Newton Polygons: around the $fg + 1$ problem

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réunion CompA

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Plan

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Motivations: the $fg + 1$ puzzle

- At the beginning, a conjecture on the number of real roots of a "sparse" polynomial:

$$f(X) = \sum_{i=1}^{p} \prod_{j=1}^{q} f_{i,j}(X)$$

- Motivation: Descartes' rule of signs gives a bound for a polynomial with t monomials
- We understand fg, but $fg + 1$ is already a puzzle...
- Here: we study a similar problem on polygons

$$\text{Real roots} \Longleftrightarrow \text{Points in a convex hull}$$

- A corollary: $\mathtt{VP} \neq \mathtt{VNP}$
  $\Longrightarrow$ an interesting problem

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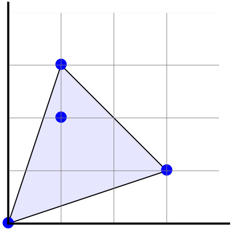
$$\text{Real roots} \Longleftrightarrow \text{Points in a convex hull}$$

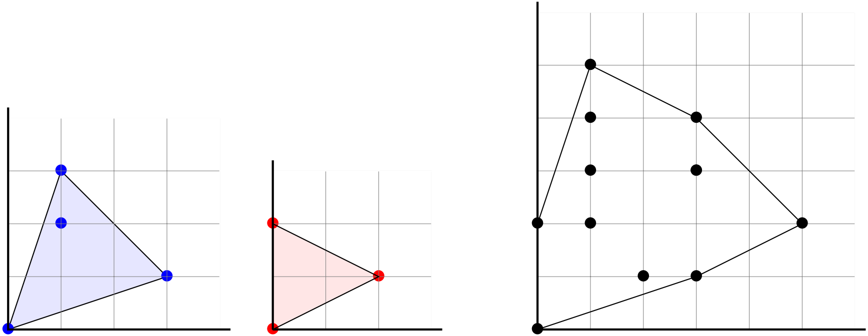
- A corollary: $\mathtt{VP} \neq \mathtt{VNP}$
$\Longrightarrow$ an interesting problem in connexion with algebraic complexity

Newton polygon

- Let $f(X, Y) = \sum_i \alpha_i X^{a_i} Y^{b_i}$
- Monomials of f: $\text{Mon}(f) = \{(a_i, b_i), \alpha_i \neq 0\}$
- Newton polygon: $\text{Newt}(f) = \text{Conv}(\text{Mon}(f))$
- Example: $f(X, Y) = 1 + 2X^3Y + XY^2 + XY^3$



Link with Minkowski sums



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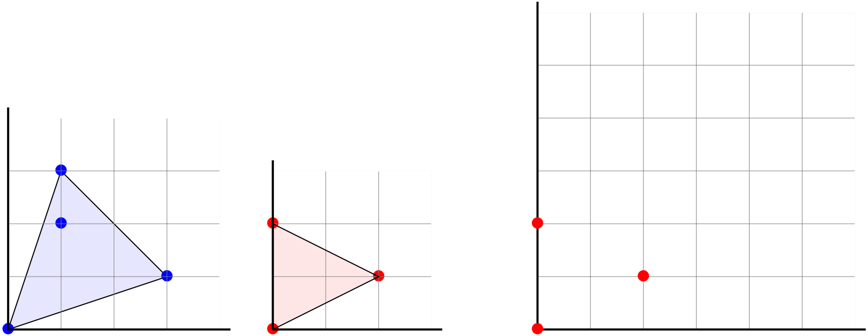

