2.1, this amounts to check that the cohomological correspondence $T_p : p_2^\star\omega^k \to p_1^!\omega^k$ vanishes at any generic point of multiplicative type in $X_0(p)_1$ if $k \geq 2$, and at any generic point of étale type in $X_0(p)_1$ if $k \leq 0$. This follows from the normalization of the correspondence as explained in the proof of proposition 3.1. $\square$

Remark 4.1. We can informally rephrase this lemma by saying that we have congruences : $T_p = U_p \mod p$ if $k \geq 2$ and $T_p = \mathrm{Frob} \mod p$ if $k \leq 0$, see remark 3.2.

Proposition 4.1. *For all $k \geq 2$, the cohomological correspondence T_p induces a map:*

$$p_2^\star(\omega^k((np+k-2)SS)) \to p_1^!(\omega^k(nSS)).$$

For all $k \leq 0$, the cohomological correspondence T_p induces a map :

$$p_2^\star(\omega^k(-nSS)) \to p_1^!(\omega^k((-np+k)SS)).$$

Proof. We first prove the first claim (when $k \geq 2$). The cohomological correspondence is supported on $X_0(p)_1^V$. The map p_1^V is totally ramified of degree p and the map p_2^V is an isomorphism. It follows that we have an equality of divisors $(p_1^V)^\star(SS) = p(p_2^V)^\star(SS)$. Let us simply denote by $SS = (p_2^V)^\star(SS)$ (this is the reduced supersingular locus divisor).

In particular, we deduce that the map $(p_2^V)^\star\omega^2 \to (p_1^V)^!\omega^2$ induces (after twisting by $\mathscr{O}_{X_0(p)_1^V}(npSS)$ a morphism : $(p_2^V)^\star(\omega^2(npSS)) \to (p_1^V)^!(\omega^2(nSS))$.

This proves the claim for $k=2$. For $k \geq 3$, we remark that the cohomological correspondence $(p_2^V)^\star \omega^k \to (p_1^V)^! \omega^k$ is the tensor product of the map $(p_2^V)^\star \omega^2 \to (p_1^V)^! \omega^2$ and the map $(p_2^V)^\star \omega^{k-2} \to (p_1^V)^\star \omega^{k-2}$. But $(p_1^V)^\star \omega_E \simeq ((p_2^V)^\star \omega_E)^p$ and the differential of the isogeny $(p_2^V)^\star \omega_E \to (p_1^V)^\star \omega_E$ identifies with the Hasse invariant and induces an isomorphism : $(p_2^V)^\star \omega_E(SS) \to (p_1^V)^\star \omega_E$. We deduce that there is therefore a map : $p_2^\star(\omega^k((np+k-2)SS)) \to p_1^!(\omega^k(nSS))$.

We now prove the second claim (for $k \leq 0$). The cohomological correspondence is supported on $X_0(p)_1^F$. The map p_2^F is totally ramified of degree p and the map p_1^F is an isomorphism. It follows that we have an equality of divisors $(p_2^F)^\star(SS) = p(p_1^F)^\star(SS)$. Let us simply denote by $SS = (p_1^F)^\star(SS)$ (this is the reduced supersingular locus divisor).

In particular, we deduce that the map $(p_2^F)^\star \mathscr{O}_X \to (p_1^F)^! \mathscr{O}_X$ induces (after twisting by $\mathscr{O}_{X_0(p)_1^V}(-npSS)$ an morphism : $(p_2^F)^\star \mathscr{O}_X(-nSS) \to (p_1^F)^! \mathscr{O}_X(-npSS)$.

This proves the claim for $k=0$. For $k \leq -1$, we remark that the cohomological correspondence $(p_2^F)^\star \omega^k \to (p_1^F)^! \omega^k$ is the tensor product of the map $(p_2^F)^\star \mathscr{O}_X \to (p_1^F)^! \mathscr{O}_X$ (the cohomological correspondence for $k=0$) and a map $(p_2^F)^\star \omega^k \to (p_1^F)^\star \omega^k$ that we now describe. Unformally, this map is deduced from the differential of the isogeny $(p_1^F)^\star E \to (p_2^F)^\star E$, after normalizing by a factor p^{-1} (one can give sense of this over the formal scheme ordinary locus). Equivalently, it is deduced from the differential of the isogeny of the dual map $(p_2^F)^\star E \to (p_1^F)^\star E$ (the Vershiebung map).

We observe that $(p_2^F)^\star \omega_E \simeq ((p_1^F)^\star \omega_E)^p$ and there is a natural isomorphism: $(p_2^F)^\star \omega_E \overset{(p_1^F)^\star Ha^{-1}}{\longrightarrow} (p_1^F)^\star \omega_E(SS)$, and therefore, for all $k \leq 0$, an isomorphism: $(p_2^F)^\star \omega^k \overset{(p_1^F)^\star Ha^k}{\longrightarrow} (p_1^F)^\star \omega^k(-kSS)$ which factors the map $(p_2^F)^\star \omega^k \overset{(p_1^F)^\star Ha^k}{\to} (p_1^F)^\star \omega^k$. We deduce that there is therefore a map : $p_2^\star(\omega^k(-nSS)) \to p_1^!(\omega^k((-np+k)SS))$.
$\square$

Corollary 4.1. (1) *The T_p operator acts on $\mathrm{R}\Gamma(X_1, \omega^k(nSS))$ for all $n \geq 0$ and $k \geq 2$, and the maps $\mathrm{R}\Gamma(X_1, \omega^k(nSS)) \to \mathrm{R}\Gamma(X_1, \omega^k(n'SS))$ are equivariant for $0 \leq n \leq n'$,*

(2) *We have commutative diagrams for all $n \geq 0$ and $k \geq 2$:*

$$\begin{tikzcd}
\mathrm{R}\Gamma(X_1, \omega^k((np+k-2)SS)) \arrow[r, "T_p"] \arrow[dr] & \mathrm{R}\Gamma(X_1, \omega^k((np+k-2)SS)) \\
\mathrm{R}\Gamma(X_1, \omega^k(nSS)) \arrow[r, "T_p"] \arrow[u] & \mathrm{R}\Gamma(X_1, \omega^k(nSS)) \arrow[u]
\end{tikzcd}$$

(3) *The T_p operator acts on $\mathrm{R}\Gamma(X_1, \omega^k(nSS))$ for all $n \leq 0$ and $k \leq 0$, and the maps $\mathrm{R}\Gamma(X_1, \omega^k(nSS)) \to \mathrm{R}\Gamma(X_1, \omega^k(n'SS))$ are equivariant for $0 \geq n' \geq n$,*

(4) *We have commutative diagrams for all $n \leq 0$ and $k \leq 0$:*

$$\begin{tikzcd}
\mathrm{R}\Gamma(X_1, \omega^k(-nSS)) \arrow[r, "T_p"] \arrow[dr] & \mathrm{R}\Gamma(X_1, \omega^k(-nSS)) \\
\mathrm{R}\Gamma(X_1, \omega^k((-np+k)SS)) \arrow[r, "T_p"] \arrow[u] & \mathrm{R}\Gamma(X_1, \omega^k((-np+k)SS)) \arrow[u]
\end{tikzcd}$$

For any k, we define as usual $\mathrm{H}^i_c(X_1^{ord}, \omega^k) = \lim_n \mathrm{H}^i(X_1, \omega^k(-nSS))$ following [Har72]. This is a profinite $\mathbb{F}_p$-vector space. We also recall that $\mathrm{H}^i(X_1^{ord}, \omega^k) = \mathrm{colim}_n \mathrm{H}^i(X_1, \omega^k(nSS))$.

Corollary 4.2. (1) *If $k \geq 2$, T_p is locally finite on $\mathrm{H}^i(X_1^{ord}, \omega^k)$.*
(2) *If $k \leq 0$, T_p is locally finite on $\mathrm{H}^i_c(X_1^{ord}, \omega^k)$.*
(3) *If $k \geq 3$, we have $e(T_p)\mathrm{H}^i(X_1^{ord}, \omega^k) = e(T_p)\mathrm{H}^i(X_1, \omega^k)$.*
(4) *If $k = 2$, we have $e(T_p)\mathrm{H}^i(X_1^{ord}, \omega^2) = e(T_p)\mathrm{H}^i(X_1, \omega^2(SS))$.*
(5) *If $k \leq -1$, we have $e(T_p)\mathrm{H}^i_c(X_1^{ord}, \omega^k) = e(T_p)\mathrm{H}^i(X_1, \omega^k)$.*
(6) *If $k = 0$, we have $e(T_p)\mathrm{H}^i_c(X_1^{ord}, \mathscr{O}_X) = e(T_p)\mathrm{H}^i(X_1, \mathscr{O}_X(-SS))$.*

Corollary 4.3. (1) *If $k \leq -1$, $e(T_p)\mathrm{R}\Gamma(X_1, \omega^k)$ is concentrated in degree* 1,
(2) *If $k \geq 3$, $e(T_p)\mathrm{R}\Gamma(X_1, \omega^k)$ is concentrated in degree* 0.

Proof. This follows from $\mathrm{H}^1(X_1^{ord}, \omega^k) = 0$ and $\mathrm{H}^0_c(X_1^{ord}, \omega^k) = 0$ because X_1^{ord} is affine □

Remark 4.2. Of course, this last corollary is obvious from the Riemann-Roch theorem, but the given proof is not using this theorem.

4.2. **The p-adic theory.**

4.2.1. *The Igusa tower.* Recall that the principal $\mathbb{G}_m$-torsor ω_E has a $\mathbb{Z}_p^\times$-reduction (in the pro-étale topology) over $\mathfrak{X}^{ord}$ given by $T_p(E)^{et}$ and the Hodge-Tate map :

$$\mathrm{HT} : T_p(E)^{et} \to \omega_E$$

which induces an isomorphism :

$$\mathrm{HT} \otimes 1 : T_p(E)^{et} \otimes_{\mathbb{Z}_p} \mathscr{O}_{\mathfrak{X}^{ord}} \to \omega_E.$$

We can form $\pi : \mathfrak{IG} = \mathrm{Isom}(\mathbb{Z}_p, T_p(E)^{et}) \to \mathfrak{X}^{ord}$, the Igusa tower. This is a p-adic formal scheme, and a pro-finite étale cover of $\mathfrak{X}^{ord}$.

Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Let $\kappa^{un} : \mathbb{Z}_p^\times \to \Lambda^\times$ be the universal character. For any $k \in \mathbb{Z}$, we have an algebraic character $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$, given by $x \mapsto x^k$ and we also denote by $k : \Lambda \to \mathbb{Z}_p$ the corresponding algebra morphism. We let $\omega^{\kappa^{un}} = (\mathscr{O}_{\mathfrak{IG}} \otimes \Lambda)^{\mathbb{Z}_p^\times}$ where the invariants are taken for the diagonal action.

