# $\mathbb{E}$xercise sheet 2 : Construction and first properties of the Brownian motion.

**Exercise 1** — *Transformations.*
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(1) Show that for any $\lambda \in \mathbb{R}_+^\star$, the process $(\lambda^{-1/2}B_{\lambda t})_{t\geq 0}$ is a Brownian motion.
(2) Show that $B_1 - B_{1-t}$ is a Brownian motion on $[0,1]$.
(3) Show that if we set $X_t = tB_{1/t}$ for $t > 0$ and $X_0 = 0$, then $(X_t)_{t\geq 0}$ is a Brownian motion, as long as one allows oneself to redefine this stochastic process on a negligible event.

**Exercise 2** — *Constructing a Brownian motion indexed by* $\mathbb{R}+$.
Let $(B^{(n)})$ be a sequence of independent Brownian motions defined on $[0,1]$. Define the following function $B : \mathbb{R}+ \to \mathbb{R}$:

$$B : t \mapsto B^{(\lfloor t \rfloor)}_{t-\lfloor t \rfloor} + \sum_{i=0}^{\lfloor t \rfloor - 1} B^{(i)}_1 .$$

Show that $B$ is a Brownian motion.

**Exercise 3** — *A nowhere continuous version of the Brownian motion.*
Given a Brownian motion, build a probability space with a random variable $B_t$ for all $t \in \mathbb{R}_+$ such that:

- for every $\omega$, the function $t \mapsto B_t(\omega)$ is nowhere continuous;
- for every $t_1 < \ldots < t_n$, the vector $(B_{t_1}, \ldots, B_{t_n})$ has the same distribution as if $B$ were a Brownian motion.

*Hint:* change the value of $B$ on a countable dense random subset of $\mathbb{R}$, so that the value at a fixed deterministic time is almost surely not changed.

**Exercise 4** — *Brownian motion is nowhere monotonous.*
Let $B$ be a Brownian motion. Show that almost surely, the function $t \mapsto B_t$ is not monotonous on any nonempty open interval.

**Exercise 5** — $L^2$ *theory and construction of the Brownian motion.*
Let $H = L^2([0,1])$ with the usual inner product. For $t \geq 0$ let $I_t = \mathbb{1}_{[0,t]} \in H$. We also set $(e_i)_{i\in\mathbb{N}}$ to be an orthonormal basis of $H$.

(1) Check that $\langle I_s, I_t \rangle = s \wedge t$.

(2) Suppose we could build a standard Gaussian random variable in $H$, that is $\xi \in H$ such that for every $x \in H$, $\langle x, \xi \rangle \sim \mathcal{N}(0, \|x\|)$. How could a Gaussian process $(B_t)_{t\in[0,1]}$ such that $\mathrm{Cov}(B_s, B_t) = s \wedge t$ be built from it ?

(3) Let $Z_i = \langle \xi, e_i \rangle$, so that $\xi = \sum_{i\in\mathbb{N}} Z_i e_i$. Show that the $(Z_i)$ are independent standard Gaussians (*Hint :* compute the characteristic function of $(Z_{i_1}, \dots Z_{i_p})$ for $p \geq 1$ and $(i_1, \dots i_p) \in \mathbb{N}^p$). Deduce that we could then write the following equality in $L^2$:

$$B_t = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s) ds \tag{†}$$

(4) Show that $\xi$ can not exist[1] (hint: compute its norm with the help of the basis $e$)

(5) Nevertheless, show that in the case of the Haar wavelet basis ef $L^2$: $h_0 = 1$ and for $n \geq 0$ and $0 \leq k < 2^n$

$$h_{k,n} := 2^{n/2} \left( \mathbb{1}_{[2k/2^{n+1},(2k+1)/2^{n+1}]} - \mathbb{1}_{[(2k+1)/2^{n+1},(2k+2)/2^{n+1}]} \right),$$

the series in (†) coincides with the Lévy construction of Brownian motion (and hence converges almost surely in $\mathcal{C}([0,1])$ to a Brownian motion).

(6) What do we obtain in (†) with the Fourier basis $e_0 = 1$, and $e_m(t) = \sqrt{2}\cos(\pi m t)$?

(7) $\star$ Show also the almost sure convergence in $\mathcal{C}([0,1])$ of this series.

[1] It is possible to build $\xi$ in the space $D'$ of distributions. It is then called a *white noise*.