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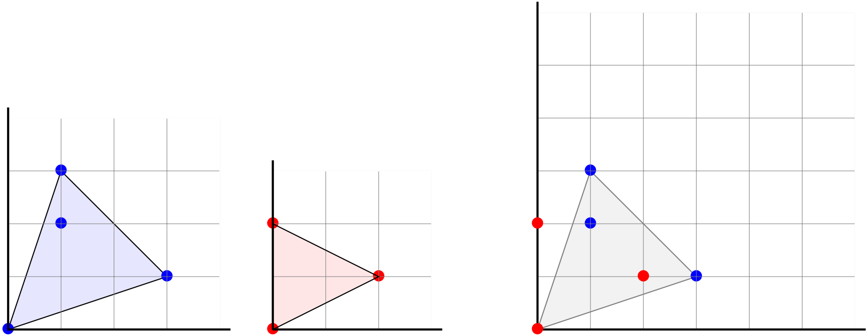

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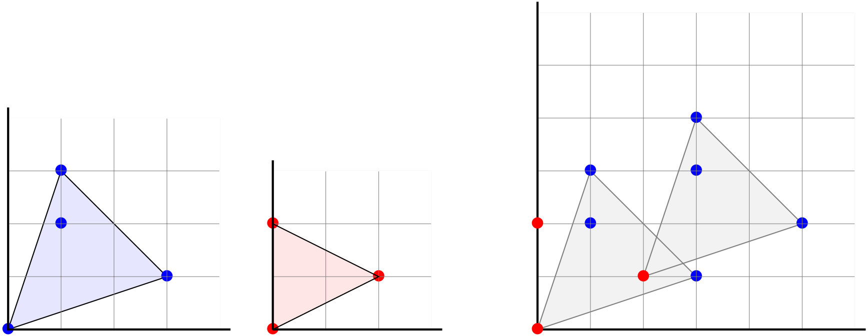

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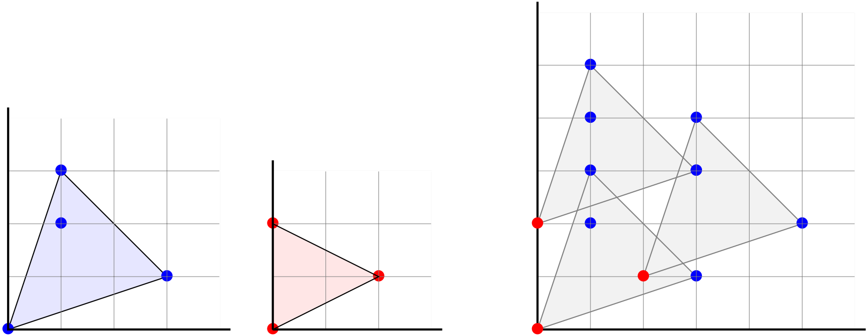

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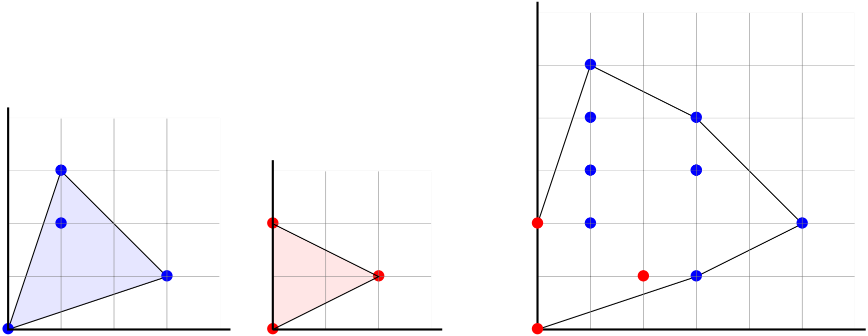