Lemma 4.2. *The sheaf $\omega^{\kappa^{un}}$ is an invertible sheaf of $\Lambda\hat{\otimes}\mathcal{O}_{\mathfrak{X}^{ord}}$-modules and for any $k \in \mathbb{Z}$, there is a canonical isomorphism of invertible sheaves over $\mathfrak{X}^{ord}$:*

$$\omega^k \to \omega^{\kappa^{un}} \otimes_{\Lambda,k} \mathbb{Z}_p.$$

4.2.2. *Cohomology of the ordinary locus.* We may now consider the following Λ-modules :

$$\mathrm{H}^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}})$$

and

$$\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}).$$

Let us give the definition of the second module. Let $\mathfrak{m}_\Lambda$ be the kernel of the reduction map $\Lambda \to \mathbb{F}_p[\mathbb{F}_p^\times]$. We first define $\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n) = \mathrm{H}^1_c(X_n^{ord}, \omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n)$ as follows : we can take any extension of $\omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n$ to a coherent sheaf $\mathscr{F}$ over X_n (this means that if $j : X_n^{ord} \to X_n$ is the inclusion, then $j^\star\mathscr{F} = \omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n$) and we let $\mathrm{H}^1_c(X_n^{ord}, \omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n) = \lim_l \mathrm{H}^1(X_n, \mathscr{I}^l\mathscr{F})$ for $\mathscr{I}$ the ideal of the complement of X_n^{ord} in X_n. We remark that this is a profinite $\Lambda/(\mathfrak{m}_\Lambda)^n$-module. The pro-finite module $\mathrm{H}^1_c(X_n^{ord}, \omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n)$ is well-defined (it does not depend on the choice of $\mathscr{F}$ as a topological module) following [Har72] section 2. We then define $\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) := \lim_n \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n)$.

4.2.3. *U_p and Frobenius.* We now define a U_p-operator and Frobenius operator on these cohomologies.

There is a Frobenius morphism $F : \mathfrak{X}^{ord} \to \mathfrak{X}^{ord}$ which is given by $E \mapsto E/H_1^{can}$ where $H_1^{can} \subset E[p]$ is the multiplicative subgroup.

This map extends to the Igusa tower into a map: $F : \mathfrak{IG} \to \mathfrak{IG}$, which is given by $(E, \psi : \mathbb{Z}_p \simeq T_p(E)^{et}) \mapsto (E/H_1^{can}, \psi' : \mathbb{Z}_p \simeq T_p(E/H_1^{can})^{et})$ where ψ' is defined by $\mathbb{Z}_p \xrightarrow{\psi} T_p(E)^{et} \tilde{\to} T_p(E/H_1^{can})^{et}$.

We deduce that there are two maps, the natural pull-back map on functions : $F^\star \mathscr{O}_{\mathfrak{IG}} \to \mathscr{O}_{\mathfrak{IG}}$ and the trace map $\mathrm{tr}_F : F_\star \mathscr{O}_{\mathfrak{IG}} \to \mathscr{O}_{\mathfrak{IG}}$.

Lemma 4.3. *We have* $\mathrm{tr}_F(F_\star \mathscr{O}_{\mathfrak{IG}}) \subset p\mathscr{O}_{\mathfrak{IG}}$.

Proof. We have $\mathrm{tr}_F(F_\star \mathscr{O}_{\mathfrak{X}^{ord}}) \subset p\mathscr{O}_{\mathfrak{X}^{ord}}$. Since $F \times \pi : \mathfrak{IG} \tilde{\to} \mathfrak{IG} \times_{\pi, \mathfrak{X}^{ord}, F} \mathfrak{X}^{ord}$, we deduce that $\mathrm{tr}_F(F_\star \mathscr{O}_{\mathfrak{IG}}) \subset p\mathscr{O}_{\mathfrak{IG}}$. $\square$

Corollary 4.4. *There are two maps* $F : F^\star \omega^{\kappa^{un}} \to \omega^{\kappa^{un}}$ *and* $U_p : F_\star \omega^{\kappa^{un}} \to \omega^{\kappa^{un}}$.

Proof. The maps $F^\star \mathscr{O}_{\mathfrak{IG}} \to \mathscr{O}_{\mathfrak{IG}}$ and $\frac{1}{p}\mathrm{tr}_F : F_\star \mathscr{O}_{\mathfrak{IG}} \to \mathscr{O}_{\mathfrak{IG}}$ are $\mathbb{Z}_p^\times$-equivariant. We can tensor with Λ and take the invariants. $\square$

4.2.4. *U_p, Frobenius and T_p.* We now describe the specialization of these maps at classical weight $k \in \mathbb{Z}$ in the spirit of section 3.1. Let $F : F^\star \omega^k \to \omega^k$ be the specialization of the map constructed in corollary 4.4. The universal isogeny $\pi : E \to E/H_1^{can}$ has a differential $\pi^\star : F^\star \omega_E \to \omega_E$.

Lemma 4.4. $F = p^{-k}(\pi^\star)^k : F^\star \omega^k \to \omega^k$.

Proof. We have a commutative diagram :

$$\begin{tikzcd}
T_p(E)^{et} \arrow[r] \arrow[d] & T_p(E/H_1^{can})^{et} \arrow[d] \\
\omega_E \arrow[r, "(\pi^D)^\star"] & \omega_{E/H_1^{can}}
\end{tikzcd}$$

from which it follows easily that $F^\star \omega^k \to \omega^k$ is given by $((\pi^D)^\star)^{-k} = p^{-k}(\pi^\star)^k$. $\square$

By duality, there is an étale isogeny $\pi^D : E/H_1^{can} \to E$ with differential $(\pi^D)^\star : \omega_E \to F^\star \omega_E$, and we can construct a map :

$$F_\star \omega^k \overset{((\pi^D)^\star)^k}{\to} F_\star F^\star \omega^k \overset{\frac{1}{p}\mathrm{tr}_F}{\to} \omega^k.$$

Lemma 4.5. *The above map coincides with the map* $U_p : F_\star \omega^k \to \omega^k$.

Proof. Similar to lemma 4.4 and left to the reader. $\square$

In section 3.1 we constructed a cohomological correspondence T_p over $X_0(p)$:

$$T_p : p_2^\star \omega^k \to p_1^! \omega^k.$$

We can consider the completion $\mathfrak{X}_0(p)$ and its ordinary part $\mathfrak{X}_0(p)^{ord}$ which is the disjoint union of two types of irreducible components :

$$\mathfrak{X}_0(p)^{ord} = \mathfrak{X}_0(p)^{ord,F} \coprod \mathfrak{X}_0(p)^{ord,V},$$

where on the first components the universal isogeny is not étale, and where it is étale on the other components.

On $\mathfrak{X}_0(p)^{ord,F}$, the map p_1 is an isomorphism and the map p_2 identifies with the Frobenius map F. On $\mathfrak{X}_0(p)^{ord,V}$, the map p_2 is an isomorphism and the map p_1 identifies with the Frobenius map F. We therefore can think of U_p and F has cohomological correspondences : $p_2^\star\omega^k \to p_1^!\omega^k$ supported respectively on $\mathfrak{X}_0(p)^{ord,V}$ and $\mathfrak{X}_0(p)^{ord,F}$.

One can also restrict T_p to a cohomological correspondence over $\mathfrak{X}_0(p)^{ord}$ and project it on the components $\mathfrak{X}_0(p)^{ord,F}$ and $\mathfrak{X}_0(p)^{ord,V}$. We denote by T_p^F and T_p^V the two projections of the correspondence T_p.

Lemma 4.6. (1) *We have* $T_p^F = p^{\sup\{0,k-1\}}F$.
(2) *We have* $T_p^V = p^{\sup\{0,1-k\}}U_p$.
(3) *If* $k \geq 1$, *we have* $T_p = U_p + p^{k-1}F$,
(4) *If* $k \leq 1$, *we have* $T_p = F + p^{1-k}U_p$.

Proof. This follows from the definitions, compare also with remark 3.2. □

We end up this discussion with duality.

Lemma 4.7. *We have* $D(F) = U_p$.

Proof. Compare with proposition 3.2. □

4.2.5. *Higher Hida theory.*

Theorem 4.1. (1) *F is locally finite on* $\mathrm{H}^1_c(\mathfrak{X}^{ord},\omega^{\kappa^{un}})$ *and* $e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord},\omega^{\kappa^{un}})$ *is a finite projective Λ-module. Moreover,*

$$e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord},\omega^{\kappa^{un}})\otimes_{\Lambda,k}\mathbb{Z}_p = e(T_p)\mathrm{H}^1(X,\omega^k)$$

if $k \leq -1$.
(2) *U_p is locally finite on* $\mathrm{H}^0(\mathfrak{X}^{ord},\omega^{\kappa^{un}})$ *and* $e(U_p)\mathrm{H}^0(\mathfrak{X}^{ord},\omega^{\kappa^{un}})$ *is a finite projective Λ-module. Moreover,*

$$e(U_p)\mathrm{H}^0(\mathfrak{X}^{ord},\omega^{\kappa^{un}})\otimes_{\Lambda,k}\mathbb{Z}_p = e(T_p)\mathrm{H}^0(X,\omega^k)$$

if $k \geq 3$.

Proof. We first need to justify why F is acting on $\mathrm{H}^1_c(\mathfrak{X}^{ord},\omega^{\kappa^{un}})$. We exhibit a continuous action on $\mathrm{H}^1_c(X_n^{ord},\omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n)$, compatible for all n. Let $X_0(p)_n^{ord}\hookrightarrow X_0(p)_n$ be the ordinary locus. We have $X_0(p)_n^{ord} = X_0(p)_n^{ord,F}\coprod X_0(p)_n^{ord,V}$, where $X_0(p)_n^{ord,F}$ is the component where the universal isogeny has connected kernel, and $X_0(p)_n^{ord,V}$ the component where the universal isogeny has étale kernel. The graph of the map $F : X_n^{ord}\to X_n^{ord}$ is $X_0(p)_n^{ord,F}$. We can therefore think of $F : F^\star\omega^{\kappa^{un}}\to\omega^{\kappa^{un}}$ has a cohomological correspondence on $X_0(p)_n^{ord}$:

$$p_2^\star\omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n \to p_1^!\omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n$$

which is precisely given by F on the component $X_0(p)_n^{ord,F}$ and by 0 on the component $X_0(p)_n^{ord,V}$.

We take a coherent sheaf $\mathscr{F}$ over X_n extending $\omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n$. If we denote by $\mathscr{I}$ the ideal of the complement of X_n^{ord} in X_n and by $j : X_n^{ord}\to X_n$ the inclusion. We have $j_\star\omega^{\kappa^{un}}/(\mathfrak{m}_\Lambda)^n = \mathrm{colim}\ \mathscr{I}^{-n}\mathscr{F}$ and we therefore have a cohomological correspondence : $p_2^\star(\mathrm{colim}_l\mathscr{I}^{-l}\mathscr{F})\to p_1^!(\mathrm{colim}_l\ \mathscr{I}^{-l}\mathscr{F})$ and it follows that there exists l

such that this map induces : $p_2^\star(\mathscr{F}) \to p_1^!(\mathscr{I}^{-l}\mathscr{F})$ and there also exists m such that $p_2^\star \mathscr{I}^m \subset p_1^\star \mathscr{I}$. Therefore, we have maps : $p_2^\star(\mathscr{I}^{km}\mathscr{F}) \to p_1^!(\mathscr{I}^{-l+k}\mathscr{F})$ for all $k \geq 0$. This provides a map $F : \mathrm{H}^i(X_n, \mathscr{I}^{km}\mathscr{F}) \to \mathrm{H}^i(X_n, \mathscr{I}^{-l+k}\mathscr{F})$. Passing to the projective limit over k, we finally get a continuous map $F \in \mathrm{End}(\mathrm{H}^1_c(X_n^{ord}, \omega^{\kappa^{un}}/\mathfrak{m}_\Lambda^n))$.

We now need to prove that F is locally finite. We first deal with $n = 1$. In that case, $\mathrm{H}^1_c(X_1^{ord}, \omega^{\kappa^{un}}/\mathfrak{m}_\Lambda) = \oplus_{k=-p+2}^{0} \mathrm{H}^1_c(X_1^{ord}, \omega^k)$ (in this last formula, we can let k go through any set of representatives of $\mathbb{F}_p^\times$ in $\mathbb{Z}$) and the isomorphism is equivariant for the action of F on the LHS, and T_p on the RHS by lemma 4.6. It follows from corollary 4.2 that F is locally finite for $n = 1$, and also that $e(F)\mathrm{H}^1_c(X_1^{ord}, \omega^{\kappa^{un}}/\mathfrak{m}_\Lambda)$ is a finite $\mathbb{F}_p$-vector space. We deal with the general case by induction, using the short exact sequences in cohomology ($\mathfrak{X}^{ord}$ is affine) and proposition 2.5 :

$$0 \to \mathrm{H}^1_c(X_{n+1}^{ord}, \omega^{\kappa^{un}} \otimes (\mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda)^{n+1})) \to \mathrm{H}^1_c(X_{n+1}^{ord}, \omega^{\kappa^{un}}/\mathfrak{m}_\Lambda^{n+1}) \to \mathrm{H}^1_c(X_1^{ord}, \omega^{\kappa^{un}}/\mathfrak{m}_\Lambda^n) \to 0$$

It follows that F is locally finite and that $e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{\kappa^{un}})$ is a finite projective Λ-module.

Finally, for all $k \leq -1$, we have an isomorphism : $N \otimes_{\Lambda,k} \mathbb{Z}_p = e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^k)$ and there is a canonical map:

$$\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^k) \to \mathrm{H}^1(X, \omega^k)$$

and by applying projectors on both side we get a map:

$$e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^k) \to e(T_p)\mathrm{H}^1(X, \omega^k)$$

of finite free $\mathbb{Z}_p$-modules, which is an isomorphism modulo p by corollary 4.2. Therefore this map is an isomorphism. The proof of the second point of the theorem follows along similar lines. $\square$

4.3. **Serre duality.** Recall that we have a residue map $\mathrm{res} : \mathrm{H}^1(X, \Omega^1_{X/\mathbb{Z}_p}) \to \mathbb{Z}_p$. Therefore, there is a natural map : $\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^2(-D) \otimes \Lambda) \to \Lambda$ which is obtained as the composite

$$\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^2(-D) \otimes \Lambda) \to \mathrm{H}^1(X, \omega^2(-D) \otimes \Lambda) \to \mathrm{H}^1(X, \omega^2(-D)) \otimes \Lambda$$

$$\xrightarrow{KS \otimes 1} \mathrm{H}^1(X, \Omega^1_{X/\mathbb{Z}_p}) \otimes \Lambda \xrightarrow{\mathrm{res} \otimes 1} \Lambda$$

Let us denote by $\omega^{2-\kappa^{un}}(-D) = \omega^2 \otimes \underline{\mathrm{Hom}}(\omega^{\kappa^{un}}, \Lambda \otimes \mathscr{O}_{\mathfrak{X}^{ord}})$. This is an invertible sheaf of $\Lambda \otimes \mathscr{O}_{\mathfrak{X}^{ord}}$-modules over $\mathfrak{X}^{ord}$.

Remark 4.3. The following character $\mathbb{Z}_p^\times \to \Lambda^\times$, $t \mapsto t^2(\kappa^{un}(t))^{-1}$ induces an automorphism $d : \Lambda \to \Lambda$. We have an isomorphism of $\mathscr{O}_{\mathfrak{X}^{ord}} \hat{\otimes} \Lambda$-modules : $\omega^{2-\kappa^{un}}(-D) = \omega^{\kappa^{un}}(-D) \otimes_{\Lambda,d} \Lambda$.

We can therefore define a pairing :

$$\langle , \rangle : \mathrm{H}^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) \times \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-\kappa^{un}}(-D)) \to \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^2(-D) \otimes_{\mathbb{Z}_p} \Lambda) \to \Lambda$$

Proposition 4.2. *For any* $(f, g) \in \mathrm{H}^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) \times \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-\kappa^{un}}(-D))$*, we have* $\langle U_p f, g \rangle = \langle f, Fg \rangle$.

Proof. We have a commutative diagram :

$$\begin{tikzcd} \mathrm{H}^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) \times \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-\kappa^{un}}(-D)) \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ \prod_{k\in\mathbb{Z}} \mathrm{H}^0(\mathfrak{X}^{ord}, \omega^k) \times \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D)) \arrow[r] & \prod_{k\in\mathbb{Z}} \mathbb{Z}_p \end{tikzcd}$$

where the vertical maps are injective. It suffices therefore to prove the identity for the pairing $\langle,\rangle_k : \mathrm{H}^0(\mathfrak{X}^{ord}, \omega^k) \times \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D)) \to \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^2(-D)) \to \mathbb{Z}_p$

We work modulo p^n and let $\mathscr{I}$ be the ideal of the complement of X_n^{ord}.

We have $\mathrm{H}^1_c(X_n^{ord}, \omega^{2-k}(-D)) = \lim \mathrm{H}^1(X_n, \mathscr{I}^n\omega^{2-k}(-D))$ and the cohomological correspondence over $X_0(p)_n$

$$F : p_2^\star \mathscr{I}^{mk}\omega^{2-k}(-D) \to p_1^! \mathscr{I}^{-l+k}\omega^{2-k}(-D)$$

Its dual is a map : $D(F) = p_1^\star \mathscr{I}^{-k+l}\omega^k \to p_2^! \mathscr{I}^{-mk}\omega^k$ which equals, on the limit over k, to the cohomological correspondence U_p by lemma 4.7.

□

This pairing restricts to a pairing :

$$\langle,\rangle : e(U_p)\mathrm{H}^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) \times e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-\kappa^{un}}(-D)) \to \Lambda.$$

Theorem 4.2. (1) *The pairing $\langle,\rangle$ is a perfect pairing,*

(2) *For any $(f, g) \in e(U_p)\mathrm{H}^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) \times e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-\kappa^{un}}(-D))$,*

$$\langle U_p f, g\rangle = \langle f, Fg\rangle,$$

(3) *The pairing $\langle,\rangle$ is compatible with the classical pairing in the sense that for any $k \in \mathbb{Z}$, we have a commutative diagram, where the bottom pairing is the one deduced from Serre duality on X :*

$$\begin{tikzcd} e(U_p)\mathrm{H}^0(\mathfrak{X}^{ord}, \omega^k) & \times & e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D)) \arrow[r] \arrow[d, "j"] & \mathbb{Z}_p \\ e(T_p)\mathrm{H}^0(X, \omega^k) \arrow[u, "i"] & \times & e(T_p)\mathrm{H}^1(X, \omega^{2-k}(-D)) \arrow[ur] & \end{tikzcd}$$

Proof. The second point follows from proposition 4.2. The first point will follow from the third point, since the map i and j are isomorphisms for integers $k \geq 3$ and the bottom pairing is perfect. Let us prove the last point. First, we consider the diagram without applying projectors :

$$\begin{tikzcd} \mathrm{H}^0(\mathfrak{X}^{ord}, \omega^k) & \times & \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D)) \arrow[r] \arrow[d, "j"] & \mathbb{Z}_p \\ \mathrm{H}^0(X, \omega^k) \arrow[u, "i"] & \times & \mathrm{H}^1(X, \omega^{2-k}(-D)) \arrow[ur] & \end{tikzcd}$$

which is commutative by construction. For any $f \in \mathrm{H}^0(X, \omega^k)$ and $g \in \mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D))$, we have $\langle i(f), g\rangle = \langle f, j(g)\rangle$. If we now assume that $f \in e(T_p)\mathrm{H}^0(X, \omega^k)$ and $g \in e(F)\mathrm{H}^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D))$, we have that

$$\langle e(U_p)i(f), g\rangle = \langle i(f), e(F)g\rangle = \langle i(f), g\rangle$$

and

$$\langle f, e(T_p)j(g)\rangle = \langle e(T_p)f, j(g)\rangle = \langle f, j(g)\rangle$$

and the conlusion follows. □

5. Higher Coleman theory

5.1. **Cohomology with support in a closed subspace.** We recall the notion of cohomology of an abelian sheaf on a topological space, with support in a closed subspace. A reference for this material is [Gro05], chapter I. Let X be a topological space. Let $i : Z \hookrightarrow X$ be a closed subspace. We denote by C_Z and C_X the categories of sheaves of abelian groups on Z and X respectively.

We have the pushforward functor $i_\star : C_Z \to C_X$ and the extension by zero functor $i_! : C_Z \to C_X$. The functor $i_!$ has a left adjoint $i^! : C_X \to C_Z$. For an abelian sheaf $\mathscr{F}$ over X, we let $\Gamma_Z(X, \mathscr{F}) = \mathrm{H}^0(X, i_\star i^! \mathscr{F})$. By definition this is the subgroup of $\mathrm{H}^0(X, \mathscr{F})$ of sections whose support is included in Z. We let $\mathrm{R}\Gamma_Z(X, -)$ be the derived functor of $\Gamma_Z(X, -)$.

Let $U = X \backslash Z$ and let $\mathscr{F}$ be an object of C_X. We have an exact triangle ([Gro05], I, corollaire 2.9):

$$\mathrm{R}\Gamma_Z(X, \mathscr{F}) \to \mathrm{R}\Gamma(X, \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}) \stackrel{+1}{\to}$$

Some properties of the cohomology with support are :

(1) (change of support)[[Gro05], I, Proposition 1.8] If $Z \subset Z'$, there is a map $\mathrm{R}\Gamma_Z(X, \mathscr{F}) \to \mathrm{R}\Gamma_{Z'}(X, \mathscr{F})$.
(2) (pull-back) If we have a cartesian diagram :

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z' \arrow[r] & X' \end{tikzcd}$$

and a sheaf $\mathscr{F}$ on X', there is a map $\mathrm{R}\Gamma_{Z'}(X', \mathscr{F}) \to \mathrm{R}\Gamma_Z(X, f^\star \mathscr{F})$,
(3) (Change of ambient space)[[Gro05], I, Proposition 2.2] If we have $Z \subset U \subset X$ for some open U of X, then the pull back map $\mathrm{R}\Gamma_Z(X, \mathscr{F}) \to \mathrm{R}\Gamma_Z(U, \mathscr{F})$ is a quasi-isomorphism.

We now discuss the construction of the trace map in the context of adic spaces and finite flat morphisms.

Lemma 5.1. *Consider a commutative diagram of topological spaces:*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z' \arrow[r] & X' \end{tikzcd}$$

with X and X' adic spaces, f a finite flat morphism of adic spaces, Z' and Z are closed subspaces of X' and X respectively. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{X'}$-modules. Then there is a map $\mathrm{R}\Gamma_Z(X, f^\star \mathscr{F}) \to \mathrm{R}\Gamma_{Z'}(X', \mathscr{F})$.