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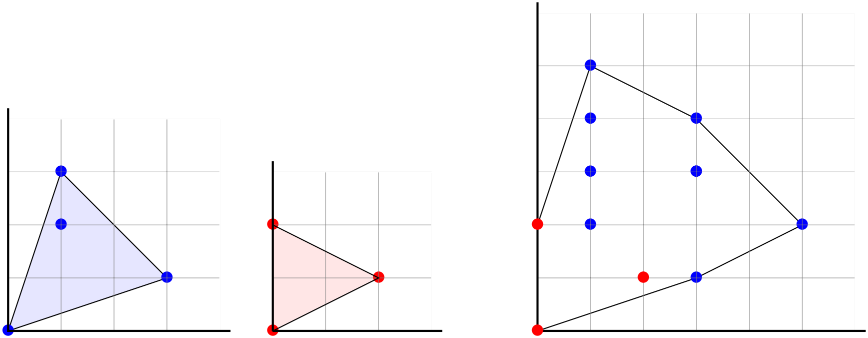

- $P + Q = \{p + q, p \in P, q \in Q\}$

Lemma (Ostrowski)

$Newt(fg) = Newt(f) + Newt(g)$

- We can build the edges of $P + Q$ from those of P and Q

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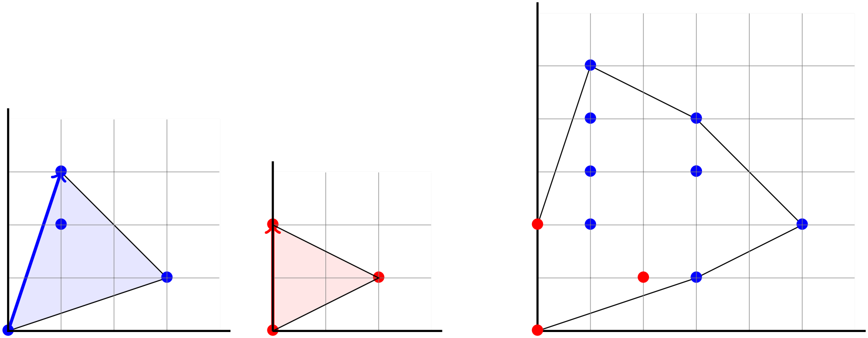

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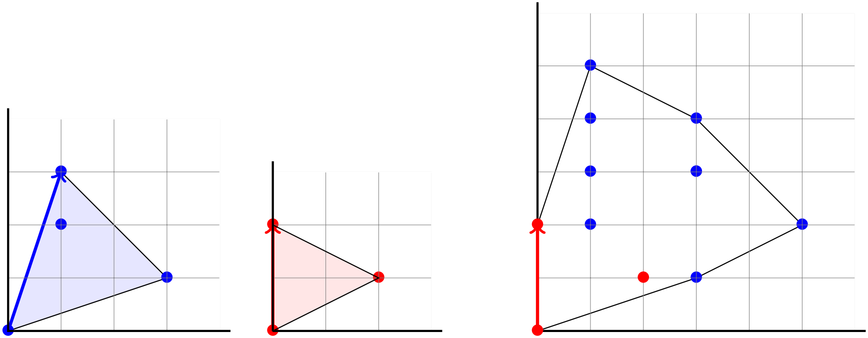

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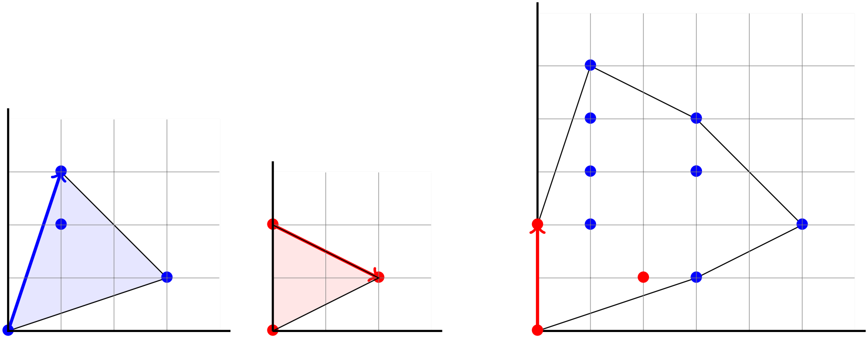