Proof. We first recall that the category of sheaves of $\mathscr{O}_T$-modules on a ringed space $(T, \mathscr{O}_T)$ has enough injectives ([Sta13], Tag 01DH). It follows that it is enough to construct a functorial map $\Gamma_Z(X, f^\star \mathscr{F}) \to \Gamma_{Z'}(X', \mathscr{F})$ for sheaves $\mathscr{F}$ of $\mathscr{O}_{X'}$-modules. We have a map $\Gamma_Z(X, f^\star \mathscr{F}) \to \Gamma_{f^{-1}Z'}(X, f^\star \mathscr{F})$. Therefore, it suffices to consider the case where $Z = f^{-1}(Z')$. We have a trace map $Tr : f_\star f^\star \mathscr{F} \to \mathscr{F}$. Let us complete the above diagram into :

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "f"] & U \arrow[l] \arrow[d, "g"] \\ Z' \arrow[r] & X' & U' \arrow[l, "j'"'] \end{tikzcd}$$

where $U' = X' \setminus Z'$ and $U = X \setminus Z$. We have a commutative diagram :

$$\begin{tikzcd} f_\star f^\star \mathscr{F} \arrow[r] \arrow[d, "Tr"] & j'_\star g_\star g^\star (j')^\star \mathscr{F} \arrow[d] \\ \mathscr{F} \arrow[r] & j'_\star (j')^\star \mathscr{F} \end{tikzcd}$$

Taking global sections and the induced map on the kernel of the two horizontal morphisms, we deduce that there is a map $\Gamma_Z(X, f^\star \mathscr{F}) \to \Gamma_{Z'}(X', \mathscr{F})$. □

5.2. **The modular curve** $X_0(p)$**.** We let $X_0(p)$ be the compactified modular curve of level $\Gamma_0(p)$ and tame level $\Gamma_1(N)$ for some prime to p integer $N \geq 3$, viewed as an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We let $H_1 \subset E[p]$ be the universal subgroup of order p.

5.2.1. *Parametrization by the degree.* Let $X_0(p)^{\mathrm{rk1}}$ be the subset of rank one points of $X_0(p)$. We define a map $\deg : X_0(p)^{\mathrm{rk1}} \to [0,1]$, which sends $x \in X_0(p)^{\mathrm{rk1}}$ to $\deg H_1 \in [0,1]$ (see [Far10], section 4, def. 3). For any rational interval $[a,b] \subset [0,1]$, there is a unique quasi-compact open $X_0(p)_{[a,b]}$ of X such that $\deg^{-1}[a,b] = X_0(p)^{\mathrm{rk1}}_{[a,b]}$. We let $X_0(p)_{[0,a[\cup]b,n]} = (X_0(p)_{[a,b]})^c$.

Remark 5.1. We remark that in this parametrization, the two extremal points 0 (resp. 1) correspond to ordinary semi-abelian schemes equipped with an étale (resp. multiplicative) subgroup H_1.

5.2.2. *The canonical subgroup.* We let X be the compactified modular curve of degree prime to p. We have an Hasse invariant Ha and we can define the Hodge height :

$$\mathrm{Hdg} : X^{\mathrm{rk1}} \to [0,1]$$

obtained by sending x to $\inf\{v_x(\tilde{\mathrm{Ha}}), 1\}$ for any local lift $\tilde{\mathrm{Ha}}$ of the Hasse invariant. For any $v \in [0,1]$, we let $X_v = \{x \in X, \mathrm{Hdg}(x) \leq v\}$ (more correctly, X_v is the quasi-compact open whose rank one points are those described as above).

We recall the following theorems :

Theorem 5.1 ([Kat73], thm. 3.1)**.** *If $v < \frac{p}{p+1}$, then over X_v we have a canonical subgroup $H_1^{can} \subset E[p]$ which is locally isomorphic to $\mathbb{Z}/p\mathbb{Z}$ in the étale topology.*

Theorem 5.2.
(1) *For any rank one point of $X_0(p)$, we have the identity $\sum_{H \subset E[p]} \deg H = 1$. Moreover, either all the degrees are equal or there exists a canonical subgroup H_1^{can} and for all $H \neq H_1^{can}$, $\deg H = \frac{1 - \deg H_1^{can}}{p}$.*
(2) *If $a < \frac{1}{p+1}$, $X_0(p)_{[0,a]}$ carries a canonical subgroup $H_1^{can} \neq H_1$ and $\deg(H_1) = \frac{\mathrm{Hdg}}{p}$.*
(3) *If $a > \frac{1}{p+1}$, $X_0(p)_{[a,1]}$ carries a canonical subgroup $H_1^{can} = H_1$, and $\deg(H_1) = 1 - \mathrm{Hdg}$.*

Proof. Voir [Pil11], section A.2 and [Far11], thm. 6. □

5.3. **The correspondence U_p.** We let C be the correspondence underlying U_p. It parametrizes isogenies of degree p : $(E \to E', H_1 \overset{\simeq}{\to} H'_1)$. We denote by $H = \mathrm{Ker}(E \to E')$. We have two projections $p_1((E, H_1, E', H'_1)) = (E, H_1)$, $p_2((E, H_1, E', H'_1)) = (E', H'_1)$.

There is actually an isomorphism $X_0(p^2) \to C$ (where $X_0(p^2)$ parametrizes $(E, H_2 \subset E[p^2])$, with H_2 locally isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$), mapping (E, H_2) to $(E/H_1, H_2/H_1, E, H_1)$ where $H_1 = H_2[p]$, and the isogeny $E/H_1 \to E$ is dual to $E \to E/H_1$.

Let us denote by $C_{[a,b]} = p_1^{-1}(X_0(p)_{[a,b]})$. If $a > \frac{1}{p+1}$, then we have a canonical subgroup of order 1 over $X_0(p)_{[a,1]}$, $H_1^{can} = H_1$. If $a < \frac{1}{p+1}$, we also have a canonical subgroup of order 1 over $X_0(p)_{[0,a]}$ and $H_1 \neq H_1^{can}$. The map p_1 : $C_{[0,a]} \to X_0(p)_{[0,a]}$ has a section given by H_1^{can}. Let $C^{can}_{[0,a]}$ be the image of this section and let $C^{et}_{[0,a]}$ be its complement, so that $C_{[0,a]} = C^{can}_{[0,a]} \coprod C^{et}_{[0,a]}$.

Proposition 5.1. (1) *If $a \geq \frac{1}{p+1}$, $p_2(C_{\{a\}}) = X_0(p)_{\{\frac{p-1}{p}+\frac{a}{p}\}}$.*
(2) *If $a < \frac{1}{p+1}$, we have that $p_2(C^{can}_{\{a\}}) = X_0(p)_{\{pa\}}$, and $p_2(C^{et}_{\{a\}}) = X_0(p)_{\{1-a\}}$.*
(3) *If $a \in]0, 1[$, we have $\overline{p_2(C_{[a,1]})} \subseteq X_{[a,1]}$.*

Proof. We do the case by case argument to check the first two points :
(1) We have $H_1 = H_1^{can}$. If $H \subset E[p]$ satisfies $H \neq H_1^{can}$, then $\deg H = \frac{1-\deg H_1}{p}$ and $\deg E[p]/H = \frac{p-1}{p} + \frac{\deg H_1}{p}$.
(2) On $C^{can}_{[0,a]}$, the isogeny $E \to E/H$ is the canonical isogeny and $\deg E[p]/H = 1 - \deg H$ where $\deg H = 1 - p \deg H_1$. On $C^{et}_{[0,a]}$ we have $H \neq H_1^{can}$, and $\deg H = \deg H_1$. Therefore $\deg E[p]/H = 1 - \deg H_1$.

We deduce that if $a \in]0, 1[$, there exists $b > a$ such that $p_2(C_{[a,1]}) \subseteq X_{[b,1]}$, and the last point follows. □

We now give a similar analysis for the transpose of U_p. It is useful to use the isomorphism $C \simeq X_0(p^2)$, for which we have $p_2(E, H_2) = (E, H_1)$ and $p_1(E, H_2) = (E/H_1, H_2/H_1)$. We let $C^{[a,b]} = p_2^{-1}(X_0(p)_{[a,b]})$.

If $\deg H_1 < \frac{p}{p+1}$, we deduce that $\deg E[p]/H_1 > \frac{1}{p+1}$ is the canonical subgroup. If $\deg H_1 > \frac{p}{p+1}$, we deduce that $\deg E[p]/H_1 < \frac{1}{p+1}$ is not the canonical subgroup, but that E/H_1 admits a canonical subgroup. We denote by $C^{[a,1],can}$ the component where H_2/H_1 is the canonical subgroup and by $C^{[a,1],et}$ its complement.

Proposition 5.2. (1) *If $a \leq \frac{p}{p+1}$, $p_1(C^{\{a\}}) = X_0(p)_{\{\frac{a}{p}\}}$.*
(2) *If $a > \frac{p}{p+1}$, we have that $p_1(C^{\{a\},can}) = X_0(p)_{\{1-p(1-a)\}}$, and $p_1(C^{\{a\},et})) = X_0(p)_{\{1-a\}}$.*
(3) *For any $0 < a < 1$, we have that $p_1(C^{[0,a[}) \subseteq \overset{\circ}{X_0(p)}_{[0,a[}$.*

Proof. We do the case by case argument for the first two points :
(1) We have $\deg E[p]/H_1 = 1 - \deg H_1$. This is the canonical subgroup. We deduce that $\deg H_2/H_1 = \frac{\deg H_1}{p}$.
(2) We have $\deg E[p]/H_1 = 1 - \deg H_1$, is not the canonical subgroup. In case $(E, H_2) \in C^{[a,1],can}$, we deduce that $\deg H_2/H_1 = 1 - p(1 - \deg H_1)$. If $(E, H_2) \in C^{[a,1],et}$, we deduce that $\deg H_1/H_2 = 1 - \deg H_1$.

We deduce that if $0 < a < 1$, there exists $b > a$ such that $p_1(C^{[0,a[}) \subseteq X_0(p)_{[0,b[}$. $\square$

5.4. **The U_p-operator.** We work over $X_0(p)$. Let $a \in]0,1[\cap\mathbb{Q}$. The cohomologies of interest are :

(1) $\mathrm{R}\Gamma(X_0(p), \omega^k)$,
(2) $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)$,
(3) $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)$.

The category of perfect Banach complexes is the homotopy category of the category of bounded complexes of Banach spaces over $\mathbb{Q}_p$.

Lemma 5.2. *These cohomologies are objects of the category of perfect Banach complexes.*

Proof. In case (1) and (2) we can take a finite affinoid covering to represent the cohomology. In case (3), the cohomology fits in an exact triangle :

$$\mathrm{R}\Gamma_{X_{[0,a[}}(X_0(p), \omega^k) \to \mathrm{R}\Gamma(X_0(p), \omega^k) \to \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k) \to$$

since the two right objects belong to the category of perfect Banach complexes, so is the last one. $\square$

5.4.1. *Constructing the action.* We define a naive cohomological correspondence U_p^{naive} has follows. The two ingredients are the differential map $p_2^\star\omega_E \to p_1^\star\omega_E$ and the trace map $(p_1)_\star\mathscr{O}_C \to \mathscr{O}_{X_0(p)}$. Putting all this together, we obtain

$$U_p^{naive} : (p_1)_\star p_2^\star\omega^k \to \omega^k.$$

Proposition 5.3. *We have an action of the U_p^{naive}-operator on $\mathrm{R}\Gamma(X_0(p), \omega^k)$, $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)$ and $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)$ for any $0 < a < 1$. The U_p^{naive}-operator is compact. Moreover, the U_p^{naive} operator acts equivariantly on the triangle :*

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k) \to \mathrm{R}\Gamma(X_0(p), \omega^k) \to \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k) \overset{+1}{\to}$$

Proof. The operator U_p^{naive} is compact on $\mathrm{R}\Gamma(X_0(p), \omega^k)$ because this later complex is a perfect complex of finite dimensional $\mathbb{Q}_p$-vector spaces. By proposition 5.1:

$$\overline{p_2(C_{[a,1]})} \subset X_0(p)_{[a,1]}.$$

Therefore, we have a map :

$$\begin{aligned}\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k) \to \mathrm{R}\Gamma(p_2(C_{[a,1]}), \omega^k) &\overset{p_2^\star}{\to} \mathrm{R}\Gamma(C_{[a,1]}, p_2^\star\omega^k) \\ \to \mathrm{R}\Gamma(C_{[a,1]}, p_1^\star\omega^k) &\overset{\mathrm{trace}}{\to} \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k).\end{aligned}$$

Therefore, U_p^{naive} acts and is compact on $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)$ because the first map $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k) \to \mathrm{R}\Gamma(p_2(C_{[a,1]}), \omega^k)$ is. Similarly, by proposition 5.2:

$$p_1(C^{[0,a[}) \subset \overset{\circ}{X_0(p)}_{[0,a[}.$$

The operator U_p^{naive} acts like the composite of the following maps :

$$\mathrm{R}\Gamma_{X_{[0,a[}}(X_0(p), \omega^k) \overset{p_2^\star}{\to} \mathrm{R}\Gamma_{C^{[0,a[}}(C, p_2^\star\omega^k) \to \mathrm{R}\Gamma_{C^{[0,a[}}(C, p_1^\star\omega^k)$$
$$\to \mathrm{R}\Gamma_{p_1^{-1}p_1(C^{[0,a[})}(C, p_1^\star\omega^k) \overset{\mathrm{trace}}{\to} \mathrm{R}\Gamma_{p_1(C^{[0,a[})}(X_0(p), \omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k).$$

We claim that the map $\mathrm{R}\Gamma_{p_1(C[0,a[)}(X_0(p),\omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k)$ is compact. This will follow if we prove that the map $\mathrm{R}\Gamma_{X_0(p)_{[0,b[}}(X_0(p),\omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k)$ for $0 \leq b < a \leq 1$ is compact. To see this, we observe that there is a map of exact triangles :

$$\begin{tikzcd}
\mathrm{R}\Gamma_{X_0(p)_{[0,b[}}(X_0(p),\omega^k) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(X_0(p),\omega^k) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(X_0(p)_{[b,1]},\omega^k) \arrow[r] \arrow[d] & +1 \\
\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k) \arrow[r] & \mathrm{R}\Gamma(X_0(p),\omega^k) \arrow[r] & \mathrm{R}\Gamma(X_0(p)_{[a,1]},\omega^k) \arrow[r] & +1
\end{tikzcd}$$

and since the two vertical maps on the right are compact, the first vertical map is also compact.

$\square$

We also note the following corollary of the proof :

Corollary 5.1. *For any $0 < a < b < 1$, the natural maps*

$$\mathrm{R}\Gamma(X_0(p)_{[a,1]},\omega^k) \to \mathrm{R}\Gamma(X_0(p)_{[b,1]},\omega^k)$$

and

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,b[}}(X_0(p),\omega^k)$$

induce quasi-isomorphism on the finite slope part for U_p^{naive}.

5.4.2. *U_p and Frobenius.* It is worth spelling out the action of U_p on $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k)$.

If $a < \frac{1}{p+1}$, we have a correspondence

$$\begin{tikzcd}
 & C^{can}_{[0,a]} \arrow[dl, "p_2^{can}"'] \arrow[dr, "p_1^{can}"] & \\
X_0(p)_{[0,pa]} & & X_0(p)_{[0,a]}
\end{tikzcd}$$

where p_1^{can} is actually an isomorphism. We can think of this correspondence as the graph of the Frobenius map $X_0(p)_{[0,pa]} \to X_0(p)_{[0,a]}$, sending (E,H_1) to $(E/H_1^{can}, E[p]/H_1^{can})$. We claim that there is an associated operator :

$$U_p^{naive,can} : \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k).$$

We first observe that $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k) = \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p)_{[0,a]},\omega^k)$.

We now construct $U_p^{naive,can}$ as the composite :

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,pa[}}(X_0(p),\omega^k) \stackrel{(p_2^{can})^\star}{\to} \mathrm{R}\Gamma_{C^{can}_{[0,a[}}(C^{can}_{[0,a]},(p_2^{can})^\star\omega^k)$$

$$\to \mathrm{R}\Gamma_{C^{can}_{[0,a[}}(C^{can}_{[0,a]},(p_1^{can})^\star\omega^k) \tilde{\to} \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k).$$

Proposition 5.4. *For $a < \frac{1}{p+1}$, we have $U_p^{naive,can} = U_p^{naive}$ on $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^k)$.*

Proof. This amounts to compare the construction of U_p^{naive} (given in the proof of proposition 5.3) and of $U_p^{naive,can}$. We see that $p_1(C^{[0,a[}) \subset X_0(p)_{[0,a]}$ and therefore,

$$p_1^{-1}p_1(C^{[0,a[}) = (p_1^{-1}p_1(C^{[0,a[}) \cap C^{can}_{[0,a]}) \coprod (p_1^{-1}p_1(C^{[0,a[}) \cap C^{et}_{[0,a]}).$$

Moreover, $p_2((p_1^{-1}p_1(C^{[0,a[}) \cap C^{et}_{[0,a]})) \subset X_0(p)_{[1-a,a]}$ and therefore, $C^{[0,a[} \hookrightarrow (p_1^{-1}p_1(C^{[0,a[}) \cap C^{can}_{[0,a]})$. We have

$$\mathrm{R}\Gamma_{p_1^{-1}p_1(C^{[0,a[})}(C, p_1^\star\omega^k) =$$

$$\mathrm{R}\Gamma_{(p_1^{-1}p_1(C^{[0,a[}) \cap C^{can}_{[0,a]})}(C, p_1^\star\omega^k) \oplus \mathrm{R}\Gamma_{(p_1^{-1}p_1(C^{[0,a[}) \cap C^{et}_{[0,a]})}(C, p_1^\star\omega^k).$$

Therefore the map $\mathrm{R}\Gamma_{C^{[0,a[}}(C, p_1^\star\omega^k) \to \mathrm{R}\Gamma_{p_1^{-1}p_1(C^{[0,a[})}(C, p_1^\star\omega^k)$ factors through the direct factor $\mathrm{R}\Gamma_{(p_1^{-1}p_1(C^{[0,a[}) \cap C^{can}_{[0,a]})}(C, p_1^\star\omega^k)$. □

5.4.3. *Slopes estimates and the control theorem.* The following lemma is the key technical input to proving Coleman's classicality theorem.

Lemma 5.3. *For any $a \in]0,1[\cap \mathbb{Q}$,*

(1) *the slopes of U_p^{naive} on $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)$ are ≥ 1.*
(2) *the slopes of U_p^{naive} on $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)$ are $\geq k$.*

Proof. We first observe that the finite slope part of the cohomology $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)^{fs}$ is independent of $a \in]0,1[$ and is therefore supported in degree 0 by affiness for a close to 1. It follows that we are left to prove that U_p^{naive} has slopes at least 1 and this is a q-expansion computation. Namely, $U_p^{naive}(\sum a_n q^n) = p\sum a_{np}q^n$. For the second cohomology, we again observe that the cohomology is independent of $a \in]0,1[$. We may therefore suppose that a is small enough and use that $U_p^{naive} = U_p^{naive,can}$ (proposition 5.4).

We first claim that for any $s \in \mathbb{Q}$,

$$\mathrm{im}(\mathrm{H}^i_{X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+}) \to \mathrm{H}^i_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)^{=s})$$

(where the supscript $= s$ means the slope s part for the action of U_p^{naive}) defines a lattice in $\mathrm{H}^i_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)^{=s}$ (an open and bounded sub-module). We can indeed represent the cohomology $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)$ by the Cech complex $C^\bullet$ relative to some open covering $\mathcal{U}$, and we can lift the U_p-operator to a compact operator $\tilde{U}_p$ on the complex. Note that $C^\bullet$ is a complex of Banach modules.

The map from Cech cohomology with respect to $\mathcal{U}$ to cohomology

$$\check{\mathrm{H}}^i_{\mathcal{U}, X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+}) \to \mathrm{H}^i_{X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+})$$

has kernel and cokernel of bounded torsion by [Pil17], lemma 3.2.2. It suffices to prove that

$$\mathrm{im}(\check{\mathrm{H}}^i_{\mathcal{U}, X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+}) \to \mathrm{H}^i_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)^{=s})$$

defines an open and bounded sub-module in $\mathrm{H}^i_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)^{=s}$. The Cech cohomology $\check{\mathrm{H}}^i_{\mathcal{U}, X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+})$ is obtained by taking the cohomology of and open and bounded sub-complex $C^{+,\bullet} \subset C^\bullet$. The image of $C^{+,\bullet}$ in $C^{\bullet,=s}$ under the continuous projection $C^\bullet \to C^{\bullet,=s}$ (the target is now a complex of finite dimensional vector spaces) is again open and bounded and the claim follows.

Moreover, over $C^{can}_{[0,a[}$, we have a universal isogeny which gives an isomorphism, $p_2^\star\omega \to p_1^\star\omega$, for which $pp_1^\star\omega^+ \subset p_2^\star\omega^+ \subset p^{1-\frac{a}{p}}p_1^\star\omega^+$. We deduce that U_p^{naive} induces a map

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+}) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), p^{k(1-\frac{a}{p})}\omega^{k,+})$$

if $k \geq 0$, and

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^{k,+}) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), p^k\omega^{k,+})$$

if $k \leq 0$. We deduce that $p^{-k(1-\frac{a}{p})}U_p^{naive}$ if $k \geq 0$ and $p^{-k}U_p^{naive}$ if $k \leq 0$ stabilize a lattice in the cohomology, and therefore have only non-negative slope. The lemma follows. □

Remark 5.2. The lemma 3.2.2 in [Pil17] depends on the main result of [Bar78], which in turn is a key technical ingredient in [Kas06].

We now define $U_p = p^{-\inf\{1,k\}}U_p^{naive}$ and we get :

Theorem 5.3. *(1) U_p has slopes ≥ 0 on $\mathrm{R}\Gamma(X_0(p), \omega^k)$,*

(2) For any $a \in]0,1[\cap\mathbb{Q}$, the map $\mathrm{R}\Gamma(X_0(p), \omega^k)^{<k-1} \to \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)^{<k-1}$ is a quasi-isomorphism,

(3) For any $a \in]0,1[\cap\mathbb{Q}$, the map $\mathrm{R}\Gamma_{X_{[0,a[}}(X_0(p), \omega^k)^{<1-k} \to \mathrm{R}\Gamma(X_0(p), \omega^k)^{<1-k}$ is a quasi-isomorphism.

Proof. We consider the triangle

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k) \to \mathrm{R}\Gamma(X_0(p), \omega^k) \to \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k) \xrightarrow{+1}$$

on which U_p acts equivariantly and apply the slope estimates of lemma 5.3. □

5.4.4. *Going down to spherical level.* For $a > \frac{1}{p+1}$, the map $p_1 : X_0(p)_{[a,1]} \to X_{1-a}$ is an isomorphism and therefore the pull back map $\mathrm{R}\Gamma(X_{1-a}, \omega^k) \to \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^k)$ is a quasi-isomorphism. There is a analogue statement for the cohomology with support that we now explain. Let $a < \frac{1}{p+1}$. We have a Frobenius map $F : X_0(p)_{[0,a]} \to X_0(p)_{[0,pa]}$ given by $(E, H_1) \mapsto (E/H_1^{can}, E[p]/H_1^{can})$. There is similarly a Frobenius map $F : X_{\frac{a}{p}} \to X_a$, given by $E \mapsto E/H_1^{can}$.