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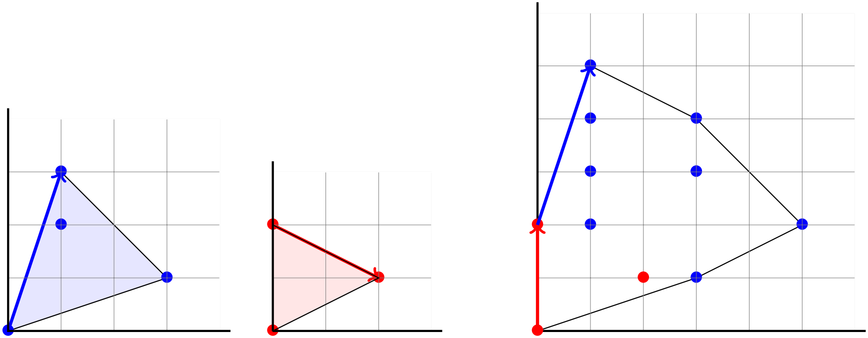

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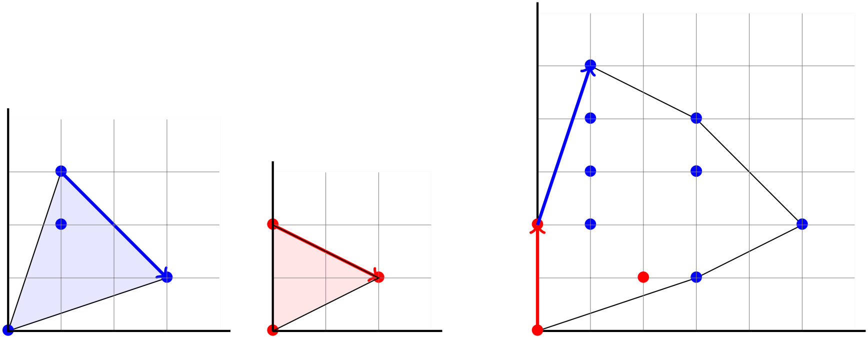

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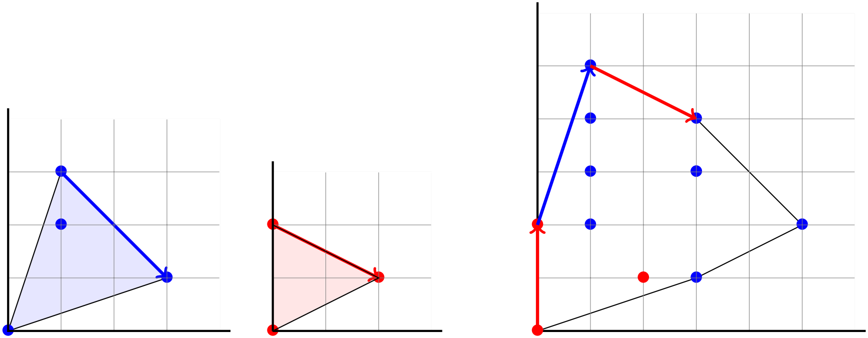

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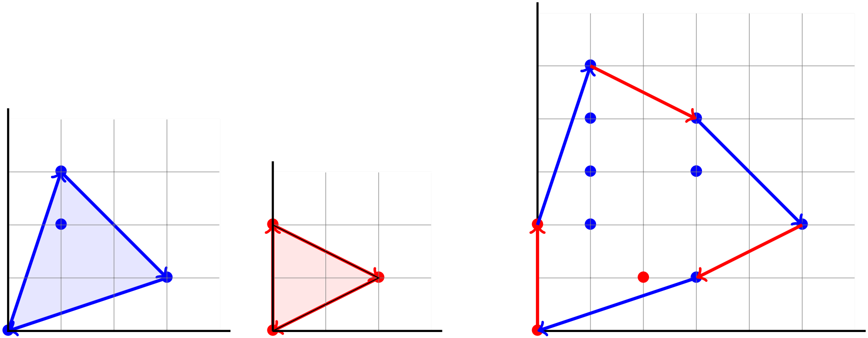