The Frobenius map fits into the following diagram :

$$\begin{tikzcd}
X_0(p)_{[0,a]} \arrow[r, "F"] \arrow[d, "p_1"] & X_0(p)_{[0,pa]} \arrow[d, "p_1"] \\
X_{\frac{a}{p}} \arrow[r, "F"] \arrow[ur] & X_a
\end{tikzcd}$$

where the diagonal map is given by $E \mapsto (E/H_1^{can}, E[p]/H_1^{can})$.

Proposition 5.5. *The pull back map $\mathrm{R}\Gamma_{X_{\frac{a}{p}}}(X, \omega^k) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^k)$ induces a quasi-isomorphism on the finite slope part.*

5.5. **p-adic variation.** We now consider the problem of interpolation of these cohomologies.

5.5.1. *Reduction of the torsor ω_E.* We recall here a construction from [Pil13] and [AIS14]. We let $\mathcal{T} = \{w \in \omega_E,\ \omega \neq 0\}$ be the $\mathbb{G}_m$-torsor associated to ω_E. We let $\mathbb{G}_a^+ = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ be the unit ball, with its additive analytic group structure. We fix a positive integer n. We have an analytic subgroup $\mathbb{Z}_p^\times(1 + p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+) \hookrightarrow \mathbb{G}_m$.

Proposition 5.6. *Let $v < \frac{1}{p^{n-1}(p-1)}$. The $\mathbb{G}_m$-torsor $\mathcal{T} \times_X X_v \to X_v$ has a natural reduction to a $\mathbb{Z}_p^\times(1 + p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+)$-torsor denoted $\mathcal{T}_v$.*

Proof. We denote by $\omega_E^+ \subset \omega_E$ the locally free sheaf of integral relative differential forms. For $v < \frac{1}{p^{n-1}(p-1)}$, there is a canonical subgroup of level n, H_n^{can} over X_v. The isogeny $E \to E/H_n^{can}$ yields a map $\omega^+_{E/H_n^{can}} \to \omega_E^+$, with cokernel $\omega^+_{H_n}$. The surjective map $r : \omega_E^+ \to \omega^+_{H_n^{can}}$ induces an isomorphism :

$$\omega_E^+/p^{n-v\frac{p^n-1}{p-1}} \tilde{\to} \omega^+_{H_n^{can}}/p^{n-v\frac{p^n-1}{p-1}}.$$

There is a Hodge-Tate map : $\mathrm{HT} : (H_n^{can})^D \to \omega^+_{H_n^{can}}$ (of sheaves on the étale site), and its linearization : $\mathrm{HT} \otimes 1 : (H_n^{can})^D \otimes \mathscr{O}^+_{X_v} \to \omega^+_{H_n^{can}}$ has cokernel killed by $p^{\frac{v}{p-1}}$. We have a diagram :

$$\begin{array}{ccc} & & \omega_E^+ \\ & & \downarrow r \\ (H_n^{can})^D & \xrightarrow{\mathrm{HT}} & \omega^+_{H_n} \end{array}$$

We now introduce a modification of ω_E^+ : let $\omega_E^\sharp = \{w \in \omega_E^+,\ r(w) \in \mathrm{im}(\mathrm{HT} \otimes 1)\} \subset \omega_E^+$. This is a locally free sheaf of $\mathscr{O}^+_{X_v}$-modules on the étale site. The Hodge-Tate map induces an isomorphism :

$$\mathrm{HT}_v : (H_n^{can})^D \otimes \mathscr{O}^+_{X_v}/p^{n-v\frac{p^n}{p-1}} \to \omega_E^\sharp/p^{n-v\frac{p^n}{p-1}}.$$

We let $\mathcal{T}_v$ be the torsor under the group $\mathbb{Z}_p^\times(1 + p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+)$ defined by

$$\mathcal{T}_v = \{\omega \in \omega_E^\sharp,\ \exists P \in (H_n^{can})^D,\ p^{n-1}P \neq 0, \mathrm{HT}_v(P) = \omega \mod p^{n-v\frac{p^n}{p-1}}\}.$$

We have a natural map $\mathcal{T}_v \hookrightarrow \mathcal{T}$, equivariant for the analytic group map : $\mathbb{Z}_p^\times(1 + p^{n-v\frac{p^n}{p-1}}\mathbb{G}_a^+) \to \mathbb{G}_m$. □

5.5.2. *Interpolation of the sheaf.* We let $\mathcal{W} = \mathrm{Spa}(\Lambda, \Lambda) \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the weight space. We let $\kappa^{un} : \mathbb{Z}_p^\times \to \mathscr{O}_\mathcal{W}^\times$ be the universal character.

We can write $\mathcal{W}$ as an increasing union of affinoids $\mathcal{W} = \cup_{0<r<1}\mathcal{W}_r$ where $r \in \mathbb{Q} \cap]0, 1[$ and each $\mathcal{W}_r$ is a finite union of balls of radius r. Over each $\mathcal{W}_r$, there is $t(r) \in \mathbb{Q}_{>0}$ such that the universal character extends to a character κ^{un} : $\mathbb{Z}_p^\times(1 + p^{t(r)}\mathscr{O}_W^+) \to \mathscr{O}_\mathcal{W}^\times$.

We now fix r and we choose v small enough and n large enough such that $t(r) \leq n - v\frac{p^n}{p-1}$ and we define a locally free sheaf $\omega^{\kappa^{un}}$ over $X_v \times \mathcal{W}_r$:

$$\omega^{\kappa^{un}} = (\mathscr{O}_{\mathcal{T}_v} \otimes \mathscr{O}_{\mathcal{W}_r})^{\mathbb{Z}_p^\times(1+p^{n-v\frac{p^n}{p-1}}\mathscr{O}^+_{X_v \times \mathcal{W}_r})}.$$

Since $\mathcal{T}_v \to X_v$ is an étale torsor, the sheaf $\omega^{\kappa^{un}}$ is a locally free sheaf of $\mathscr{O}_{X_v \times \mathcal{W}_r}$-modules in the étale topology. It is actually a locally free sheaf of $\mathscr{O}_{X_v \times \mathcal{W}_r}$-modules in the Zariski topology by the main result of [BG98].

5.5.3. *Interpolation of the cohomology.* For $a \in]0, \frac{v}{p}]$, we have a map $p_1 : X_0(p)_{[0,a]} \to X_v$ and we can therefore pull back the sheaf $\omega^{\kappa^{un}}$ to an invertible sheaf over $X_0(p)_{[0,a]} \times \mathcal{W}_r$.

If $a \in [1-v, 1[$, we have a map $p_1 : X_0(p)_{[a,1]} \to X_v$ and we can pull back the sheaf $\omega^{\kappa^{un}}$ to an invertible sheaf over $X_0(p)_{[a,1]} \times \mathcal{W}_r$.

We consider the cohomologies :

(1) $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^{\kappa^{un}})$
(2) $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{un}})$.

Note that the first cohomology group is well defined because $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^{\kappa^{un}}) = \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p)_{[0,a]}, \omega^{\kappa^{un}})$.

These cohomologies belong to the category of perfect complexes of Banach spaces over $\mathscr{O}_{\mathcal{W}_r}$ which is the homotopy category of the category of bounded complexes of projective Banach modules over $\mathscr{O}_{\mathcal{W}_r}$.

5.5.4. *The U_p-operator on* $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{un}})$. We need to consider the U_p-correspondence on $X_0(p)_{[a,1]}$ where it reduces to a correspondence

$$\begin{tikzcd} & C_{[a,1]} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X_0(p)_{[\frac{a}{p},1]} & & X_0(p)_{[a,1]} \end{tikzcd}$$

Lemma 5.4. *There is a natural isomorphism $p_2^\star\omega^{\kappa^{un}} \to p_1^\star\omega^{\kappa^{un}}$, and we can define a cohomological correspondence $U_p : p_{1\star}p_2^\star\omega^{\kappa^{un}} \to \omega^{\kappa^{un}}$ which specializes in weight $k \geq 1$ to U_p.*

Proof. Over $C_{[a,1]}$, the universal isogeny $\pi : p_1^\star E \to p_2^\star E$ induces an isomorphism on canonical subgroups $p_1^\star H_n^{can} \simeq p_2^\star H_n^{can}$, and therefore there is a canonical isomorphism :

$$\begin{tikzcd} p_2^\star\omega_E^\sharp \arrow[r] \arrow[d] & p_1^\star\omega_E^\sharp \arrow[d] \\ p_2^\star\omega_E^\sharp/p^{n-v\frac{p^n}{p-1}} \arrow[r] & p_1^\star\omega_E^\sharp/p^{n-v\frac{p^n}{p-1}} \\ p_2^\star(H_n^{can})^D \arrow[r, "\pi^D"] \arrow[u, "\mathrm{HT}"] & p_1^\star(H_n^{can})^D \arrow[u, "\mathrm{HT}"'] \end{tikzcd}$$

This clearly induces an isomorphism $p_1^\star\mathcal{T}_v \to p_2^\star\mathcal{T}_v$ and from this we get an isomorphism :

$$p_2^\star\omega^{\kappa^{un}} \to p_1^\star\omega^{\kappa^{un}}$$

which specializes to the natural isomorphism $p_2^\star\omega^k \to p_1^\star\omega^k$ at weight k.

We now define $U_p : p_{1\star}p_2^\star\omega^{\kappa^{un}} \to p_{1\star}p_1^\star\omega^{\kappa^{un}} \xrightarrow{\frac{1}{p}\mathrm{Tr}_{p_1}} \omega^{\kappa^{un}}$.

□

Corollary 5.2. *The operator U_p is compact on* $\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{un}})$.

Proof. The operator U_p factors as :

$$\mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{un}}) \to \mathrm{R}\Gamma(X_0(p)_{[\frac{a}{p},1]}, \omega^{\kappa^{un}}) \to \mathrm{R}\Gamma(X_0(p)_{[a,1]}, \omega^{\kappa^{un}}).$$

$\square$

5.5.5. *The U_p-operator on $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^{\kappa^{un}})$.* We need to consider the U_p-correspondence on $X_0(p)_{[0,a]}$ where actually only the canonical part of the correspondance is relevant and therefore, it reduces to a correspondence :

$$\begin{tikzcd} & C^{can}_{[0,a]} \arrow[dl, "p_2^{can}"'] \arrow[dr, "p_1^{can}"] & \\ X_0(p)_{[0,pa]} & & X_0(p)_{[0,a]} \end{tikzcd}$$

Lemma 5.5. *There is a natural isomorphism $(p_2^{can})^\star\omega^{\kappa^{un}} \to (p_1^{can})^\star\omega^{\kappa^{un}}$, and a cohomological correspondence $U_p : (p_1^{can})_\star(p_2^{can})^\star\omega^{\kappa^{un}} \to \omega^{\kappa^{un}}$ specializes in weight $k \leq 1$ to U_p^{can}.*

Proof. Over $C^{can}_{[0,a]}$, the dual universal isogeny $\pi^D : (p_2^{can})^\star E \to (p_1^{can})^\star E$ induces an isomorphism on canonical subgroups $(p_2^{can})^\star H_n^{can} \simeq (p_1^{can})^\star H_n^{can}$, and therefore there is a canonical isomorphism :

$$\begin{tikzcd} (p_1^{can})^\star\omega_E^\sharp \arrow[r, "(\pi^D)^\star"] \arrow[d] & (p_2^{can})^\star\omega_E^\sharp \arrow[d] \\ (p_1^{can})^\star\omega_E^\sharp/p^{n-v\frac{p^n}{p-1}} \arrow[r] & (p_2^{can})^\star\omega_E^\sharp/p^{n-v\frac{p^n}{p-1}} \\ (p_1^{can})^\star(H_n^{can})^D \arrow[r, "\pi"] \arrow[u, "\mathrm{HT}"] & (p_2^{can})^\star(H_n^{can})^D \arrow[u, "\mathrm{HT}"'] \end{tikzcd}$$

This clearly induces an isomorphism $(p_2^{can})^\star\mathcal{T}_v \to (p_1^{can})^\star\mathcal{T}_v$ and from this we get an isomorphism :

$$(p_2^{can})^\star\omega^{\kappa^{un}} \to (p_1^{can})^\star\omega^{\kappa^{un}}$$

or rather more naturally its inverse :

$$(p_1^{can})^\star\omega^{\kappa^{un}} \to (p_2^{can})^\star\omega^{\kappa^{un}}$$

which specializes to the natural isomorphism $(p_1^{can})^\star\omega^k \to (p_2^{can})^\star\omega^k$ given by $(\pi^D)^\star$, and its inverse $(p_2^{can})^\star\omega^k \to (p_1^{can})^\star\omega^k$ is $p^{-k}(\pi^\star)$.

Recall that p_1 is an isomorphism, and we therefore get $U_p : (p_1^{can})_\star(p_2^{can})^\star\omega^{\kappa^{un}} \to \omega^{\kappa^{un}}$ which specializes in weight $k \leq 1$ to $p^{-k}U_p^{can,naive} = U_p^{can}$.

$\square$

Remark 5.3. We therefore find that this is really U_p and not U_p^{naive} that can be interpolated over the weight space.

Corollary 5.3. *The U_p-operator is compact on $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p), \omega^{\kappa^{un}})$.*

Proof. The operator U_p factors as :

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^{\kappa^{un}}) \to \mathrm{R}\Gamma_{X_0(p)_{[0,pa[}}(X_0(p),\omega^{\kappa^{un}}) \to \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^{\kappa^{un}}).$$

□

5.6. **Construction of eigencurves.** We use these cohomologies to construct the eigencurve, following the method of [Col97].

5.6.1. *First construction.* The cohomology $\mathrm{R}\Gamma(X_0(p)_{[a,1]},\omega^{\kappa^{un}})$ is concentrated in degree 0, and is represented by $\mathrm{H}^0(X_0(p)_{[a,1]},\omega^{\kappa^{un}})$ which is a projective Banach module over $\mathscr{O}_{\mathcal{W}_r}$ (see [Pil13], corollary 5.3).

The U_p-operator acts compactly on this space. We let $\mathcal{P} \in \mathscr{O}_{\mathcal{W}_r}[[T]]$ be the characteristic series of U_p. This is an entire series. We let $\mathcal{Z} = V(\mathcal{P}) \subset \mathbb{G}_m^{an} \times \mathcal{W}_r$. We have a weight map $w : \mathcal{Z} \to \mathcal{W}_r$ which is quasi-finite, partially proper, and locally on the source and the target a finite flat map.

Over $\mathcal{Z}$ we have a coherent sheaf $\mathcal{M}$ which is the universal generalized eigenspace. This is a locally free $w^{-1}\mathscr{O}_{\mathcal{W}_r}$-module and for any $x = (\kappa,\alpha) \in \mathcal{Z}$, $x^\star\mathcal{M} = \mathrm{H}^0(X_0(p)_{[a,1]},\omega^{\kappa})^{U_p=\alpha^{-1}}$.

We let $\mathscr{O}_{\mathcal{C}} \subset \mathrm{End}_{\mathcal{Z}}(\mathcal{M})$ be the subsheaf of $\mathscr{O}_{\mathcal{Z}}$-modules generated by the Hecke operators of level prime to Np, and we let $\mathcal{C} \to \mathcal{Z}$ be the associated analytic space.

The construction of $(\mathcal{M},\mathcal{C},\mathcal{Z})$ is compatible when r changes (and does not depend on auxiliary choices like a, v, n...). We now let r tend to 1, glue everything, and with a slight abuse of notation we have $\mathcal{C} \to \mathcal{Z} \to \mathcal{W}$ and a coherent sheaf $\mathcal{M}$ over $\mathcal{C}$. This is the eigencurve of [CM98].

5.6.2. *Second construction.* We can perform a similar construction, using instead the cohomology $\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^{2-\kappa^{un}}(-D))$ where $\omega^{2-\kappa^{un}}(-D) = (\omega^{\kappa^{un}})^\vee \otimes \omega^2(-D)$. The introduction of this twist is motivated by Serre duality. Recall that the character $\mathbb{Z}_p^\times \to \Lambda^\times$, $t \mapsto t^2\kappa^{un}(t)^{-1}$ induces an automorphism $d : \Lambda \to \Lambda$, and therefore:

$$\mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^{2-\kappa^{un}}(-D)) = \mathrm{R}\Gamma_{X_0(p)_{[0,a[}}(X_0(p),\omega^{\kappa^{un}}(-D)) \otimes^L_{\mathscr{O}_{\mathcal{W}_r},d} \mathscr{O}_{\mathcal{W}_r}.$$

This cohomology is a perfect complex of projective Banach modules over $\mathscr{O}_{\mathcal{W}_r}$ and for any morphism $\mathrm{Spa}(A,A^+) \to \mathcal{W}_r$, the cohomology $\mathrm{R}\Gamma_{[0,a[}(X_0(p),\omega^{2-\kappa^{un}}(-D))\hat{\otimes}A)$ is supported in degree 1.

We choose a representative $N^\bullet$ for this cohomology, as well as a compact representative $\tilde{U}_p$ representing the action of U_p. We let $\mathcal{Q}_i$ be the characteristic series of $\tilde{U}_p$ acting on N^i. We let $\tilde{\mathcal{Q}} = \prod \mathcal{Q}_i$ and we let $\mathcal{X} = V(\tilde{\mathcal{Q}}) \subset \mathbb{G}_m^{an} \times \mathcal{W}_r$. We have a weight map $w : \tilde{\mathcal{X}} \to \mathcal{W}_r$ which is quasi-finite locally on the source and the target and a bounded complex of coherent sheaves "generalized eigenspaces" $\tilde{\mathcal{N}}^\bullet$ over $\tilde{\mathcal{X}}$. This is a perfect complex of finite projective $w^{-1}\mathscr{O}_{\mathcal{W}_r}$-modules. Moreover, $\tilde{\mathcal{N}}^\bullet$ has cohomology only in degree 1, and we deduce that $\mathrm{H}^1(\mathcal{N}^\bullet) = \mathcal{N}$ is a locally free $w^{-1}\mathscr{O}_{\mathcal{W}_r}$-module. We let $\mathcal{Q} = V(\prod_i(-1)^i\mathcal{Q}_i)$ and we set $\mathcal{X} = V(\mathcal{Q}) \subset \tilde{\mathcal{X}}$. The module $\mathcal{N}$ is supported on $\mathcal{X}$. We let $\mathscr{O}_{\mathcal{D}} \subset \mathrm{End}_{\mathcal{X}}(\mathcal{N})$ be the subsheaf generated by the Hecke algebra of prime to Np level, and we let $\mathcal{D} \to \mathcal{X}$ be the associated analytic space. We can now let r tend to 1, and we have $\mathcal{D} \to \mathcal{X} \to \mathcal{W}$ and the sheaf $\mathcal{N}$ over $\mathcal{D}$. This is a second eigencurve.

5.7. **The duality pairing.** In this last section, we prove that $\mathcal{Z}$ and $\mathcal{X}$ are canonically isomorphic, that $\mathcal{M}$ and $\mathcal{N}$ are canonically dual to each other and that $\mathcal{C}$ and $\mathcal{D}$ are canonically identified under the pairing between $\mathcal{M}$ and $\mathcal{N}$.

5.7.1. *Preliminaries.* We are going to use the theory of dagger spaces [GK00]. Let $X^\dagger$ be a dagger space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, smooth of pure relative dimension d. Let $\mathscr{F}$ be a coherent sheaf on $X^\dagger$. Then one can define the cohomology groups $\mathrm{H}^i(X^\dagger, \mathscr{F})$ and $\mathrm{H}^i_c(X^\dagger, \mathscr{F})$. Moreover, these cohomology groups carry canonical topologies ([vdP92], sect. 1.6).

By [GK00], thm 4.4, there is a residue map :

$$\mathrm{res}_X : \mathrm{H}^d_c(X^\dagger, \Omega^d_{X/\mathbb{Q}_p}) \to \mathbb{Q}_p.$$

This residue map has the following two important properties. Let $f : Y^\dagger \to X^\dagger$ be an open immersion. Then the diagram :

$$\begin{tikzcd} \mathrm{H}^d_c(Y^\dagger, \Omega^d_{Y/\mathbb{Q}_p}) \arrow[r, "f_\star"] \arrow[dr, "\mathrm{res}_Y"'] & \mathrm{H}^d_c(X^\dagger, \Omega^d_{X/\mathbb{Q}_p}) \arrow[d, "\mathrm{res}_X"] \\ & \mathbb{Q}_p \end{tikzcd}$$

is commutative. See [Bey97], coro. 4.2.12 (although the author is working here in the "dual" setting of wide open spaces)).

Let $f : Y^\dagger \to X^\dagger$ be a finite flat map. Then the diagram :

$$\begin{tikzcd} \mathrm{H}^d_c(Y^\dagger, \Omega^d_{Y/\mathbb{Q}_p}) \arrow[r, "\mathrm{tr}_f"] \arrow[dr, "\mathrm{res}_Y"'] & \mathrm{H}^d_c(X^\dagger, \Omega^d_{X/\mathbb{Q}_p}) \arrow[d, "\mathrm{res}_X"] \\ & \mathbb{Q}_p \end{tikzcd} \tag{1}$$

is commutative. See [Bey97], coro. 4.2.11 (although the author is working here again in the "dual" setting of wide open spaces).

Let $\mathscr{F}$ be a locally free sheaf of finite rank over $X^\dagger$ which is assumed to be affinoid, smooth of pure dimension d. We let $D(\mathscr{F}) = \mathscr{F}^\vee \otimes \Omega^d_{X/\mathbb{Q}_p}$.

Then the residue map induces a perfect pairing ([GK00], thm. 4.4) :

$$\mathrm{H}^0(X^\dagger, \mathscr{F}) \times \mathrm{H}^d_c(X^\dagger, D(\mathscr{F})) \to \mathbb{Q}_p$$

for which both spaces are strong duals of each other.

Remark 5.4. Since the topological vector space $\mathrm{H}^0(X^\dagger, \mathscr{F})$ is a compact inductive limit of Banach spaces, we deduce from the theorem that the topological vector space $\mathrm{H}^d_c(X^\dagger, D(\mathscr{F}))$ is a compact projective limit of Banach spaces.