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The $fg + 1$ problem

- $\text{Newt}(fg) = \text{Newt}(f) + \text{Newt}(g)$
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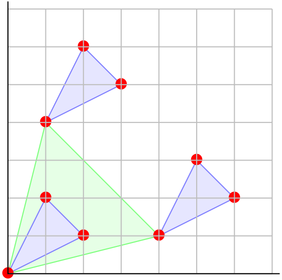

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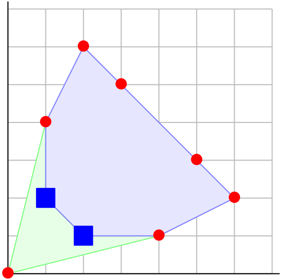

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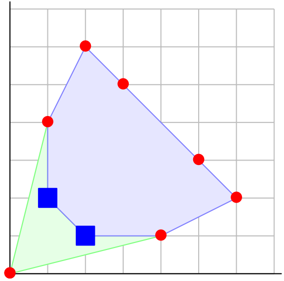

Trivial bound: t^2

Better bound: $\mathcal{O}(t^{4/3})$

Expectation: linear bound...

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- Main idea: decompose $P = \text{Mon}(f)$, $Q = \text{Mon}(g)$, and the set of new points in the Newton polygon, and then study the differents "interactions"
- We will use the following theorem:

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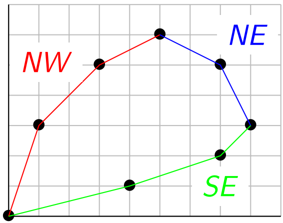
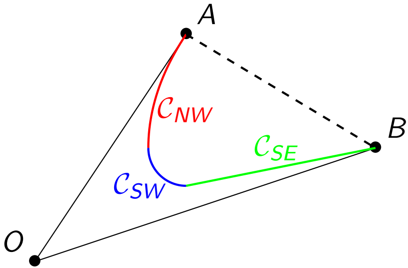
- Suppose that P and Q are convex

First structural step

- We decompose P and Q into convex chains: P_{NW}, P_{SE}, etc.





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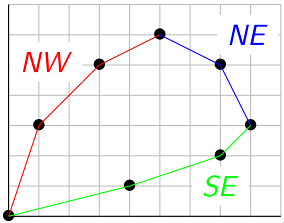

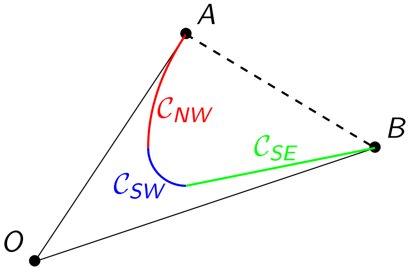

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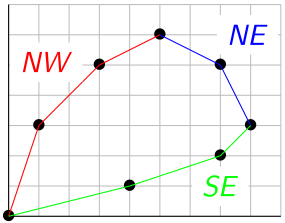
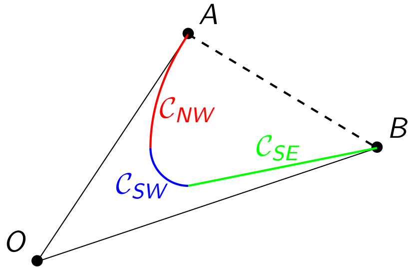
- Goal: bound the $|(P_d + Q_{d'}) \cap \mathcal{C}_{d''}|$ separately
  $\Rightarrow$ different arguments

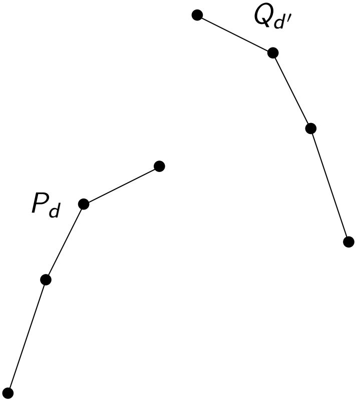




Argument 1: "$P + P$ convex"