5.7.2. *The classical pairing.* We denote by w the Atkin-Lehner involution over $X_0(p)$ and by $\langle p \rangle$ the diamond operator given by multiplication by p. We recall that $w \circ w = \langle p \rangle$. We recall that C is the Hecke correspondence underlying U_p. We can think of it as the moduli space of (E, H, H_1) where H, H_1 are distinct subgroups of $E[p]$. We have the projection $p_1(E, H, H_1) = (E, H_1)$. We also have a projection $p_2((E, H, H_1) = (E/H, E[p]/H)$. Exchanging the roles of H and H_1 yields an automorphism $\iota : C \to C$ and we let $q_i = p_i \circ \iota$. Now one checks easily that $w \circ p_1 = q_2$

and $w \circ p_2 = \langle p\rangle \circ q_1$. We have a residue map $\mathrm{H}^1(X_0(p), \Omega^1_{X_0(p)/\mathbb{Q}_p}) \to \mathbb{Q}_p$ and there is a perfect pairing:

$$\langle , \rangle_0 : \mathrm{H}^0(X_0(p), \omega^k) \times \mathrm{H}^1(X_0(p), \omega^{2-k}(-D)) \to \mathbb{Q}_p$$

where we use the Kodaira-Spencer isomorphism $\Omega^1_{X_0(p)/\mathbb{Q}_p} \simeq \omega^2(-D)$. We modify this pairing, and set :

$$\langle , \rangle = \langle ., w^\star . \rangle_0.$$

Lemma 5.6. *For any $(f, g) \in \mathrm{H}^0(X_0(p), \omega^k) \times \mathrm{H}^1(X_0(p), \omega^{2-k}(-D))$, we have: $\langle f, U_p g\rangle = \langle U_p f, g\rangle$.*

Proof. For any $(f, g) \in \mathrm{H}^0(X_0(p), \omega^k) \times \mathrm{H}^1(X_0(p), \omega^{2-k}(-D))$, we have: $\langle U_p^{naive} f, g\rangle_0 = \langle f, (U_p^{naive})^t g\rangle$ where $(U_p^{naive})^t$ is the operator associated to the transpose of C, and is obtained as follows:

$$\mathrm{R}\Gamma(X_0(p), D(\omega^k)) \xrightarrow{p_1^\star} \mathrm{R}\Gamma(C, p_1^\star D(\omega^k)) \to \mathrm{R}\Gamma(C, p_2^\star D(\omega^k)) \xrightarrow{\mathrm{tr}_{p_2}} \mathrm{R}\Gamma(X_0(p), D(\omega^k)).$$

We observe that the determination of the adjoint of U_p^{naive} as the operator associated to the transpose of C uses the compatibility property of diagram 1. We have a commutative diagram :

$$\begin{tikzcd}
\mathrm{R}\Gamma(X_0(p), D(\omega^k)) \arrow[r, "p_1^\star"] & \mathrm{R}\Gamma(C, p_1^\star D(\omega^k)) \arrow[r] & \mathrm{R}\Gamma(C, p_2^\star D(\omega^k)) \arrow[r, "\mathrm{tr}_{p_2}"] & \mathrm{R}\Gamma(X_0(p), D(\omega^k)) \\
\mathrm{R}\Gamma(X_0(p), D(\omega^k)) \arrow[r, "q_2^\star"] \arrow[u, "w^\star"] & \mathrm{R}\Gamma(C, q_2^\star D(\omega^k)) \arrow[r] \arrow[u, "\iota^\star"] & \mathrm{R}\Gamma(C, q_1^\star D(\omega^k)) \arrow[r, "\mathrm{tr}_{q_1}"] \arrow[u, "\iota^\star"] & \mathrm{R}\Gamma(X_0(p), D(\omega^k)) \arrow[u, "w^\star \langle p\rangle^{-1}"]
\end{tikzcd}$$

We see that $\langle U_p^{naive} f, wg\rangle_0 = \langle f, w^\star U_p^{naive} g\rangle_0$. We now check that the normalizing factors are correct so that $\langle U_p f, g\rangle = \langle f, U_p g\rangle$. □

5.7.3. *The p-adic pairing.* We now work again over $\mathscr{O}_{\mathcal{W}_r}$. We let $X_0(p)^{m,\dagger} = \mathrm{colim}_{a\to 1} X_0(p)_{[a,1]}$. We let $X_0(p)^{et,\dagger} = \mathrm{colim}_{a\to 0} X_0(p)_{[0,a]}$. The Atkin-Lehner map is an isomorphism $w : X_0(p)^{m,\dagger} \to X_0(p)^{et,\dagger}$ and there is an isomorphism $w : w^\star \omega^{\kappa^{un}} \to \omega^{\kappa^{un}}$ (compare with sections 5.5.4 and 5.5.5).

Lemma 5.7. *We have a canonical perfect pairing $\langle , \rangle_0 : \mathrm{H}^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{un}}) \times \mathrm{H}^1_c(X_0(p)^{m,\dagger}, \omega^{2-\kappa^{un}}(-D)) \to \mathscr{O}_{\mathcal{W}_r}$. Moreover, $\mathrm{H}^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{un}})$ is a compact inductive limit of projective Banach spaces over $\mathscr{O}_{\mathcal{W}_r}$, $\mathrm{H}^1_c(X_0(p)^{m,\dagger}, \omega^{2-\kappa^{un}}(-D))$ is a compact projective limit of projective Banach spaces over $\mathscr{O}_{\mathcal{W}_r}$, and both spaces are strong duals of each other.*

Proof. The pairing is obtained as follows:

$$\mathrm{H}^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{un}}) \times \mathrm{H}^1_c(X_0(p)^{m,\dagger}, \omega^{2-\kappa^{un}}(-D))$$
$$\to \mathrm{H}^1_c(X_0(p)^{m,\dagger}, \Omega^1_{X_0(p)^m/\mathbb{Q}_p} \hat{\otimes}_{\mathbb{Q}_p} \mathscr{O}_{\mathcal{W}_r}) \xrightarrow{\mathrm{res}_{X_0(p)^m}} \mathscr{O}_{\mathcal{W}_r}.$$

We prove the remaining claims. Let $\mathcal{W}^i$ be the connected component of the character $i : x \mapsto x^i$ for $i = 0, \cdots, p-2$ in $\mathcal{W}$. Then for all i, $\omega^{\kappa^{un}}|_{\mathcal{W}^i_r} = \omega^i \hat{\otimes}_{\mathbb{Q}_p} \mathscr{O}_{\mathcal{W}^i_r}$ is an "isotrivial" sheaf. This follows from the existence of the Eisenstein family (see [Pil13], the final discussion below proposition 6.2). Therefore, all the statements of the lemma are reduced to similar statement for classical invertible sheaves, and they follow from [GK00], thm. 4.4. □

We deduce form this lemma that there is a canonical pairing :

$$\langle , \rangle : \mathrm{H}^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{un}}) \times \mathrm{H}^1_c(X_0(p)^{et,\dagger}, \omega^{2-\kappa^{un}}(-D)) \to \mathscr{O}_{\mathcal{W}_r}$$

by putting $\langle , \rangle = \langle , w^\star . \rangle_0$. For this pairing, $\langle U_p ., . \rangle = \langle ., U_p . \rangle$. We have by definition that $\mathrm{H}^0(X_0(p)^{m,\dagger}, \omega^{\kappa^{un}}) = \mathrm{colim}_{a \to 1} \mathrm{H}^0(X_0(p)_{[a,1]}, \omega^{\kappa^{un}})$.

Lemma 5.8. *We have a canonical isomorphism :* $\mathrm{H}^1_c(X_0(p)^{et,\dagger}, \omega^{2-\kappa^{un}}(-D)) = \lim_{a \to 0} \mathrm{H}^1_{X_0(p)_{[0,a[}}(X_0(p), \omega^{2-\kappa^{un}}(-D))$.

Proof. We have a short exact sequence for $0 < a < b$ small enough :

$$0 \to \mathrm{H}^0(X_0(p)_{[0,b]}, \omega^{2-\kappa^{un}}(-D)) \to \mathrm{H}^0(X_0(p)_{[a,b]}, \omega^{2-\kappa^{un}}(-D))$$
$$\to \mathrm{H}^1_{X_0(p)_{[0,a[}}(X_0(p)_{[0,b]}, \omega^{2-\kappa^{un}}(-D)) \to 0.$$

Passing to the limit as $a \to 0$ proves the lemma. □

Therefore the operator U_p is compact on both cohomology groups. We deduce from the pairing that the characteristic series of U_p are the same in degree 0 and degree 1, so that $\mathcal{X} = \mathcal{Z}$. We have a canonical perfect pairing $\langle , \rangle : \mathcal{M} \times \mathcal{N} \to w^{-1}\mathscr{O}_{\mathcal{W}_r}$, for which $\langle zf, g \rangle = \langle f, zg \rangle$ for all $(z, f, g) \in \mathscr{O}_{\mathcal{Z}} \times \mathcal{M} \times \mathcal{N}$ and $\langle hf, g \rangle = \langle f, hg \rangle$ for any Hecke operator h of level prime to Np. We deduce that the eigencurves $\mathcal{C}$ and $\mathcal{D}$ are canonically isomorphic.

References

[AIS14] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens, *Overconvergent modular sheaves and modular forms for* $\mathbf{GL}_{2/F}$, Israel J. Math. **201** (2014), no. 1, 299–359.

[Bar78] Wolfgang Bartenwerfer, *Die erste "metrische" Kohomologiegruppe glatter affinoider Räume*, Nederl. Akad. Wetensch. Proc. Ser. A **40** (1978), no. 1, 1–14.

[BCGP18] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni, *Abelian surfaces over totally real fields are potentially modular*, preprint, 2018.

[Bey97] Peter Beyer, *On Serre-duality for coherent sheaves on rigid-analytic spaces*, Manuscripta Math. **93** (1997), no. 2, 219–245.

[BG98] Siegfried Bosch and Ulrich Görtz, *Coherent modules and their descent on relative rigid spaces*, J. Reine Angew. Math. **495** (1998), 119–134.

[BG14] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187.

[CM98] R. Coleman and B. Mazur, *The eigencurve*, Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, Cambridge, 1998, pp. 1–113.

[Col97] Robert F. Coleman, *P -adic Banach spaces and families of modular forms*, Inventiones Mathematicae **127** (1997), 417–479.

[DR73] Pierre Deligne and Michael Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 143–316. Lecture Notes in Math., Vol. 349.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[Far10] Laurent Fargues, *La filtration de Harder-Narasimhan des schémas en groupes finis et plats*, J. Reine Angew. Math. **645** (2010), 1–39.

[Far11] ———, *La filtration canonique des points de torsion des groupes p-divisibles*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 6, 905–961, With collaboration of Yichao Tian.

[FP19] Najmuddin Fakhruddin and Vincent Pilloni, *Hecke operators and the coherent cohomology of shimura varieties*, 2019.

[GK00] Elmar Grosse-Klönne, *Rigid analytic spaces with overconvergent structure sheaf*, J. Reine Angew. Math. **519** (2000), 73–95.

[Gro05] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 4, Société Mathématique de France, Paris, 2005, Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966.

[Har72] ———, *Cohomology with compact supports for coherent sheaves on an algebraic variety*, Math. Ann. **195** (1972), 199–207.

[Hid86] Haruzo Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 2, 231–273.

[Kas06] Payman L. Kassaei, *A gluing lemma and overconvergent modular forms*, Duke Math. J. **132** (2006), no. 3, 509–529.

[Kat73] Nicholas M. Katz, *p-adic properties of modular schemes and modular forms*, 69–190. Lecture Notes in Mathematics, Vol. 350.

[Pil11] Vincent Pilloni, *Prolongement analytique sur les variétés de Siegel*, Duke Math. J. **157** (2011), no. 1, 167–222.

[Pil13] ———, *Overconvergent modular forms*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 1, 219–239.

[Pil17] ———, *Higher coherent cohomology and p-adic modular forms of singular weight*, preprint, 2017.

[Sta13] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2013.

[vdP92] Marius van der Put, *Serre duality for rigid analytic spaces*, Indag. Math. (N.S.) **3** (1992), no. 2, 219–235.