Lemma

If $d \neq d'$, then $|(P_d + Q_{d'}) \cap \mathcal{C}| \leq 2(|P_d| + |Q_{d'}|)$



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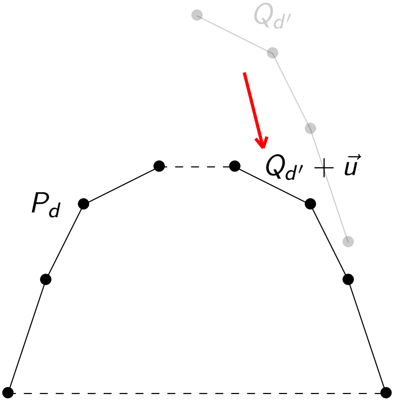

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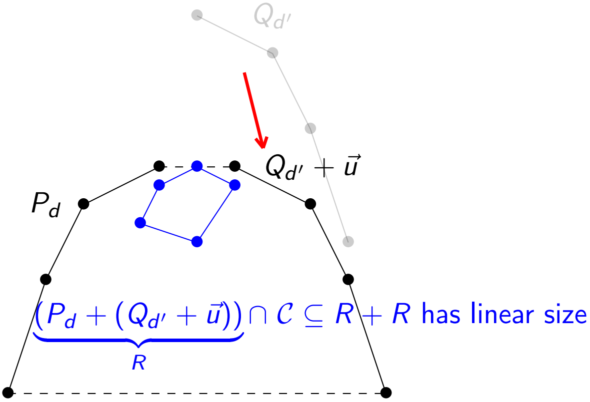

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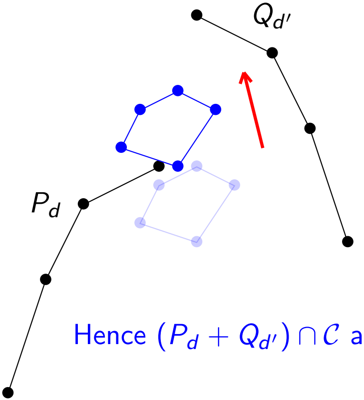

Hence $(P_d + Q_{d'}) \cap \mathcal{C}$ also

Argument 2: chains

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- We build the graph $G = (V, E)$ with $V = P_d \cup Q_d$ and $(p, q) \in E \Leftrightarrow p + q \in \mathcal{C}_{d'}$
- If p has three neighbors q_1, q_2 and q_3, they form a chain...

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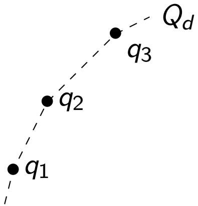

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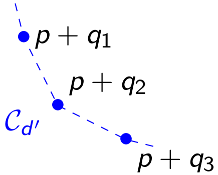

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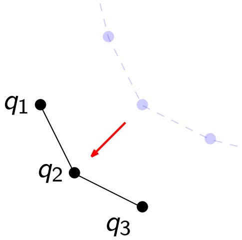

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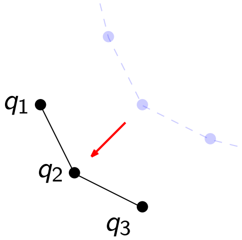

- of type d because $q_i \in Q_d$
- of type d' by translation of a subset of $\mathcal{C}_{d'}$
- Contradiction since $d \neq d'$

Argument 3: slopes

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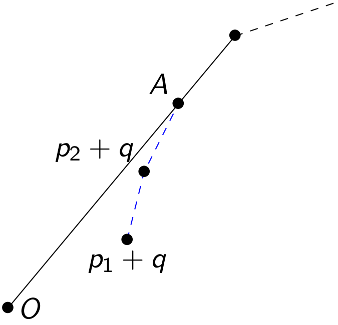

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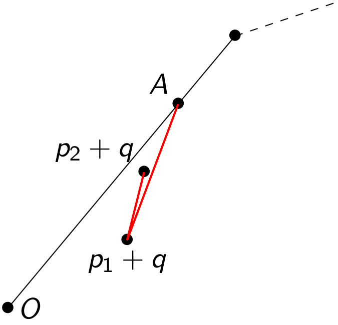

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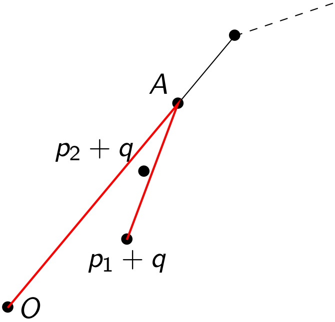

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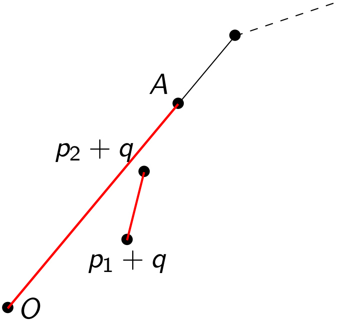

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- If q has two neighbors p_1 and p_2...
- $\text{slope}(p_1, p_2) > \text{slope}(p_1 + q, A)$
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Argument 3: slopes

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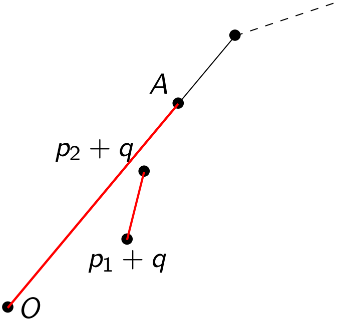

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Theorem

If $Mon(f)$ and $Mon(g)$ are convex, then $|Newt(fg + 1)|$ is linear

Generalizations

- P convex: we can extend the second and third arguments:
  - If $d \neq d'$, $|(P_d + Q_d) \cap \mathcal{C}_{d'}| \leq |P_d| + |Q_d|$
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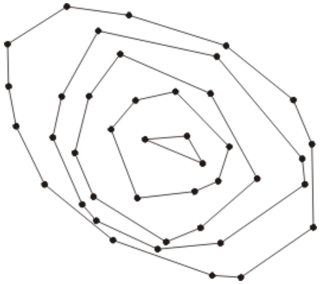
Plan

Onion Peeling

- Decompose a set of points into "layers"



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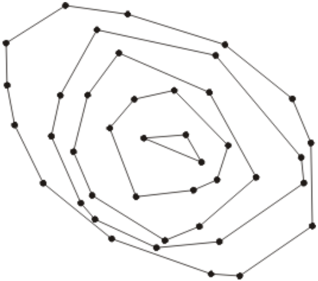

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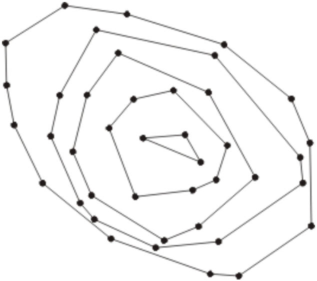

- Motivation: a $fg - h$ problem
- Goals: in the $P + Q$ case: size of the layers, number of layers, structural properties

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- If P and Q are any point set, we get a $\mathcal{O}(k\ n \log n)$ bound by studying the links between the layers of P, Q and $P + Q$

Plan

Conclusion

- About the $fg + 1$ problem: a linear bound when one of the set of monomials is convex
- First generalization to study: if P and Q have two layers
  The problem: $(P_{NW,2} + Q_{NW,2}) \cap \mathcal{C}_{NW}$
- Lower bounds ? Even $\alpha \cdot n$ with $\alpha > 2$ seems hard...

Questions?